# Market Liquidity and Asset Prices under Costly Participation*

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**Abstract**

In this paper, we develop an equilibrium model for market liquidity and its impact on asset prices when constant participation in the market is costly. We show that, even when investors' trading needs are perfectly matched, costly participation prevents them from synchronize their trades, which gives rise to the need for liquidity in the market. Fluctuations in liquidity needs cause asset prices to deviate from the fundamentals. Moreover, these price deviations tend to have large magnitudes in absence of any aggregate shocks, resembling what can be called "liquidity crashes", and lead to fat tails in return distributions. We also show that the lack of coordination among investors in the demand and the supply of liquidity generates negative externalities, and the loss in social welfare can out-weight the savings on participation costs.

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# 1. Introduction

It is well recognized that liquidity is of critical importance to the stability and the efficiency of the financial market.[1] Yet, there is little consensus about exactly what liquidity is, what determines it and how it impacts asset prices. Market frictions such as transactions costs have been considered as important determinants of liquidity and asset prices.[2] But the precise nature of this link and its significance is not well understood due to the difficulty in analyzing the complex interactions among diverse market participants in the presence of transactions costs. Most of the existing analysis examine how market frictions affect the provision of liquidity and the resulting prices, taking as given the liquidity needs that often arise from exogenous changes in preferences or portfolio positions.[3] Such an approach ignores the fact that it is the same transactions costs that give rise to the need for liquidity in the first place. To study liquidity, it is essential to understand how transactions costs give rise to the need for liquidity, what drives the liquidity need and how liquidity impact prices.

In this paper, we study how transactions costs lead to investors' needs for liquidity, and how such needs affect asset prices. We focus our attention on a specific form of transactions costs, namely, the cost to participate in the market. We show that participation costs prevent investors from being present in the market at all times. Their infrequent presence in the market leads to non-synchronization in their trades even when their underlying trading needs are perfectly matched. This endogenous non-synchronous trading gives rise to order imbalances and needs for liquidity in certain states, which can cause large price movements in absence of any aggregate shocks to the fundamentals. The high likelihood of these large price movements also gives rise to fat-tails in returns.

Two elements are essential to liquidity: the need to trade and the cost to trade. In the absence of any trading needs, there will be no need for liquidity. In the absence of

[1] See, for example, Allen and Gale (1994), Grossman and Miller (1988), Kiyotaki and Moore (1997), Holmstrom and Tirole (1996, 1998).

[2] See, for example, Amihud and Mendelson (1986), Constantinides (1986), Campbell, Grossman, and Wang (1993), Vayanos (1998), Vayanos (2004), Chordia, Roll, and Subrahmanyam (2000), Pastor and Stambaugh (2003), Acharya and Pedersen (2004), Lo, Mamaysky, and Wang (2004).

[3] See, for example, Grossman and Miller (1988), Allen and Gale (1994), Huang (2003), Kyle and Xiong (2001). In most of the market micro-structure literature, which has liquidity as one of its central focus, the need for liquidity, as described by the order flow process, is often taken as given. See, for example, Glosten and Milgrom (1985), Kyle (1985), Stoll (1985, 1989). There are a few authors who have modelled the liquidity needs endogenously, such as Vayanos (1998) and Lo, Mamaysky, and Wang (2004).

transactions costs, investors can trade in the market at all times in response to exogenous shocks and prices adjust to match the buyers with the sellers. Although in this case the equilibrium price may well adjust to equalize the demand and supply, the market is perfectly liquid in the sense the market prices reflect the “fair value” of the assets. Actual markets do not function in this “gigantic town meeting” style, as Grossman and Miller (1988) called it, where all potential buyers and sellers are present at all times and trades are conducted to balance the full demand and supply. Costs prevent potential buyers and sellers from constantly participating in the market. Instead, only a subset of buyers and sellers are in the market at any given instant. When a trader arrives at the market, he only faces a “partial” demand/supply. Adjustments in price fail to attract all potential buyers and sellers and to synchronize their trades. It is this non-synchronization in trading due to transactions costs that gives rise to the need for liquidity and its impact on asset prices.

We start with an economy in which potential traders have idiosyncratic risk exposures and desire to transact in a competitive financial market to share these risks. In the absence of participation costs, they are in the market at all times, representing the full demand/supply of the asset. The market price adjusts to perfectly coordinate all buyers and sellers. Each trader is getting the best price possible for his trade, reflecting its full market value. Moreover, the price is fully determined by the “fundamentals” of the asset, namely, its future payoffs. In particular, the price does not depend on the idiosyncratic trading needs of individual traders. In the presence of participation costs, potential traders will no longer be in the market at all times. Each trader comes to the market only when he wants to trade. He cannot be sure who else will be in the market and only faces a partial demand/supply. The price fails to coordinate all potential buyers and sellers. This lack of coordination among the potential market participants leads to temporary order imbalances and need for liquidity in the market, which then causes the price to deviate from its fundamental value.

We show that the order imbalance and the need for liquidity is highly nonlinear in the idiosyncratic shocks which drive the diversity in investors’ risk exposure and their trading needs. First, given the level of aggregate risk exposure, the order imbalance is always onesided. In particular, when the aggregate risk exposure is positive, investors with higher than average risk exposures always have higher gains from trading and participate more in the market. Their sell orders will overwhelm the buy orders in the market and drive down the asset price. Next, the price drop from the order imbalance tends to occur at finite sizes, giving what can be called "liquidity crashes". This is because when the magnitude of idiosyncratic shocks is very small or large, investors' participation decisions are highly correlated. Either they all participate when gains from trading are large (for large idiosyncratic shocks) or not when gains from trading are small (for small idiosyncratic shocks). Only for intermediate magnitudes of idiosyncratic shocks, there is less correlation in investors' participation decisions and more non-synchronization in their trades, which lead to the negative impact on prices of finite magnitudes. As a result, when liquidity need arises in the market, it is often of finite sizes causing prices to drop discretely in absence of any shocks to the fundamentals. Moreover, such a non-linear price impact of liquidity gives rise to the fat tails as well as high volatility in asset returns.

The added price sensitivity to liquidity trades provides a quantitative measure of illiquidity in the market. We show that this measure of illiquidity increases significantly with the level of aggregate risk of the economy. With the lack of liquidity, individual market participants can no longer achieve efficient risk sharing, thus have to bear certain idiosyncratic risks, in addition to the aggregate risk. As the level of aggregate risk increases, each market participant's tolerance for any idiosyncratic risk decreases. Thus, the slope of his demand for the asset also increases.

The lack of coordination among participants to trade generates negative externality. Withdrawal from the market by a trader for cost saving not only gives up his own gains from trading but also reduces the gains from trading for other traders. This discourages other traders from participating in the market, which in turn further discourages the trader himself. As a result of this negative externality, the overall welfare loss can be very high. In particular, it is possible, as we show in the paper, that traders can be made better off if they are all forced to pay the cost and participate in the market, rather than having the individual choice to participate.

Our analysis is closely related to Grossman and Miller (1988), who consider the role of market makers in providing liquidity and reducing price volatility. They take as given the non-synchronization in trades and examine the impact of market makers who face participation costs. For their purpose, which is to analyze the role of market makers in liquidity provision, such a simplification is not crucial. For our purpose, which is to examine the fundamental link between transactions costs and liquidity, it is essential to explicitly model how transactions costs endogenously lead to the need for liquidity in the first place and how liquidity needs generate additional price fluctuations. Like Grossman and Miller (1988), most of the literature on liquidity focuses on the supply of liquidity, assuming certain demand for liquidity (see, for example, Allen and Gale (1994) and Glosten and Milgrom (1985), among others). Our analysis provides a more comprehensive analysis for demand for liquidity as well as the supply of liquidity.

Our model also shares many features with the model of Lo, Mamaysky, and Wang (2004), who consider the impact of transactions costs on trading volume and the level of asset prices. The main distinction is that we explicitly model the possible imbalance in liquidity needs and its impact on prices while they do not. They avoid the coordination problem among traders by allowing the cost to be allocated endogenously so that the trading needs of different market participants are always synchronized in equilibrium. Thus, there is no imbalance between liquidity provision and consumption and asset prices are not affected by liquidity needs. But in practice, the imbalance in liquidity needs, despite the presence of market makers, is a key element in determining the importance of liquidity and its impact on asset prices, which we examine in this paper.

The paper proceeds as follows. Section 2. describes the basic model. Section 3. solves for the optimal participation and trading decisions of different market participants and the intertemporal equilibrium of the economy. In Section 5., we discuss the measure of liquidity, how it is driven by the underlying trading needs of different market participants and how it affects the behavior of asset prices. In Section 6., we analyze the welfare implications of liquidity. Section 7. concludes. Proofs are in the appendix.

# 2. The Model

We hope to construct a model that captures several important factors in analyzing liquidity, especially, the need to trade and the cost to constantly participate in the market. For tractability, we consider a finite-horizon setting, although it can be embedded in an infinite horizon setting. We maintain parsimony to avoid any unnecessary complexities in our analysis and return to the discussion of extensions later in the paper.

## 2.1. Economy

We consider a finite horizon economy with four dates, $t = 0, 1, 2, 3$.

**Securities Market**

There are two securities traded in a competitive securities market, a risk-free bond and a risky stock. The risk-free bond gives a sure terminal payoff of 1 at date $t = 3$ and the stock gives a risky terminal payoff $V_3$, given by

$$V_3 = V + v_3 \tag{1}$$

where $V$ is a positive constant and $v_3$ is a normal random variable with a mean of zero and a volatility of $\sigma_v$. The bond will be used as the numeraire. Hence, its price remains at one over time. The price of the stock at date $t$ is denoted by $P_t$. Clearly, $P_3 = V_3$.

**Agents**

There is a continuum of agents in the economy, represented by the interval $[0,\ \mu + 2\nu]$. The agents consist of two types, with population weight $\mu$ and $2\nu$, respectively, who have different trading needs and face different transactions costs in the securities market. The first type of agents, denoted by $m$, are "market makers". They have no inherent trading needs, but are present in the securities market at all times, ready to trade with others. The second type of agents are "traders", who have inherent trading needs and have to pay a cost to participate in the market. There are two equal subgroups of traders, indexed by $i = a, b$, respectively, each with population weight $\nu$. They arrive at the market at date $t = 1$.

Each agent is initially endowed with zero units of the bond and zero shares of the stock. Agent $i$ receives a non-traded income $N_3^i$ at the terminal date $t = 3$, where $i = a, b, m$. The non-traded income is given by

$$N_3^i = (Y + Z_1^i + Z_2^i)n_3, \quad i = a, b, m \tag{2a}$$

$$Z_t^m = 0,\ Z_t^a = Z_t,\ Z_t^b = -Z_t, \quad t = 1, 2 \tag{2b}$$

where $Y$, $Z_1$, $Z_2$, and $n_3$ are mutually independent, normal random variables with mean of zero and volatility of $\sigma_y$, $\sigma_{z_1}$, $\sigma_{z_2}$, $\sigma_n$, respectively. The risk of the non-traded income, $n_3$, is assumed to be correlated with the risk of the stock payoff $v_3$. For simplicity, we let $n_3 = v_3$.

