Implied Volatility Smirk

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This paper studies implied volatility smirk quantitatively. We first propose a new concept of smirkness, which is defined as a triplet of at-the-money implied volatility, skewness (slope at the money) and smileness (curvature at the money) of implied volatility – moneyness curve. Empirical evidence from S&P 500 (SPX) index options shows that a quadratic function with both skewness and smileness fits the market implied volatility smirk very well. The risk-neutral probability density function can be recovered analytically from a "smirked" implied volatility. A new maturity- and liquidity-based procedure is proposed to calibrate option pricing models.

Keywords: Option pricing; Implied volatility smirk; Smirkness; Term structure

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Abstract

This paper studies implied volatility smirk quantitatively. We first propose a new concept of smirkness, which is defined as a triplet of at-the-money implied volatility, skewness (slope at the money) and smileness (curvature at the money) of implied volatility – moneyness curve. Empirical evidence from S&P 500 (SPX) index options shows that a quadratic function with both skewness and smileness fits the market implied volatility smirk very well. The risk-neutral probability density function can be recovered analytically from a "smirked" implied volatility. A new maturity- and liquidity-based procedure is proposed to calibrate option pricing models.

1 Introduction

Option market has been growing rapidly in the last decade. It becomes one of the most important financial markets worldwide. Based on a report of the Chicago Board Options Exchange (CBOE), the total global market value of cash market was 39.75 trillion US dollars in 2000, that of exchange-traded index options was 17.28 trillion US dollars, and that of exchange-traded stock options was 4.93 trillion US dollars.

The market price of an option is observable, and it reflects investor's expectation of underlying stock return. Somehow the study on the information content of option prices is very limited, partially because of our lack of appropriate quantities to describe the information. So far there is only one index, i.e., VIX volatility index reported by the CBOE, that is generated from the S&P index option prices.

There are two versions of volatility index, an old one and a new one. The old VIX, renamed to be VXO in 2003, was introduced by the CBOE in 1993. It represents the implied volatility of a hypothetical at-the-money S&P 100 index (OEX) option with 30 calendar days to maturity. In September 2003, the CBOE began disseminating a new VIX index, which is computed based on the prices of a portfolio of one-month out-of-the-money S&P 500 index (SPX) options with weights being inversely proportional to the squared strike price. Since 1993, the VIX (VXO) has quickly become the benchmark for stock market volatility and is often referred to as the "investor fear gauge". New financial products, VIX futures and options, were proposed by the CBOE to provide retail investors a vehicle to trade market volatility. They were starting to be listed in the CBOE on March 26, 2004. The new VIX is a fair expectation of volatility one month in the future. It is certainly a very important quantity, but it is only one piece of information distilled from the S&P index option market.

In this paper, we propose additional quantities that can be and should be extracted from the option market. We first propose a new concept of smirkness, which is defined as a triplet of at-the-money implied volatility, skewness (slope at the money) and smileness (curvature at the money) of implied volatility – moneyness curve. The moneyness is the logarithm of the strike price over the forward price, normalized by the standard deviation of expected return on maturity. With these new quantities well-defined, we are able to study the implied volatility smirk quantitatively.

Empirical evidence from S&P 500 index options shows that a quadratic function with both skewness and smileness fits the market implied volatility smirk very well. The volume weighted error can be smaller than the smallest bid-ask spread of traded options. Theoretical analysis shows that the smirkness triplet is related to the cumulants of the risk-neutral probability of the underlying asset return.