Let $X_1^i = Z_1^i$ and $X_2^i = Z_1^i + Z_2^i$. For agent $i$, $Y + X_t^i$ defines his exposure to the risk of the non-traded income at date $t$. Since $\sum_{i=a,b,m} X_t^i = 0$, $Y$ gives the aggregate exposure to the non-traded risk, which is distributed equally among the agents. The idiosyncratic exposure to the non-traded risk is defined by $X_t^i$ at date $t$, $t = 1, 2$. The market makers inherit no idiosyncratic exposure and the two groups of traders inherit offsetting idiosyncratic exposures. In the absence of idiosyncratic exposures, all agents are exposed to the same amount of aggregate exposure and there will be no trading needs among the agents. In the presence of idiosyncratic exposures, however, the two groups of traders will want to trade among themselves for risk sharing. In particular, given the correlation between the non-traded risk and the stock payoff risk, they want to adjust their stock positions in order to hedge their non-traded risk. Thus, changes in the traders' idiosyncratic exposures to the non-traded risk give rise to their inherent trading needs.[4]

By definition, the idiosyncratic exposure to the non-traded risk sum to zero. That is, $X_t^a = -X_t^b$, as assumed. Thus, the traders' underlying trading needs are perfectly matched. If all traders are present in the market at all times, a seller is always matched with a buyer and there is perfect synchronization in agents' trades.

For tractability, we assume that all investors have constant absolute risk aversion (CARA) utility over their terminal wealth $W_3^i$ at date $t = 3$:

$$\mathrm{E}\left[-e^{-\alpha W_3^i}\right], \qquad i = a, b, m, \tag{3}$$

and all agents have the same risk aversion $\alpha$. In addition, we impose the following parameter constraint:

$$\alpha^2 \left(\sigma_y^2 + \sigma_{z_1}^2 + \sigma_{z_2}^2\right) \sigma_v^2 < 1 \tag{4}$$

for the model to behave properly.[5]

[4] In our setting, heterogeneity in risk exposure is merely a device to introduce the need to trade for risk-sharing, as in Wang (1994), Huang and Wang (1997), Lo, Mamaysky, and Wang (2004). Other forms of heterogeneity among agents can also generate risk-sharing trading needs, such as difference in preferences (e.g., Dumas (1992) and Wang (1996)) or beliefs (e.g., Detemple and Murthy (1994)). Our modeling choice here is mainly motivated by tractability.

[5] In appendix A, we show that if the constraint does not hold, the expected utility of traders becomes infinitely negative, which is unreasonable.

### Transactions Costs

The market makers can always trade in the securities market at no cost.[6] The traders, however, face a fixed cost $c \geq 0$ to participate in the market. There is a one-period lag between paying the cost and being able to trade in the market. This form of transactions costs is consistent with the costs of entering a market by setting up a trading operation, constantly monitoring the market, gathering and incorporating the information into their trading activities. We consider only the fixed setup cost occurred before traders transact in the market since it is sufficient to illustrate our main point of non-synchronized trading among different trader groups.

### Time Line

At dates $t = 0$, only the market makers are present in the market, and they stay in the market at all dates. At date $t = 1$, trader $i$, $i = a, b$, arrives. Although he cannot trade in the market, he can decide whether or not to pay a cost $c$ in order to trade in the next period. At date $t = 2$, those traders who have paid the participation cost can trade with both the market maker and other participating traders, and those who haven't paid the cost stay out of the market.

Let $\eta_1^i$ be the discrete choice variable at $t = 1$ for trader $i$ of whether or not to participate, where $\eta_1^i = 1$ denotes participation and $\eta_1^i = 0$ denotes no participation. We use a continuous variable $\omega_1^i$ in the range of $[0, 1]$ to denote the fraction of type $i$ traders who pay the participation cost (i.e., choose $\eta_1^i = 1$) at date $t = 1$.

We use $\theta_t^i$ to denote the number of stock shares that agent $i$ ($i = a, b, m$) holds after trading at date $t$. Before date 2, no trader is allowed to trade, and all agents simply hold their initial endowment, hence, $\theta_t^i = 0$, $i = a, b, m$, at $t = 0$ or 1. At date $t = 2$, $\theta_t^i$ is a function of date 1 participation decision. Specifically, $\theta_2^i(\eta_1^i = 0) = 0$ for the nonparticipating traders, and $\theta_2^i(\eta_1^i = 1)$ for the participating traders needs to be solved in equilibrium. Without much risk of confusion, we use $\theta_2^i$ to denote $\theta_2^i(\eta_1^i = 1)$ in the future. The date line of the economy is illustrated in Figure 1.

[6] We later generalize the setting by introducing a set up cost for market makers and endogenously determine the participation decision of market makers.

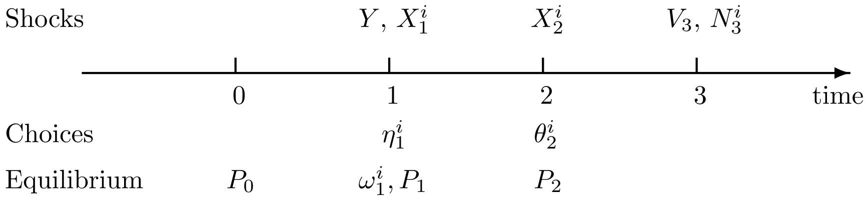



Figure 1: **The time line of the economy.**

## 2.2. Definition of Equilibrium

The equilibrium of the economy defined above requires three conditions. First, taking prices as given, all agents optimize with respect to their participation and portfolio decisions. Second, agents' participation reaches an equilibrium. Third, the securities market clears. We now specify these conditions explicitly.

**Agents' Optimization Problem**

For agent $i$, we use $F_3^i$ to denote his financial wealth at date 3, which is the total value of his bond and stock holdings. $F_3^i$ is given by

$$F_3^i = \sum_{t=1}^{2} \theta_t^i (P_{t+1} - P_t) - c^i \eta_1^i \tag{5}$$

where $c^i = c$ for $i = a, b$ and $c^i = 0$ for $i = m$. Agent $i$'s terminal wealth is then given by

$$W_3^i = F_3^i + N_3^i. \tag{6}$$

A market maker ($i = m$) can choose his optimal portfolio $\theta_t^m$ at all dates, $t = 0, 1, 2$. A trader ($i = a, b$) cannot trade at date $t = 0, 1$ and hence are forced to hold $\theta_t^i = 0$. At date 2, however, traders who decided to participate at date 1 ($\eta_1^i = 1$) can trade in the market to choose their stock holdings $\theta_2^i$, while those who decided not to participate at date 1 ($\eta_1^i = 0$) are excluded from the market and forced to set $\theta_2^i = \theta_1^i = 0$.

Let $J_t^i$ denote the value function of agent $i$ at date $t$, $i = a, b, m$. We can express his optimization problem as

$$J_t^i = \max_{\{\theta_t^i, \eta_t^i\}} \mathrm{E}_t \left[ J_{t+1}^i \right] \tag{7}$$

where $J_3^i = -e^{-\alpha W_3^i}$.

**Participation and Market Equilibrium**

At date 0 and 1, only the market makers are present in the market. They determine the competitive equilibrium price $P_0$ and $P_1$ through the market clearing condition $\theta_t^m = 0$ for $t = 0, 1$.

At date 1, traders decide whether or not to pay a cost in order to trade next period. Let $\omega_t^i$ denote the fraction of type $i$ agents who participate in the market at date $t$, $i = a, b, m$. A participation equilibrium is reached if either all traders within the same group choose identical participation decisions (i.e., $\omega_1^i = 0$, or 1), or they are indifferent between participating or not.

At date $t = 2$, both the market maker and the participating traders are present in the market. The total number of stock shares brought to the market by participating agents is $\mu\, \theta_1^m + \nu \sum_{i \in \{a,b\}} \omega_1^i \theta_1^i$, where $\theta_1^i$ is the stock holding of trader $i$ at date 1, which is assumed to be zero for all traders. Market clearing at date 1 implies $\theta_1^m = 0$. Hence, the clearing of the securities market at date 2 requires that

$$\mu\, \theta_2^m + \nu \sum_{i \in \{a,b\}} \omega_1^i \theta_2^i \;=\; 0 \tag{8}$$

which determines the securities market equilibrium at 2.

## 2.3. Equilibrium with Costless Participation

Before solving the equilibrium for the model defined above, we describe the case when participation costs are zero for all agents. This case serves as a benchmark when we examine the impact of participation costs on market liquidity and stock prices. If $c^i = 0\ \forall\ i$, all traders and market makers always participate in the market, $\omega_1^i = 1\ \forall\ i$. The equilibrium price and agents' optimal trading policies are:

$$\begin{array}{ll} P_3 = V_3, & \theta_3^i = 0 \\ P_2 = V - \alpha\sigma_v^2 Y, & \theta_2^i = -X_2^i \\ P_1 = V - \alpha\sigma_v^2 Y, & \theta_1^i = 0 \\ P_0 = V, & \theta_0^i = 0 \end{array} \tag{9}$$

where $i = a, b, m$.

We can interpret the quantity $\alpha\sigma_v^2$ as the risk premium per unit of aggregate risk exposure, which is constant over time. Clearly, $P_t$ is determined by its expected payoff $V$ and the aggregate exposure $Y$ to the non-traded risk, which we call the "fundamentals". Especially, the price does not depend on the idiosyncratic liquidity exposure $X_t^i$. It is important to point out that all traders choose to participate, and their order flows $\theta_2^i$ at date 2 depend on $X_2^i$. Hence, the market is perfectly liquid in the sense that order flows have no price impact. Moreover, the market makers perform no role in providing liquidity since their holdings are always $\theta_t^m = 0$. The lack of needs for liquidity provision is due to the fact that the traders' liquidity needs and the resulting trades exactly offset each other, hence, there is no liquidity demand at the aggregate level.

# 3. Equilibrium

We now solve for the equilibrium of the economy. As discussed earlier, the market makers always trade in the market, and the traders make their participation decision at date 1 and trading decisions at date 2 if they have chosen to participate. We solve the equilibrium backwards. First, taking the participation decisions at $t = 1$ as given, we solve the market equilibrium at $t = 2$. Next, we solve for the traders' participation decisions ($\eta_1^i$, $i = a, b$) and the participation equilibrium at $t = 1$ ($\omega_1^i$, $i = a, b$), given the market equilibrium at $t = 2$. Finally, we solve the market equilibrium at $t = 1$ and $t = 0$.

## 3.1. Market Equilibrium at $t = 2$

At $t = 2$, the state variables are $\{\omega_1^i, \theta_1^i, Y, X_2^i; i = a, b, m\}$, where $\omega_1^i$ is the fraction of type $i$ agents who will participate in the market at date 2, $\theta_1^i$ is their stock holding, $Y$ is the realization of aggregate exposure to the non-traded risk, and $X_2^i$ is the realizations of the idiosyncratic exposure for agent $i$ at date 2. For convenience, we define

$$\delta_1 \equiv \frac{\nu(\omega_1^a - \omega_1^b)}{\mu + \nu(\omega_1^a + \omega_1^b)}, \quad \text{and} \quad \delta_1^a = -\delta_1^b = \delta_1. \tag{10}$$

The quantity $\delta_1^i$ denotes the excess participation of type $i$ traders normalized by the total population weight of agents who participate in the market at date $t = 2$. For example, when $\delta_1^a > 0$, there are more group $a$ traders than group $b$ traders participating in the market.

**Proposition 1.** *Let $\delta_1$ be the difference in market participation levels for the two groups of traders, the equilibrium price at date $t = 2$ is*

$$P_2 = V - \alpha\sigma_v^2 Y - \alpha\sigma_v^2 \delta_1 X_2. \tag{11}$$

*The optimal stock holding for participating agent $i$ is*

$$\theta_2^i = \delta_1 X_2 - X_2^i, \quad i = a, b, m. \tag{12}$$

When $\delta_1 = 0$, the participation of the two groups of traders is symmetric. Since their trading decisions are driven only by $X_2^i$, which always offset each other, there is a perfect match in their buy and sell orders. As a result, the equilibrium price is not affected by the idiosyncratic shocks. Moreover, market makers hold zero position and perform no role in providing liquidity. When $\delta_1 \neq 0$, the participation of the two trader groups becomes asymmetric. Proposition 1 indicates that the idiosyncratic liquidity shock $X_2$ can affect the equilibrium price. Consider, for example, the case when $\delta_1 > 0$, i.e., more group-$a$ traders participate than group-$b$ traders. If the realization of liquidity shock is $X_2 > 0$, $X_2^a = X_2 > 0$ and $X_2^b = -X_2 < 0$; trader $a$ wants to sell while trader $b$ wants to buy. Given that there are more group-$a$ traders in the market, there will be more sellers than buyers. Consequently, the stock price has to decrease in order to attract the market makers as well as the participating group-$b$ traders to absorb the selling orders. Thus, even though traders face offsetting shocks, their asymmetric participation can give rise to mismatch in their trades, inducing market makers to provide liquidity, and causing the price to change in response to these shocks.

## 3.2. Optimal Individual Participation Decision at $t = 1$

Given the market equilibrium at $t = 2$, we now consider the participation equilibrium at date 1, which can be solved in two steps. First, taking as a given the participation decision of all other traders, we derive the optimal participation decision of an individual trader $i$ ($i = a$ or $b$). Then we find the competitive equilibrium for participation decisions.

An individual trader $i$ makes his participation decision after observing the date 1 liquidity shock $X_1^i$. He chooses to participate if and only if the gain from trading at $t = 2$ exceeds the cost to participate. Since the current liquidity shock is persistent, the current realization $X_1^i$ of the liquidity shock predicts future trading needs and the expected gain from trading. Therefore, trader $i$ is more likely to participate after receiving a larger liquidity shock.

Let $J_1^i(\eta_1^i = 1)$ and $J_1^i(\eta_1^i = 0)$ denote the value function of an individual group $i$ trader who chooses to participate or not to participate at date $t = 1$, respectively. Taking as given the difference $\delta_1$ in the market participation rate for the two groups of traders, the certainty equivalence wealth gain from participating for a trader $i$, net of the cost, can be defined as

$$g(\theta_1^i, Y, X_1^i, \delta_1^i) = -\frac{1}{\alpha} \ln \frac{J_1^i(\eta_1^i = 1)}{J_1^i(\eta_1^i = 0)} \tag{13}$$

where the minus sign on the right-hand-side adjusts for the fact that $J_1^i < 0$. The optimal decision for trader $i$ is to participate if and only if the net gain from participating is positive, or $g(\cdot) \geq 0$. The following proposition describes the optimal participation policy for an individual trader.

**Proposition 2.** *Given $Y$, $X_1^i$, $\delta_1^i$, and $\theta_1^i$, trader $i$'s participation gain is given by*

$$g(\theta_1^i, Y, X_1^i, \delta_1^i) = g_1(\delta_1^i, \theta_1^i) + g_2(\delta_1^i) - c \tag{14}$$

*where*

$$g_2(\delta_1^i) = \frac{1}{2\alpha} \ln \left[1 + (1-\delta_1^i)^2 k/(1-k)\right] \tag{15a}$$

$$g_1(\delta_1^i, \theta_1^i) = h(\delta_1^i)\big(\theta_1^i - \widehat{\theta}_1^i\big)^2 \tag{15b}$$

*and*

$$k = \alpha^2 \sigma_{z_2}^2 \sigma_v^2, \quad \widehat{\theta}_1^i = -\frac{1-\delta_1^i}{1-k\delta_1^i}(kY + X_1^i), \quad h(\delta_1^i) = \frac{\alpha\sigma_v^2(1-k\delta_1^i)^2}{2(1-k)[1-k+k(1-\delta_1^i)^2]}.$$

*Trader $i$ chooses to participate at date 1 iff*

$$\big(\theta_1^i - \widehat{\theta}_1^i\big)^2 > \big[c - g_2(\delta_1^i)\big]/h(\delta_1^i). \tag{16}$$

The gain from participating consists of three terms. The first term, $g_1(\delta_1^i, \theta_1^i)$, represents the trading benefit of adjusting his existing positions. This term is zero when $\theta_1^i = \widehat{\theta}_1^i$. Thus, $\widehat{\theta}_1^i$

can be interpreted as trader $i$'s desired stock holding at $t = 1$, given the market condition $\delta_1^i$ and his risk exposure $Y$ and $X_{i1}$. The second term, $g_2(\delta_1^i)$, is the expected trading benefit by sharing future idiosyncratic liquidity shocks, $Z_2^i$. This term depends both on future market condition $\delta_1^i$ (through its impact on $P_2$) and on the future trading needs $k$, which increases with the volatility of future shocks and the risk aversion of traders. The last term, $-c$, is simply the cost of participation.

Given the decomposition of trading gains, the participation decision is straightforward. When the participation cost is smaller than the expected future gain from trading, i.e., $g_2(\delta_1^i) \geq c$, (16) is always satisfied and trader $i$ always chooses to participate. The more interesting case is when $c > g_2(\delta_1^i)$, trader $i$ chooses to participate only if the gain from adjusting his current position is sufficiently large, which happens when his holding is sufficiently far away from the desired level, i.e., when $|\theta_1^i - \widehat{\theta}_1^i| > \sqrt{\big[c - g_2(\delta_1^i)\big]/h(\delta_1^i)}$.

## 3.3. Participation Equilibrium at $t = 1$

Given the individual participation decision, we can solve for the participation equilibrium. Let index $i$ denote the trader group whose $X_1^i$ has the same sign as $Y$ and $-i$ the other group whose $X_1^{-i}$ has the opposite sign as $Y$. The following proposition describes the participation equilibrium at date 1.

**Proposition 3.** *Given state variables $Y$ and $X_1$ and the initial holdings $\theta_1^i = 0$, we define the gain from trading for trader $i$ as*

$$g^i(\delta_1^i) \equiv g(0, Y, X_1^i, \delta_1^i). \tag{17}$$

*Let $\bar{\delta} \equiv \frac{\nu}{\mu+\nu}$, the participation equilibrium at date 1 is fully specified by the following five cases:*

*A. If $0 \leq g^{-i}(0) \leq g^i(0)$, $\omega_1^i = \omega_1^{-i} = 1$ and all traders participate.*

*B. If $g^{-i}(0) < 0$ and $g^i(0) \leq 0$, $\omega_1^i = \omega_1^{-i} = 0$ and no trader participates.*

*C. If $g^{-i}(-\bar{\delta}) < 0 < g^i(\bar{\delta})$, $\omega_1^i = 1$ and $\omega_1^{-i} = 0$.*

*Otherwise, let $\delta^*$ be the minimum $\delta \in (0, \bar{\delta}\,]$ that violates $g^{-i}(-\delta) < 0 < g^i(\delta)$.*

*D. If $g^{-i}(-\delta^*) = 0 \leq g^i(\delta^*)$, $\omega_1^i = 1$ and $\omega_1^{-i} \in (0,\ 1)$.*

*E. If $g^{-i}(-\delta^*) < 0 = g^i(\delta^*)$, $\omega_1^i \in (0,\ 1)$ and $\omega_1^{-i} = 0$.*

*The partial participation levels $\omega_1^{-i}$ and $\omega_1^i$ in cases D and E are given in Appendix A.*[7]

In case A, all traders have (weakly) positive gains from trading under symmetric market participation ($\delta_1^i = 0$). Thus, they all choose to enter the market and the market participation is indeed symmetric. Similarly in case B, the trading gain is negative for all traders when participation is symmetric. Hence, no participation is the only equilibrium outcome.

Since $\bar{\delta}$ corresponds to the value of $\delta_1^i$ when $\omega_1^i = 1$ and $\omega_1^{-i} = 0$, the case $\delta_1^i = \bar{\delta}$ reflects a market situation when only traders from group $i$ participates, hence is the least favorable market condition for trader $i$. The only trading benefit comes from risk sharing with the market makers at date 2. In contrast, it also is the most favorable market condition for an individual trader $-i$ since he can unload all his idiosyncratic risks to group $i$ traders in the market. Case C corresponds to a situation where traders of group $i$ have positive gains under the least favorable market condition while traders of group $-i$ have negative gains even under the most favorable market condition. Naturally, the only equilibrium outcome is that all group $i$ traders participate and no one from group $-i$ participates.

Cases D and E describe situations when there is partial participation from one group. For example, in case D, all traders of group $i$ participate while there is partial participation from group $-i$. Since case C corresponds to the situation when the condition $g^{-i}(-\delta) < 0 < g^i(\delta)$ is satisfied at $\bar{\delta}$, it has to be violated in cases D and E. Hence, there always exists a $\delta^* \in (0, \delta]$ that violates the condition. Moreover, since $g^{-i}(0) < 0 < g^i(0)$ in cases D and E, we know that the condition is satisfied at $\delta = 0$. Given that $g^i(\delta_1^i)$ (and $g^{-i}(-\delta_1^i)$) are monotonically decreasing (and increasing) continuous functions in $\delta_1^i$, as we gradually increase $\delta$ from 0 until the condition is violated, either case D or E has to be true. In case D, if the market participation difference is $\delta^*$, group $i$ traders are better off participating, and the only equilibrium is a corner solution where all of them participate, i.e., $\omega_1^i = 1$. Traders in group $-i$ are indifferent between participating or not. They reach a partial participation level $\omega^{-i}$ so that the difference between the two groups is exactly $\delta_1^i = \delta^*$, confirming the overall participation equilibrium. Similarly, case E corresponds to the situation when traders of group $i$ break even and traders of group $-i$ are worse off participating. The equilibrium

[7] In the case of A and B, when the gain from trading is zero for a group of traders, their participation rate may not be unique, in which case we choose a particular value for convenience.

outcome is partial participation from group $i$ and no participation from group $-i$. In all cases other than cases A and B, more traders from group $i$, the group with higher trading gains, enter the market despite the fact that fewer counter-parties will do so. Their entry creates the imbalance in trades and need for liquidity, which is provided by the market makers.