The term structure of at-the-money implied volatility (ATM-IV) of November S&P 500 index options is upward sloping on November 4, 2003. The value of implied volatility ranges from 14% for two-week short term options to 18% for long term options with an observable horizon of two-years. The term structure of skewness is almost flat especially for long term options. The value of skewness is about -0.17, very stable for options with maturity ranging from two months up to two years. The negative value of implied volatility skewness implies the negative skewness of the risk-neutral distribution of S&P 500 index return. The implied volatility smirk does not flatten out as maturity increases up to almost two years. This pattern has been documented and explained with a finite moment log stable process by Carr and Wu (2003). The term structure of smileness is oscillating between 0 and 0.04. We also observe that the term structure of the ATM-IV does not change much on a daily basis. It has a more regular pattern than the other two. It indicates that investors have a better understanding about the ATM-IV than the skewness and smileness. Empirical result also shows that the time series of the smirkness for options with a fixed maturity date is quite stationary.

Our method of documenting the shapes and evolution of the term structures, and the dynamics of the smirkness triplet seems to be new and has not been proposed before in the literature[2]. Based on the information of the three term structures distilled from option

[2]Foresi and Wu (2002) document the phenomenon of implied volatility smirk for the major equity indexes in twelve countries. Dennis and Mayhew (2002) empirically study the cross-sectional behavior of implied volatility smirk, in particular the relation between implied volatility skewness of individual stocks and their betas. Our proposal of quantifying and documenting implied volatility smirk can be applied to

market, we propose an effective and efficient way of calibrating option pricing models.

The paper is organized as follows. Section 2 uses the market prices of out-of-money options to illustrate the concept of the smirkness triplet, i.e., ATM-IV, skewness and smileness, and study the price impact of the smirkness. Section 3 recovers the risk-neutral probability distribution of underlying index return and links the smirkness with the variance, skewness and kurtosis of the risk-neutral probability. Section 4 documents the term structure of the smirkness and its evolution in three days. Section 5 documents the time-change dynamics of the smirkness from September 25, 1998 to September 5, 1999 for SPX options with a fixed maturity on September 16, 1999. Section 6 proposes a new way of calibrating option pricing models by using the term structure of the smirkness. Section 7 concludes.

2 Quantify Implied Volatility Smirk

Most of the existing research on implied volatility smirk are qualitative in nature, partially because we lack appropriate quantities to describe the implied volatility – strike price curve. For the convenience of quantifying an implied volatility smirk, we need to define a few concepts: *implied forward price*, *moneyness*, *skewness*, *smileness* and *smirkness*.

Following Duffie, Pan and Singleton (2000), Ait-Sahalia, Wang and Yared (2001), and Eraker (2004), we are using daily CBOE closing option prices to study the phenomenon of implied volatility smirk and the performance of different option-pricing models. The data of November 4, 2003, used here to demonstrate the definition of a few quantities, are downloaded from the CBOE website. The data of the treasury yield curve rates are downloaded from the website of U.S. Department of Treasury.

2.1 Implied Forward Price and Put-call Parity

We introduce implied forward price and study put-call parity by using S&P 500 index (SPX) options.

Table 1 is the market data of the option prices on November 4, 2003. The closing

any options, including index, stock and exchange rate options. Bollen and Whaley (2004) study the source of implied volatility smirk, in particular the effect of net buying pressure.

index on the day was $S_0 = 1053.25$. All the options are sorted in ascending order by strike price ranging between 850 and 1125. The lowest strike price is selected from the first out-of-the-money put with a non-zero bid price. The highest strike is selected from the first out-of-the-money call with a non-zero bid price.[3] There is a notable difference between the last sale price and mid-value of bid and ask. Some of the last sale prices even fall outside of the range between bid and ask. This is because the last sale happened sometime before the market was closed. Since it is difficult to determine the time instance of the last sale, we do not use the last sale price to compute implied volatility, we use the mid-value of closing bid and ask instead.