## 3.4. Market Equilibrium at $t = 1$

At $t = 1$, only market makers are present in the market. Given the realizations of state variables $Y$ and $X_1$, the participation equilibrium is fully determined in Proposition 3, which in turn determines the market equilibrium at $t = 2$. Thus, we can easily compute the market equilibrium at $t = 1$, which is given below.

**Proposition 4.** *Given the difference $\delta_1$ in equilibrium participation rates, the equilibrium price at date $t = 1$ is*

$$P_1 = V - \alpha\sigma_v^2 Y - \alpha\sigma_v^2 \frac{\delta_1}{1+k\,\delta_1^2} X_1. \tag{18}$$

Clearly, $\delta_1$ measures the degree of asymmetric participation between the two groups of traders, and is fully determined by the realizations of $Y$ and $X_1$. When the participation is asymmetric between the two groups of traders, i.e., when $\delta_1 \neq 0$, date 1 price $P_1$ depends not only on the fundamentals, i.e., $V$ and $Y$, but also on the idiosyncratic risk $X_1$.

## 3.5. Market Equilibrium at $t = 0$

At date 0, again only market makers are present in the market. Given the equilibrium at $t = 1$, we can easily compute the equilibrium at $t = 0$.

**Proposition 5.** *The equilibrium price at date $t = 0$ is $P_0 = V$.*

# 4. Limited Participation and Liquidity

The equilibrium under costly participation shows several striking results. First, despite the fact that the two groups of traders have perfectly matching trading needs, their actual trades are not synchronized when participation in the market is costly. The non-synchronization in their trades gives rise to the need for liquidity in the market. A group of traders may bring their orders to the market while traders with off-setting trading needs are absent, creating an imbalance of orders and a shortage of liquidity. The stock price will have to adjust in response to the order imbalance and to attract the market makers to accommodate the orders. As a result, the price of the stock not only depends on the fundamentals (i.e., its expected future payoffs and total risk), but also depends on idiosyncratic shocks market participants face. In this section, we examine in more detail these results and the economic intuition behind them.

## 4.1. Gains from Trading and Individual Participation Decisions

We start with the individual participation decisions and the participation equilibrium. The traders' participation decision depends on the trade off between the cost to be in the market and the gains from trading. Due to the lag between the entry decision (at date 1) and the time to trade (at date 2), the gain from trading consists of two parts. The first part comes from trading to hedge the risk from the current shocks, $Y$ and $X_1^i = Z_1^i$, which persists into the future. The second part comes from trading to hedge the risk from future idiosyncratic shock $X_2^i - X_1^i = Z_2^i$. From Proposition 2, these two parts are captured by $g_1(\delta_1^i, \widehat{\theta}_1^i)$ and $g_2(\delta_1^i)$, respectively.

Given any market participation rate $\delta_1^i$, the gain $g_1$ is equal to $h(\delta_1^i)(\theta_1^i - \widehat{\theta}_1^i)^2$, where $\theta_1^i$ is the trader's current asset holding and $\widehat{\theta}_1^i$ is his desired holding given the current shocks. Clearly, the larger the difference between the two, $|\theta_1^i - \widehat{\theta}_1^i|$, the larger the benefit from trading. The desired holding, $\widehat{\theta}_1^i = (kY + X_1^i)(1-\delta_1^i)/(1-k\delta_1^i)$, depends on the current shocks $Y$ and $X_1^i$, thus so does the gain $g_1$. On the other hand, $g_2$ does not depend on the current shocks.

If the gain from trading against future shocks is larger than the entry cost along, i.e., $g_2(\delta_1^i) \geq c$, trader $i$ will choose to participate in the market independent of the current shocks. If this is not the case, then traders will enter only if the benefit from trading against the current shocks is large enough. In this case, their entry decision will depend on the current state variables $Y$ and $X_1^i$.

What is important is that in general the gain from trading is different between the traders. This is true even though the idiosyncratic shocks among the two groups of traders are perfectly offsetting, i.e., $X_t^a = -X_t^b$ $(t = 1, 2)$. In order to see this, let us consider the simple situation when $\delta_1^i = 0$, i.e., the participation in the market is symmetric between the two groups. In this case, $\widehat{\theta}_1^i = -(kY+X_1^i)$. Given that the current holding is zero, i.e., $\theta_1^i = 0$ for both groups of traders, $(\theta_1^i - \widehat{\theta}_1^i)^2$ is higher for traders whose $X_1^i$ has the same sign as $Y$. This is intuitive. A shock $X_1^i$ in the same direction as $Y$ increases the distance between his current position and his desired posiition, which is $kY+X_1^i$. This makes him more willing to trade in order to reduce his total risk exposure than traders whose idiosyncratic shock has the opposite sign. Obviously, such a difference in the gains from trading among the traders is present in general when $\delta_1^i \neq 0$ and $\theta_1^i \neq 0$. Thus, we have the following result:

**Result 1.** *The gain from trading is in general different between traders even when their trading needs are perfectly matched.*

It is important to recognize that Result 1 is a general phenomenon when trading is costly. The reason is as follows. When the traders can trade continuously, they will constantly maintain at the optimal position and the gains from trading is always symmetric for small deviations from the optimal position. Let $u(\theta)$ denote the utility from holding $\theta$ and $\theta^*$ the optimal holding. Then, $u'(\theta^*) = 0$. For a small deviation from the optimum $x = \theta - \theta^*$, the gain from trading is given by $u(\theta^*) - u(\theta) \simeq -u''(\theta^*)(\theta-\theta^*)^2/2$, which is symmetric for an opposite deviation $-(\theta-\theta^*) = -x$. Thus, at the margin, traders with offsetting shocks or trading needs always have the same gain from trading. This is no longer the case when trading is costly. Facing a cost to trade, traders no longer trade constantly. They only trade when the deviation from the optimal is sufficiently large. But for a finite deviation $x = \theta–\theta^*$, $u(\theta^*) - u(\theta^*+x) \neq u(\theta^*) - u(\theta^*-x)$ and the gains from trading are different between traders with perfectly matching trading needs.[8] Naturally, the asymmetry in gains from trading can in general lead to asymmetric participation between the traders.

A trader's gain from trading also depends on how many other traders are in the market. This is obvious since $g_2(\delta_1^i)$ and $h(\delta_1^i)$ both depend on $\delta_1^i$. Moreover, $g_2'(\delta_1^i) \leq 0$ and $h'(\delta_1^i) > 0$. Thus, holding the deviation from the desired position $|\theta_1^i - \widehat{\theta}_1^i|$ constant, the gain from trading for an individual trader $i$ decreases with $\delta_1^i$. Thus, we have the following result:

**Result 2.** *The benefit from trading for a trader decreases with the population of participating*

[8]It is worth pointing out that the gain from trading also depend on the initial position $\theta_1^i$. In our setting, $\theta_1^i = 0$. In a stationary intertemporal setting, $\theta^i$ should be the optimal holding determined by the trader's dynamic optimization problem. In a discrete setting like the one in this paper, $\theta_1^i$ is always different from $\widehat{\theta}_1^i$ since the latter depends on the current shocks while the former does not. In setting similar to ours, Lo, Mamaysky, and Wang (2004) show that even in continuous-time the gain from trading is asymmetric around the optimal holding due to the fact that traders only trade infrequently.

*traders from the same group and increases with the population of participating traders from the different group.*

The fact that the gain from trading depends on the participation of other traders gives rise to the externality of a trader's participation decision. As we will see later, this externality is an important driving force in the determination of participation equilibrium, liquidity and prices.

## 4.2. Non-Synchronized Trading and Need for Liquidity

Given the individual trader's entry policy, we now examine the participation equilibrium. From the discussion above and Proposition 3, we know that if the participation cost is sufficiently small, i.e., $c \leq g_2(0)$, traders will all enter unconditionally. The more interesting case is when $c > g_2(0)$ and the participation equilibrium will depend on the state variables, $Y$ and $X_1$, as stated in Proposition 3.

(a) Participation Rate $\omega_1^a$ and $\omega_1^b$ (b) Difference in Participation Rate $\delta_1$

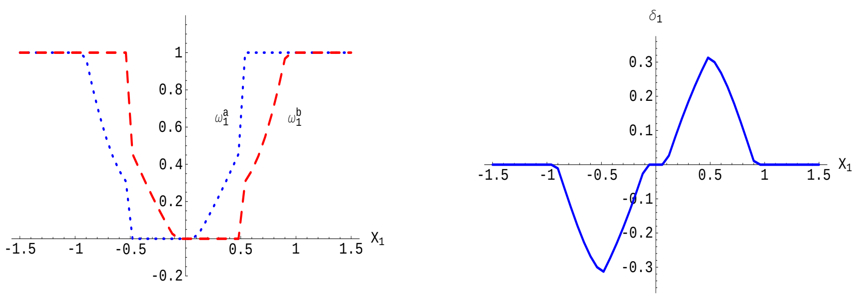



Figure 2: **Equilibrium Participation.** The figure plots the equilibrium participation rate for the two trader groups for different values of idiosyncratic shock $X_1$, while the aggregate shock is held constant at $Y = 1$. Panel A reports the equilibrium fraction of type $i$ traders who choose to participate, where the dotted line refers to trader $a$ ($\omega_1^a$) and the dashed line refers to trader $b$ ($\omega_1^b$). Panel B reports the difference in participation decisions, $\delta_1 = \nu(\omega_1^a - \omega_1^b)/[\mu + \nu(\omega_1^a + \omega_1^b)]$. Other Parameters are set at the following values: $\alpha = 4$, $\nu = \mu = \frac{1}{3}$, $c = 0.1$, $\sigma_v = 0.2$, and $\sigma_{z_2} = 0.8$.

Figures 2 shows the equilibrium participation decisions as a function of the idiosyncratic shock $X_1$ for a given realization of aggregate shock $Y$, which is set to $Y = 1$. Panel (a) reports the fraction $\omega_1^i$ of traders within each group who choose to participate. The dotted line plots $\omega_1^a$ and the dashed line plots $\omega_1^b$. Panel (b) reports the difference in participation ratio between the two groups of traders $\delta_1$, defined in equation (10), as a function of $X_1$. For a range of $X_1$ around 0, $\omega_1^a = 0$, that is, trader $a$ choose not to participate simply because the benefit from trading is too small. This is the no-participation region. It is worth pointing out that the no-participation region is not symmetric about 0, reflecting the fact that a trader's gain from trading is asymmetric between positive and negative idiosyncratic shocks. As $X_1$ falls outside the no-participation region, $\omega_1^a$ starts to increase, indicating that more and more group $a$ traders choose to participate. Moreover, $\omega_1^a$ is larger when $X_1$ is positive (in the same direction as $Y$). When $X_1$ exceeds certain thresholds, the gain from trading dominates the cost, all group $a$ traders choose to participate and $\omega_1^a$ reaches 1. The participation rate of group $b$ traders, $\omega_1^b$, behaves similarly. In fact, $\omega_1^b$ is simply the mirror image of $\omega_1^a$ around the vertical axis because trader $a$ and $b$ face opposite idiosyncratic shocks. When $X_1 < 0$, the risk exposure of group-$a$ traders is below the average (which is $Y = 1$), their gains from trading is lower than that of group-$b$ traders. Thus, $\omega_1^a \leq \omega_1^b$ and $\delta_1 < 0$, indicating that group $a$ traders are less likely to participate. When $X_1 > 0$, their idiosyncratic shock has the same sign as the aggregate shock, and group $a$ traders are more likely to participate, or $\delta_1 > 0$.

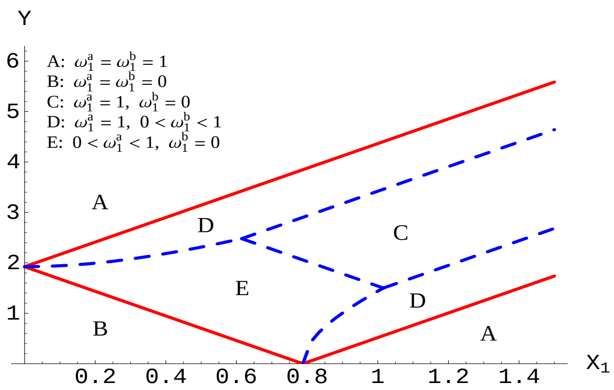



Figure 3: **Participation equilibrium.** The figure shows the different participation rates of the two groups of traders for different realizations of aggregate risk exposure $Y$ and idiosyncratic risk exposure $X_1$. Other parameters are set at the following values: $\alpha = 4$, $\nu = \mu = 1/3$, $c = 0.15$, $\sigma_{z_2} = 0.8$, $\sigma_v = 0.2$.

In general, the equilibrium participation depends on both $Y$ and $X_1$. Figure 3 shows the nature of the participation equilibrium in the space of state variables, or the "phase diagram". We only plot the first quadrant of the phase diagram since the other quadrants are symmetric.

As stated in Proposition 3, there are five regions in the state space, corresponding to the five cases, A, B, C, D, and E in Proposition 3, respectively. When both $Y$ and $X_1^i$ are small, i.e., in the area close to the origin and below the downward sloping solid-line, the initial holding $\theta_1^i = 0$ is close to the desired position $\widehat{\theta}_1^i = kY + X_1^i$ and the gain from trading is small. As a result, no traders choose to enter the market. This corresponds to case B in the proposition. For large values of $Y$ or $X_1$, i.e., the areas above the top upward sloping solid-line and below the bottom upward sloping solid-line, $\theta_1^i$ is significantly different from $\widehat{\theta}_1^i$ and the gain from trading is large for both groups of traders. In equilibrium, all traders enter the market, i.e., $\omega_1^a = \omega_1^b = 1$ and we have Case A.

For values of $X_1$ close to $kY$, i.e., in the area between the three solid lines, the situation becomes more interesting. When $X_1$ is close to $kY$, $\widehat{\theta}_1^a = kY + X_1$ is large and far away from $\theta_1^a = 0$ and the gain from trading is large for group-$a$ traders. But $\widehat{\theta}_1^b = kY - X_1$ is small and close to $\theta_1^b = 0$, and the gain from trade is small for group-$b$ traders. Thus, we have the situation where group $a$ traders are willing to enter the market while group-$b$ traders are less eager.

There are two possibilities under this situation. When $kY + X_1$ is sufficiently large, in regions C and D, group-$a$ traders see large gains from trading, even if only with the market makers, and they will choose to enter, i.e., $\omega_1^a = 1$. The high participation rate of group-$a$ traders increases the gains from trading for group-$b$ traders, given their offsetting trading needs. In region C, $kY - X_1$ is too small and such an inducement does not increase the gain from trading for group-$b$ traders sufficiently. As a result, they remain out of the market. However, in region D where $kY - X_1$ is not too small, the participation of group-$a$ traders increases the gains from trading sufficiently for group-$b$ traders so that at least some of them choose to participate as well. When $kY + X_1$ is modest as in region E, the gain from trading is only large enough to induce a fraction of group-$a$ traders to participate. It is not enough, even with the participation of some group-$a$ traders, to induce any group-$b$ traders to enter the market. In this case, we have $0 < \omega_1^a < 1$ and $\omega_1^b = 0$.

Clearly, even with perfectly matching trading needs, the traders fail to synchronize their trades under costly participation. In particular, for certain realizations of the shocks $Y$ and $X_1$, the participation is asymmetric, causing a mismatch in the buy and sell orders in the market. This creates the need for liquidity in the market. Thus, we have the following result.

**Result 3.** *In equilibrium, participation can be asymmetric among traders even when their trading needs are perfectly matched, giving rise to non-synchronization in their trades and* *the endogenous need for liquidity. Moreover, the liquidity need depends on the idiosyncratic shocks traders face as well as the aggregate risk exposure.*

From Proposition 3, we observe that $\delta_1^i$ is always positive for trader $i$ whose idiosyncratic shock $X_1^i$ has the same sign as $Y$, the aggregate risk exposure. Hence, $\delta_1 X_1 = \delta_1^i X_1^i$ always has the same sign as $Y$. This gives the following result:

**Result 4.** *In equilibrium, traders with risk exposures exceeding the average, i.e., whose idiosyncratic risk exposure is in the same direction as the aggregate exposure, participate more in the market.*

# 5. Liquidity and Equilibrium Stock Price

Our analysis above suggests that participation costs prevent traders from always being in the market, and the benefit from trading is different for different traders. As a result, self interest fails to coordinate their participating decisions and synchronize their trades even when their trading needs perfectly match. This non-synchronization in trades gives rise to imbalances in asset demand and the need for liquidity. Such exogenous order imbalances are the starting point of Grossman and Miller (1988) and market microstructure models like Glosten and Milgrom (1985) and Stoll (1985, 1989). In our model, by explicitly modeling the motives and the cost to be in the market, we endogenously derive the order imbalance and its dependence on market conditions. In particular, we show that the order imbalance is always in the same direction as the aggregate risk exposure. This correlation between trade imbalances and the aggregate risk exposure leads to interesting implications on equilibrium prices, which we now turn our attention to.

From Equations (11) and (18), the equilibrium stock price consists of two components: the risk-adjusted "fundamental value", $V - \alpha\sigma_v^2 Y$, and the liquidity effect. Naturally, we focus on the second component, which is defined by

$$\tilde{p}_t \equiv P_t - (V - \alpha\sigma_v^2 Y) = -\lambda_t X_t \tag{19}$$

where $t = 1, 2$, $\lambda_1 = \lambda_2/(1+k\delta_1^2)$ and $\lambda_2 = \alpha\sigma_v^2\delta_1$. In the absence of participation costs, all traders participate in the market and there is no need for liquidity. The stock price equals the fundamental value (thus $\tilde{p}_t = 0$, $t = 1, 2$) and does not depend on the idiosyncratic shocks individual traders face. In the presence of participation cost, partial participation leads to non-synchronized trades among traders and the need for liquidity. The stock price has to adjust to attract the market makers to provide the liquidity and to accommodate the trades. In general, $\tilde{p}_t \neq 0$ and the stock price becomes dependent on the idiosyncratic shocks of individual traders.

To fix ideas, let us consider the case when $Y > 0$, i.e., the aggregate risk exposure is positive. As Result 4 states, traders whose risk exposure is higher than the average are more willing to enter the market. In this case, they are the traders with a positive $X_1^i$, which has the same sign as $Y$, and who want to sell the stock. Thus, the order imbalance, as captured by $-\delta_1 X_1$, is negative, which leads to a negative $\tilde{p}_t$, $t = 1, 2$. It is important to note that the sign of $\tilde{p}_t$ is independent of the sign of $X_1$. In other words, it does not depend on the distribution of idiosyncratic shocks among the traders, but only depends on the aggregate risk exposure. Since a positive aggregate risk exposure (i.e., $Y > 0$) leads to a discount on the stock price, we have the following result:

**Result 5.** *The impact of liquidity on the asset price always magnifies the impact of the aggregate risk on the price.*

The magnitude of the liquidity effect on price depends on $X_1$. Figure 4(a) plots $\tilde{p}_1$ against $X_1$ for two different values of $Y$: the solid line for $Y = 1$ and the dashed line for $Y = 0.5$. First, we note that at $t = 1$, both groups of traders receive their idiosyncratic shocks but are not trading in the market. However, in anticipation of their trades at date 2 and the need for liquidity due to the order imbalances, the current price already adjusts. Second, given that $Y$ is positive, the liquidity effect on the price is always negative, as mentioned before. More importantly, the impact of liquidity on the stock price is not linear in $X_1$, the idiosyncratic shocks to the traders. In particular, for small values of $X_1$, gains from trading are small for all traders and they do not enter the market. As a result, there is no need for liquidity and price equals the fundamental. For large values of $X_1$, gains from trading are sufficiently large for all traders and they all enter the market. As a result, their traders are synchronized and there is no need for liquidity. The stock price also equals the fundamental. For intermediate ranges of $X_1$, the gains from trading are large enough for some traders to enter the market, but not for all traders. It is in this case when trades are non-synchronized and liquidity is needed in the market, which will in turn affect the stock price. As Figure

4 shows, the price impact of liquidity reaches the maximum for a certain magnitude of the idiosyncratic shock.

(a) Liquidity Factor in Price ($\tilde{p}_1|Y$) (b) Probability Distribution of $\tilde{p}_1|Y$

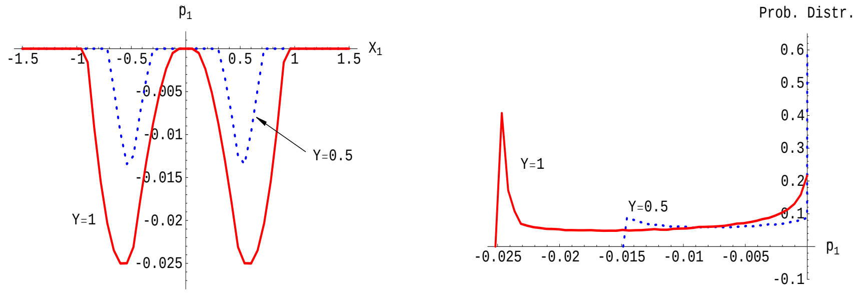



Figure 4: **The conditional liquidity factor.** The figure reports the date $t = 1$ conditional liquidity factor, $\tilde{p}_1|Y \equiv P_1 - (V - \pi Y)$, which captures the price movement in excess of the "fundamentals". The solid line corresponds to the case when the realization of aggregate shock is $Y = 1$, and the dotted line is when $Y = 0.5$. Panel (a) plots the conditional liquidity factor as a function of the idiosyncratic liquidity shocks ($X_1$). Panel (b) reports the conditional probability density function of $\tilde{p}_1|Y$, except at the point $\tilde{p}_1|Y = 0$ where the value corresponds to the total probability mass at the point (since the density function should be infinity at the point). For ease of exposition, except at the point of $\tilde{p}_1 = 0$, we scale down the probability density function by a factor of 400 in the plot. Other parameters are set at the following values: $\alpha = 4$, $\nu = \mu = \frac{1}{3}$, $c = 0.1$, $\sigma_v = 0.2$, $\sigma_{z_1} = 0.5$, and $\sigma_{z_2} = 0.8$.

The results that the price impact of liquidity is one-sided (always negative when the aggregate risk exposure is positive) and highly non-linear arise from the fact that liquidity needs are endogenous in our model. In most of the existing models of liquidity, such as Grossman and Miller (1988), liquidity needs are exogenously specified. Consequently, its price impact is linear in the exogenous liquidity needs and symmetrically distributed. Our analysis shows that modelling the liquidity needs endogenously is important to understand the impact of liquidity on prices. After all, it is the same economic factor, the cost to participating in the market, that drives both the liquidity needs of the traders and the liquidity provision of market makers.

It is also worth pointing out that the magnitude of the price impact of liquidity depends on the value of $Y$, the level of aggregate risk exposure. For large values of $Y$, individual traders bear a significant amount of aggregate risk. Their idiosyncratic shocks create a larger dispersion in their gains from trading, which raises the likelihood of asymmetric participation, non-synchronization in trading and the need for liquidity. As shown in Figure 4(a), the price impact of liquidity is smaller for $Y = 0.5$, shown by the dashed line, than for $Y = 1$, shown by the solid line.

The non-linearity in the price impact of liquidity leads to another interesting result: large and frequent price movements in absence of any aggregate shocks. Figure 4(b) plots the probability distribution of $\tilde{p}_1$, conditional on a given value of $Y$. The solid line is for $Y = 1.0$ and the dashed line is for $Y = 0.5$. Even though the underlying idiosyncratic shocks that drive the individual traders' trading needs are normally distributed, their price impact as measured by $\tilde{p}_1$ has a fat-tailed distribution. Aside from a non-zero probability mass at the origin, the distribution peaks at a finite and negative value. This simply reflects the fact that liquidity becomes important and affects the price only for a range of finite shocks. Moreover, the impact of liquidity gives rise to the possibility of a large price movement in absence of any shocks to the fundamentals of the stock. Since such a price movement is associated with a large imbalance in trades and a surge of liquidity needs, it can be called "liquidity crashes". Summarizing the results above, we have the following:

**Result 6.** *The impact of liquidity on the stock price is nonlinear in the idiosyncratic shocks among traders and is more important for a range of finite idiosyncratic shocks. Especially, it can lead to "liquidity crashes" in which large price movements occur in the absence of shocks to the fundamentals.*

The discussion above takes the aggregate risk exposure as given and examines the impact of liquidity from individuals' trading needs on prices. We now examine how liquidity can affect the relation between aggregate risk exposure and the price. For this purpose, we consider the average $\tilde{p}_t$ over possible values of $X_t$:

$$\bar{p}_t \equiv \mathrm{E}[P_t|Y] - (V - \alpha\sigma_v^2 Y) = -\mathrm{E}[\lambda_t X_t|Y] \tag{20}$$

where $t = 1, 2$. Clearly, $\bar{p}_t$ captures how liquidity enhances the impact of aggregate risk on the stock price. In absence of any aggregate risk, there is also no liquidity effect. Trading needs are perfectly matched between traders and so are their gains from trading. Their participation is always symmetric and trades are synchronized. There is no need for liquidity and the price of the stock equals its fundamental value, which is simply $V$. In the presence of aggregate risk exposure, gains from trading becomes different among traders, depending on the direction of the idiosyncratic shocks to their risk exposure relative to the aggregate risk exposure. Thus, the presence of aggregate risk in the fundamentals is an important source of liquidity needs in our model.

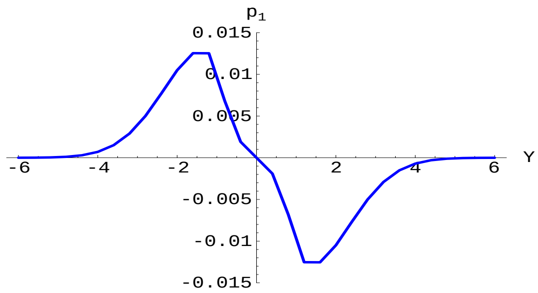



Figure 5: **The conditional mean of the liquidity factor.** The figure reports the expected liquidity factor $\bar{p}_1 \equiv \mathrm{E}[\tilde{p}_1|Y]$ as a function of the aggregate risk exposure $Y$. Other parameters are set at the following values: $\alpha = 4$, $\nu = \mu = \frac{1}{3}$, $c = 0.1$, $\sigma_v = 0.2$, $\sigma_{z_1} = 0.5$, $\sigma_{z_2} = 0.8$, and $\sigma_y = .8$.

Figure 5 plots $\bar{p}_1$ as a function of $Y$. (The plot for $\bar{p}_2$ is similar and hence is omitted.) Since the aggregate risk exposure gives a risk discount on the stock equal to $\alpha\sigma_v^2 Y$ and $\bar{p}_1$ has the same sign as $Y$, the effect of liquidity magnifies the impact of aggregate risk on the stock price. In other words, as the aggregate level of risk increases, not only the risk discount on the stock price increases. The potential need for liquidity also increases, which leads to an impact on the price in the same direction as the risk discount. As shown in Figure 5(a), the impact of liquidity on the stock price is highly nonlinear in $Y$. It diminishes when $Y$ approaches zero or infinity since liquidity needs are minimal in these cases as participation is symmetric. However, it is significant for intermediate ranges of $Y$. As we have seen earlier, such a feature is shared by the impact of idiosyncratic shock $X_1$ on the price. The fact that the liquidity effect tends make the stock price more dependent on the shocks to both the aggregate and idiosyncratic risk exposure and such a dependence is stronger for shocks of finite sizes leads to the following result:

**Result 7.** *The impact of liquidity increases the price volatility of the stock and leads to fat tails in its returns.*

Figure 6 plots the probability distribution of $\tilde{p}_1$. In absence of the liquidity effect, i.e., when the participation cost is zero, the distribution is a delta-function at zero (which is equivalent to a normal density function with zero volatility).

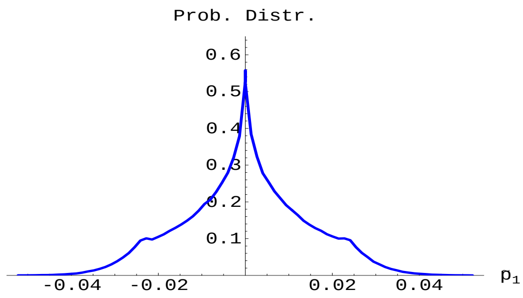



Figure 6: **"Fat tail" in the return distribution.** The figure reports the probability density function of the unconditional liquidity factor $\tilde{p}_1$, which captures the price movement in excess of the "fundamentals". We report the total probably mass at the point $\bar{p}_1 = 0$ since the density function should be infinity at the point. For ease of exposition, except at the point of $\bar{p}_1 = 0$, we scale down the probability density function by a factor of 40 in the plot. Other parameters are set at the following values: $\alpha = 4$, $\nu = \mu = \frac{1}{3}$, $c = 0.1$, $\sigma_v = 0.2$, $\sigma_{z_1} = 0.5$, $\sigma_{z_2} = 0.8$, and $\sigma_y = .8$.

# 6. Externality in Trading and Welfare

As stated in Proposition 3 and shown in Figure 3, the participation equilibrium has an interesting feature, that is, it is always a corner equilibrium. In other words, the participation rate is either zero or one for at least one trader group. This reflects the externality in the traders' entry decisions. In particular, the withdrawal of a trader from the market also takes away the opportunity for other traders to trade with him. Thus, it reduces the liquidity in the market and the gains from trading for other traders. Furthermore, it reduces the incentive for other traders to participate in the market.

When entry is costly, individual traders make their participation decision based on their own gains from trading. Such an optimizing behavior at the individual level ignores the externality in each trader's participation decisions. In this section, we examine the welfare implications of such an externality in the traders' participation decisions.

## 6.1. Externality of Individual Participation

To analyze the externality of individual participation, we plot in Figure 7 the gains from trading for the two groups of traders as a function of $\delta_1$, the relative excess participation of group-$a$ traders over group-$b$ traders, where the solid line is for group-$a$ traders and the dashed line is for group-$b$ traders. It clearly shows that the gain from trading, say for group-$a$ traders, decreases as more of them participates (in excess of group-$b$ traders), i.e., when $\delta_1$ becomes more positive, but increases as more of group-$b$ traders participates, i.e., when $\delta_1$

becomes more negative. The fact that the participation of one group of traders can increase the participation of the other group of traders reflects the externality of their participation decisions. Such an externality drives the market towards a corner equilibrium.

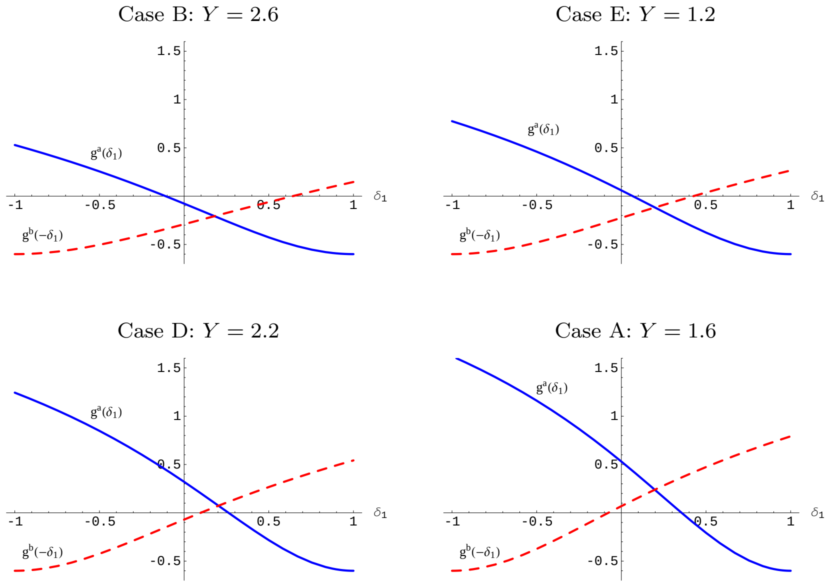



Figure 7: **Individual gains from trading.** The figure reports the trading gain as functions of $\delta_1$, where the solid lines report the gain for trader $a$, $g^a(\delta_1) \equiv g(0, Y, X_1, \delta_1)$, and the dashed lines report gains for trader $b$, $g^b(-\delta_1) \equiv g(0, Y, -X_1, -\delta_1)$, respectively. The gain $g(\cdot)$ is defined in Equation (14). The only parameter that changes between difference cases is $Y$. Other parameters are set at the following values: $X_1 = .2$, $\alpha = 4$, $\nu = \mu = \frac{1}{3}$, $c = 0.15$, $\sigma_v = 0.2$, and $\sigma_{z_2} = 0.8$.

In the upper left panel of Figure 7, at $\delta_1 = 0$ the gain from trading is negative for both groups of traders. Consequently, no traders choose to enter the market (Case B in Proposition 3). In the upper right panel, at $\delta_1 = 0$ the gain from trading is positive for group-$a$ traders but negative for group-$b$ traders. Thus, a subset of group-$a$ traders will choose to enter the market. However, for $\delta_1$ sufficiently large, the gain from trading remains negative for group-$b$ traders and they remain out of the market. Given that no group-$b$ traders enters the market to provide additional liquidity, the gain from trading quickly diminishes as more group-$a$ traders enter the market to consume the limited amount of liquidity the market makers provide. As a result, we have the situation in which only a fraction of group-$a$ traders enter to trade with the market makers while no group-$b$ traders enter. This corresponds to Case E in Proposition 3.

The lower left panel of Figure 7 best illustrates the positive participation externality between the two groups. At $\delta_1 = 0$, the gain from trading is positive for group-$a$ traders and negative for group-$b$ traders. However, as group-$a$ traders enter the market, the gain from trading starts to increase for group-$b$ traders and quickly becomes positive. This induces group-$b$ traders to enter the market. Moreover, their participation in the market keeps the gain from trading positive for group-$a$ traders, which lures more group-$a$ traders into the market. The positive feedback stops only when all group-$a$ are in the market. The equilibrium is reached when enough group-$b$ traders enter the market such that the gain from trading becomes zero. This corresponds to Case D in which $\omega_1^a = 1$ and $0 < \omega^b \leq 1$. The positive externality is reflected by the fact that the equilibrium is reached at a level of participation higher than the level when either group is alone in the market. Obviously, without the participation of group-$a$ traders, it is not feasible for any group-$b$ traders to participate since their gain only becomes positive when there are more group-$a$ traders in the market (when $\delta_1^a > 0$). Interestingly, without the participation of group-$b$ traders, it is not feasible for *all* group-$a$ traders to participate either, since the trading gains for type-$a$ trader is negative at the point $\delta_1^a = \overline{\delta} = \frac{1}{2}$, which corresponds to the case when $\omega_1^a = 1$ and $\omega_1^b = 0$. The lower right panel shows the simple situation when gain from trading is positive for both groups at $\delta_1 = 0$, in which case they all participate in the market and we have Case A in which $\omega_1^a = \omega_1^b = 1$.

Thus, we conclude the following: A trader's participation decreases the gain from trading for traders with similar trading needs but increases the gain for traders with offsetting trading needs. When the latter effect dominates, as in the case shown in the lower left panel of Figure 7, a trader generates a net positive externality by participating in the market. His participation attracts more traders with offsetting trading needs, which in turn will attract more traders of his own type.

## 6.2. Welfare Implications

When individual traders make decentralized participation decisions based only on their trading benefits, it is possible that they choose not to participate even though the social gain of their participation is large. To understand the welfare implication of the participation externality, we compare our equilibrium outcome with the equilibrium when all traders are forced to participate. In the latter case, instead of allowing each trader to optimally make their participation decisions, we assume that all traders are forced to pay the cost $c$ at date

1 so that they can all trade at date 2. We solve for the market equilibrium at date 1 and 2 in the case of forced participation and compute the value function at date 0 for traders and market makers, which is denoted by $J^i_{0,FP}$, where $i = a, b, m$. We then compare this value function with the value function of agent $i$ achieved in the equilibrium with optimal individual participation, $J^i_0$. Given the form of the value function in both cases, which is negative exponential in financial wealth, we can define the certainty equivalence of agent $i$'s value function as follows:

$$CE^i = -\ln\left(-J^i_0\right), \quad CE^i_{FP} = -\ln\left(-J^i_{0,FP}\right), \quad i = a, b, m. \tag{21}$$

For investor $i$, we define his "welfare gain under forced participation" as the difference between his certainty equivalent wealth levels in the equilibrium under forced participation and in the equilibrium under optimal participation.

$$\Delta^i \equiv CE^i_{FP} - CE^i, \quad i = a, b, m. \tag{22}$$

Since the certainty equivalence measures the ex ante expected utility, they are the same for both types of traders. Of course, they are different between traders and market makers.

Market makers do not pay transactions costs and always participate in the market. If all traders are forced to participate, there is a perfect match in their trades and the market makers have no role to play. If, instead, traders optimally choose when to participate, as we have seen in the previous discussions, non-synchronous participation arises, leading to mismatch in their trades. Market makers then provide liquidity service and are compensated for it. Thus, they are worse off when traders are forced to participate, i.e., $\Delta^m \leq 0$.

Since $CE^i_0$ measures the welfare achieved when traders choose their participation optimally, one might expect that they will be worse off when forced to participate unconditionally, i.e., $\Delta^i$ should always be negative for $i = a, b$. This is certainly true if we ignore the externality generated by the traders' participation decisions. After observing their own shocks, traders can avoid the participation cost if the gain from trade is small. However, the withdrawal from the market by many traders will have a negative impact on market liquidity and the gains from trading for traders who participate. In particular, since the gains from trading are asymmetric between the two groups of traders, their incentives to participate are different. Despite the price adjustment, the potential gain from trading cannot always be fully internalized and appropriately allocated among traders so that they all choose to participate when it is socially optimal to do so. In fact, for certain parameter values, we find that $\Delta^a$ and $\Delta^b$ become positive, indicating that traders are ex-ante better off when they are all forced to participate rather than leaving the choice to themselves. Moreover, the welfare gains for the traders can significantly out weight the losses for the market makers. In summary, we have the following result.

**Result 8.** *Individual participation choices can lead to a lack of coordination among traders to be in the market, which generates a lack of liquidity and negative externalities, and the social welfare loss can out-weight total participation costs.*

Figure 8 plots the welfare gains for individual agents under forced participation relative to that under optimal participation. In the two panes, A and B, the plot on the left reports the welfare gains for a trader ($\Delta^a = \Delta^b$), the solid line, and a market maker ($\Delta^m$), the dotted line. The plot on the right reports the social welfare improvement, $2\nu\Delta^a + \mu\Delta^m$, which is the weighted average of welfare gains for all agents.

Panel (a) reports the welfare gain of forced participation as a function of the participation cost for traders. The difference between the forced and the optimal participation equilibrium comes from the balance between paying the participation cost when ex-post trading needs are small and the additional risk sharing benefit with more counter-parties when ex-post trading needs are large. As shown in Proposition 3 and discussed in Section 4., when the cost is small, all traders will choose to participate unconditionally. The equilibrium under optimal participation is identical to that under forced participation. Thus, $\Delta^m = \Delta^a = 0$, as Figure 8.A shows. When the cost is prohibitively high, traders are clearly worse off if forced to participate, and $\Delta^a < 0$. The interesting case is for intermediate levels of participation costs. Some traders optimally choose not to participate when the ex-post gain from trading is small. Their withdrawal from the market reduces the risk sharing capacity in the market. In the figure, there exists a range of participation costs when the welfare gain of forced participation is positive, indicating that the benefit of improved risk sharing when traders are forced to participate dominates the cost of participation. Traders are better off when all of them are forced to participate. Although, as previously discussed, the market makers are always worse off, socially it is still beneficial to force participation when the gains to the traders are sufficiently large. This is shown in the left the plot on the right in Figure 8. The positive social welfare gain of forced participation illustrates a situation of market failure. Optimal behavior at the individual level within a conventional market setting does not necessarily lead to optimal risk-sharing.

Panel (b) considers the impact of idiosyncratic needs, measured by the volatility of idiosyncratic risk exposure $\sigma_z$, on the welfare gain of forced participation. When idiosyncratic trading needs are small, so are the gains from trading. Hence, forcing traders to participate reduces individual and social welfare, as indicated by the negative $\Delta$ for small $\sigma_x$. As the idiosyncratic trading needs increase, gains from trading become larger and traders are more willing to participate. However, the ex-post trading gains are generally different for traders with offsetting needs, and they may not choose to participate at the same time. The market can fail to coordinate their entry and synchronize their trades, which lead to a net social welfare loss. We see from Figure 8.B that, as $\sigma_x$ increases and the trading needs become more dispersed, the market failure is more likely and the agents are better off if all of them are forced to participate.

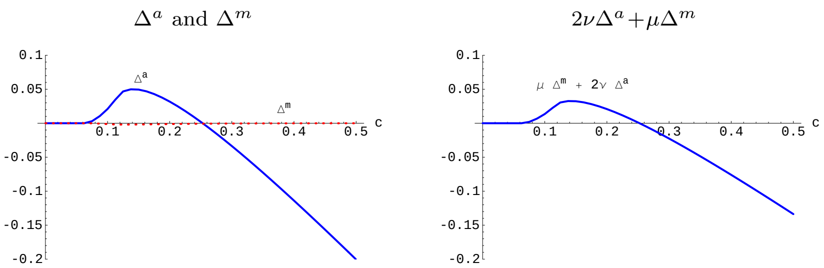
Despite our restrictive model assumptions, the mechanism we have identified for a market failure in coordinating the trades of potential traders under costly participation seems rather general: Each market participant not only benefits from trading on his own but also brings liquidity to the market. Under costly participation, while bearing the full cost along, each trader may not be able to sufficiently internalize the benefit he creates for the market as a whole. In other words, other participants can free-ride on the extra liquidity he brings to the market. As a result, the traders' participation decisions, while optimal at the the individual level, may well be socially sub-optimal.[9]

# 7. Conclusion

In this paper, we show that frictions such as participation costs can induce non-synchronization in agents' trades even when their trading needs are perfectly matched. Each trader, when

[9] Allen and Gale (1988), Allen and Gale (1994) considered situations when agents need to pay a fixed cost to create market for new securities. Duffie and Jackson (1989) considered the introduction of new futures contracts. In these cases, they also show that the free-rider problem can lead to sub-optimal outcomes of the economy. However, the situation considered in these papers are different from the situation here since there decisions by the agents, what securities to introduce into the market, are macroscopic by nature. The externality involved is more explicit as changes in market structure can drastically change the equilibrium and the allocation. In our case, the participation decision of individual traders is microscopic by nature. The externality is more implicit and endogenous. It arises from the interaction among the agents through the feedback of their individual actions on each other.

(a) Welfare Gains as a Function of Participation Costs ($c$)




(b) Welfare Gains as a Function of Liquidity Needs ($\sigma_{z_2}$)

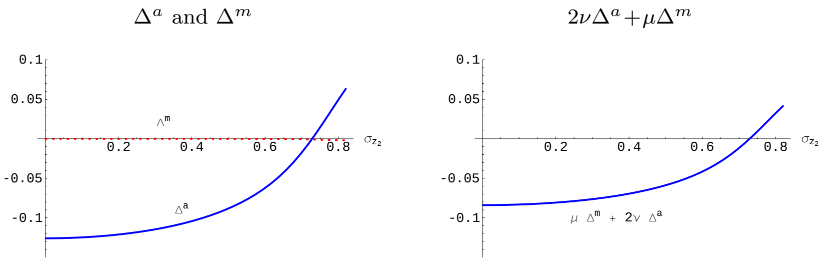



Figure 8: **Welfare improvement from forced participation.** The plot on the left in each panel reports the welfare gains from forced participation for the market makers ($\Delta^m$) with the dotted lines and the traders ($\Delta^a$) with the solid lines. The plot on the right reports the population-weighted total social welfare gain, $2\nu\Delta^a + \mu\Delta^m$. Panel (a) and (b) report welfare gains as a function of transactions costs $c$ and the volatility of individual liquidity shock $\sigma_{z_2}$, respectively. In panel (a), the volatility of date 2 trading needs is set at $\sigma_{z_2} = 0.8$, with the transactions cost $c$ varying in the range of $[0, .5]$. In panel (b) the transactions cost is set at $c = 0.15$ with varying date 2 trading needs $\sigma_{z_2} \in [0, 0.82]$. Other parameters are set at the following values: $\alpha = 4$, $\nu = \mu = \frac{1}{3}$, $\sigma_v = 0.2$, $\sigma_{z_1} = 0.5$, and $\sigma_y = .8$.

arriving at the market, faces only a partial demand/supply of the asset. The mismatch in the timing and the size of trades in the market creates temporary order imbalance and the need of liquidity which causes asset prices to deviate from the fundamentals. In particular, purely idiosyncratic liquidity shocks can affect prices, introducing additional price volatility. Moreover, the price deviations tend to be highly skewed. In the setting we consider and when the aggregate risk exposure is positive, the shortage of liquidity always cause the price to decrease and when it happens, the price tends to drop significantly, resembling a crash due to a sudden surge in liquidity needs. We further show that partial participation in the market by a subset of traders can have important welfare implications. In particular, the withdrawal of a subset of traders from the market reduces market liquidity and the incentives for others to participate in the market. The fact that participating agents cannot fully internalize the benefit from their liquidity provision leads to sub-optimal allocations of the economy despite the optimizing behavior at the individual level.

# A Appendix: Proofs

**Proof of Proposition 1.**

Given any stock price $P_2$, the participating agent $i$'s expected utility is given by

$$J_2^i(\eta_1^i = 1) = \max_{\theta_2^i} \ \mathrm{E}_2\left[-e^{-\alpha W_3^i}\right] = \max_{\theta_2^i} -e^{-\alpha\left[F_2^i - c^i + N_2^i + \theta_2^i(V-P_2) - \frac{1}{2}\alpha\sigma_v^2(\theta_2^i + X_2^i)^2\right]} .$$

His optimal stock holding is easily calculated by solving the first order condition with respect to $\theta_2^i$ for all agents,

$$\theta_2^i = \frac{1}{\alpha\sigma_v^2}(V - P_2) - Y - X_2^i, \qquad i = a, b, m.$$

The market clearing condition (8) implies an equilibrium price at date 2 of

$$P_2 = V - \alpha\sigma_v^2 Y - \frac{\alpha\sigma_v^2}{\omega_1}\left(\nu \sum_{i\in\{a,b\}} \omega_1^i X_2^i\right)$$

where $\omega_1$ is the total amount of participating agents defined in (10). Further plugging in the definition of liquidity shock $X_2^a = -X_2^b = X_2$ and $X_2^m = 0$ yields the result.

**Proof of Proposition 2.**

Calculating the expected value function for the participating and non-participating traders, $J_2^i(\eta_1^i = 1)$ and $J_2^i(\eta_1^i = 0)$, and plugging into equation (13) yield the result. Trader $i$ chooses to participate in the market if and only if gains from trading is positive, or $g(\theta_1^i, Y, X_1^i, \delta_1^i) > 0$.

**Proof of Proposition 3.**

Since all investors start with zero initial stock holding, $\theta_1^i = 0$, the participation gain $g(\theta_1^i, Y, X_1^i, \delta_1^i)$ in equation (14) can be simplified. Specifically, we have

$$g^i(\delta_1^i) \equiv g(0, Y, X_1^i, \delta_1^i) = g_2(\delta_1^i) - c + \frac{1}{2}\frac{\alpha\sigma_v^2(1-\delta_1^i)^2}{(1-k)\left(1-k+k(1-\delta_1^i)^2\right)}\left(kY + X_1^i\right)^2$$

Since $\delta_1^i \leq 1$, both $g_2(\cdot)$ and $g^i(\cdot)$ are monotonically decreasing in $\delta_1^i$. Let $s_1^i$ be the positive root of the equation

$$g^i(s_1^i) = 0.$$

Then the equilibrium fraction of type $i$ traders ($\omega_1^i$) who participate in the market can be determined as follows:

Case 1: If $\min[s_{a1}, s_{b1}] > 0$, then both traders always participate and $\omega_1^a = \omega_1^b = 1$.

Case 2: If $\max[s_{a1}, s_{b1}] < 0$, then neither trader participates and $\omega_1^a = \omega_1^b = 0$.

Case 3: If $\min[s_{a1}, s_{b1}] < 0 < \max[s_{a1}, s_{b1}]$, denote trader $i$ as the trader with $s_{i1} = \max[s_{a1}, s_{b1}]$, and $j$ as the other trader with $s_{j1} = \min[s_{a1}, s_{b1}]$, then

- if $s_{a1} + s_{b1} > 0$, $\omega_1^i = 1$ and $\omega_1^j = \max\left[0,\ \min\left[1,\ \frac{\mu+2\nu}{\nu}(\frac{1}{1-s_1^j}-1)+1\right]\right]$
- if $s_{a1} + s_{b1} < 0$, $\omega_1^i = \max\left[0, \min\left[1,\ \frac{\mu}{\nu}(\frac{1}{1-s_{i1}}-1)\right]\right]$ and $\omega_1^j = 0$
- if $s_{a1} + s_{b1} = 0$, any linear combination of the above two solutions is a solution.

To understand the result, assume that $s_1^a$ solves equation $g^a(s_1^a) = 0$, and that $\omega_1^a$ fraction of class $a$ traders choose to participate. Moreover, we assume that given the participation ratio $\omega_1^b$ of class $b$ traders, $\delta_1 = \frac{\mu(\omega_1^a - \omega_1^b)}{\omega_1} < s_{a1}$. Since the utility gain for class $a$ traders is monotonically decreasing in $\delta_1$, we must have $g^i(\delta_1) > 0$, implying that individual traders within class $i$ would find it optimal to participate. As a matter of fact, they would find it optimal to participate until $\omega_1^a$ increases to the level such that $\delta_1 = s_1^a$, at which time their gains from participation decreases to zero. Therefore, taking as a given the participation decision of the type $b$ traders, the optimal fraction of class $a$ traders who choose to participate ($\omega_1^a$) must solve $\delta_1 \mid_{\omega_1^b} = s_1^a$. Similarly for the class $b$ traders. The optimal response functions for traders a and b can be written as

$$\begin{aligned}
\omega_1^a &= \max\left[0, \min\left[1,\ \frac{\mu}{\nu}\left(\frac{s_1^a}{1-s_1^a}\right) + \left(\frac{1+s_1^a}{1-s_1^a}\right)\omega_1^b\right]\right], \\
\omega_1^b &= \max\left[0, \min\left[1,\ \frac{\mu}{\nu}\left(\frac{s_1^b}{1-s_1^b}\right) + \left(\frac{1+s_1^b}{1-s_1^b}\right)\omega_1^a\right]\right].
\end{aligned}$$

Ignoring the constraints, the solution to the above set of equations is $\omega_1^a = \omega_1^b = -\frac{\mu}{2\nu}$. If we plot both response functions in the same figure with $\omega_1^a$ as the x-axis and $\omega_1^b$ as the y-axis, we have two lines passing through the same point $\left(-\frac{\mu}{2\nu}, -\frac{\mu}{2\nu}\right)$, with a slope of $(\frac{1+s_1^b}{1-s_1^b})$ for the response function of $\omega_1^b$, and $(\frac{1-s_1^a}{1+s_1^a})$ for the response function of $\omega_1^a$.

From the above equations, we note that $\omega_1^i$ increases in $s_1^i$, therefore, a higher $s_1^i$ implies higher participation ratio for trader $i$. With constraints, if both $s_1^a > 1$ and $s_1^b > 1$, then both traders participate, or $\omega_1^a = \omega_1^b = 1$. When both $s_1^a < 1$ and $s_1^b < 1$, only $\omega_1^a = \omega_1^b = 0$ emerges as a solution, or no one participates. Otherwise, the solution depends both the relative size of $s_1^i$, and on the slope of the two lines. Specifically, if $s_1^a > s_1^b$, then trader a is more likely to participate. And if the slope of the response function for $\omega_1^b$ is higher, or

$$\frac{2}{1 - s_1^b} - 1 > \frac{1}{\frac{2}{1-s_1^a} - 1} \iff s_1^a + s_1^b > 0,$$

then $\omega_1^a = 1$ and $\omega_1^b$ solve the above equation. All other cases follows similarly.

**Proof of Proposition 4.**

Given the realization of shocks $Y$ and $X_1$, individual participation decisions are fully determined in Proposition 3, and so is the equilibrium price $P_2$ in equation (11). We can calculate the expected value function of the market maker at date $t = 1$ if they choose to hold $\theta_1^m$ stocks:

$$\mathrm{E}[J_2^m \mid \theta_1^m, P_1, Y, X_1] = -\frac{1}{\sqrt{1 + k\delta_1^2}} e^{-\alpha \;\; F_1^m + \theta_1^m (V - P_1) - \frac{1}{2}\alpha\sigma_v^2 (\theta_1^m)^2 + \frac{\frac{1}{2}\alpha\sigma_v^2(\theta_1^m - \delta_1 X_1)^2}{1+k\delta_1^2}}$$

where $F_1^m$ is total financial wealth at beginning of period. Taking first order condition with respect to $\theta_1^m$ and applying market clearing condition $\theta_1^m = 0$ yields the equilibrium price $P_1$.

**Proof of Proposition 5.**

Let $\lambda_1 \equiv \frac{\lambda_2}{1+k\,\delta_1^2}$ and $\lambda_2 \equiv \alpha\sigma_v^2\delta_1$ be the coefficient in front of idiosyncratic shocks $X_t$ in (18) and (11) for equilibrium stock prices at date 1 and 2, the equilibrium price at date $t = 0$ conditional on the realization of $Y$ is the solution to the following equation

$$P_0|_Y = V - \frac{\mathrm{E}\left[\sqrt{\lambda_1/\lambda_2}\,(\lambda_1 X_1)\, e^{-\frac{1}{2}\alpha\left[-\alpha\sigma_v^2 Y + \lambda_1\delta_1 X_1^2\right]}\right]}{\mathrm{E}\left[\sqrt{\lambda_1/\lambda_2}\, e^{-\frac{1}{2}\alpha\left[-\alpha\sigma_v^2 Y + \lambda_1\delta_1 X_1^2\right]}\right]}$$

Since $\delta_1(Y) = -\delta_1(-Y)$, $\lambda_1(Y) = -\lambda_1(-Y)$ and $\lambda_2(Y) = -\lambda_2(-Y)$, it is easy to verify that $P_0|_Y - V = -(P_0|_{-Y} - V)$. Given the symmetry between the case of $Y$ and $-Y$, the only feasible price is $P_0 = V$.

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