The *implied forward price*, F_0, is determined based on at-the-money option prices. The at-the-money strike is the strike price at which the difference between the call and put prices is smallest. As shown in Table 1, the difference between the call and put prices is smallest at the 1055 strike on November 4, 2003. The forward index level is calculated by following formula

$$\begin{aligned} F_0 &= \text{Strike price} + e^{r\tau} \times (\text{Call price} - \text{Put price}) \\ &= 1055 + e^{0.9743\% \times 17/365}(11.9 - 14.2) = 1052.70, \end{aligned}$$

where the risk-free rate, r, is determined by the U.S. treasury yield curve rates on November 4, 2003, provided in Table 2. Since the yield curve starts from 1 month rate and the term of the options is only 17 days, an extrapolation technique is used to compute the rate for 17 days, i.e.,

$$r = r_{1\text{mth}} - (r_{3\text{mth}} - r_{1\text{mth}}) \times \frac{30-17}{61} = 0.97\% - (0.95\% - 0.97\%) \times \frac{30-17}{61} = 0.9743\%.$$

The forward index level, $F_0 = 1052.70$, is slightly smaller than the closing index, $S_0 = 1053.25$. This means that the expected dividend yield, q, is slightly larger than the risk-free rate, r.

With the implied forward index, we define the *time value of an option* (call or put) to be the difference between the option price and its intrinsic value. For a given maturity,

[3] We follow the practice set up by the CBOE (2003) in computing the new volatility index, VIX.

the time value of an option is a function of strike. For example, the time value of a call is defined as

$$c_{tv}(K) = c_0(K) - e^{-r\tau}\max(F_0 - K, 0), \tag{1}$$

and the time value of a put is defined as

$$p_{tv}(K) = p_0(K) - e^{-r\tau}\max(K - F_0, 0). \tag{2}$$

If put-call parity is true for a call and a put with time to maturity, τ, and strike, K, i.e.,

$$c_0(K) - p_0(K) = (F_0 - K)e^{-r\tau}, \tag{3}$$

then the time values of the call and the put will be the same,

$$c_{tv}(K) = p_{tv}(K). \tag{4}$$

This can be easily verified by a simple algebra with equations (1), (2) and (3).

Figure 1 shows the time values of SPX index options on November 4, 2003 for both calls and puts that mature on November 17, 2003. The time values are computed by using the formula (1) and (2) from the market prices of options listed in Table 1. The two time value functions from calls and puts almost collapse each other. This means that the put-call parity is valid for the market prices of options with different strikes on the day.

Since put-call parity holds, the market data of either call or put gives the same value of the implied volatility. Following Carr and Wu (2003) and the practice set up by the CBOE (2003) in computing the new VIX index, we use out-of-the-money options to compute the implied volatilities for different strikes. We choose put option for strike that is below forward index, $K < F_0$, choose call option for strike that is above forward index, $K > F_0$. The exclusive use of out-of-the-money options is an industry convention that arises from their greater liquidity and model sensitivity than their in-the-money counterparts.

2.2 Implied Volatility Smirk

We now study the phenomenon of implied volatility smirk, i.e., implied volatility as a function of moneyness.

The *moneyness* is defined as the logarithm of the strike price over the forward price, normalized by the standard deviation of expected return on maturity, i.e.,

$$\xi \equiv \frac{\ln(K/F_0)}{\bar{\sigma}\sqrt{\tau}}, \tag{5}$$

where $\bar{\sigma}$ denotes some measure of the average volatility of the index. We use the volatility index VIX reported by the CBOE as a proxy of the average volatility. This definition of moneyness follows Carr and Wu (2003). It has been widely used in financial industry.

The implied volatility, σ_{I}, is computed by equating the Black-Scholes formula to the market price of an option. For example, on a particular day, the implied forward price, F_0 can be computed. For a call option, given time to maturity, τ, the discount rate over the period, r, can be determined by using yield curve. We solve the following equation

$$F_0 e^{-r\tau} N\left(\frac{\ln(F_0/K) + \frac{1}{2}\sigma_{\mathrm{I}}^2\tau}{\sigma_{\mathrm{I}}\sqrt{\tau}}\right) - Ke^{-r\tau} N\left(\frac{\ln(F_0/K) - \frac{1}{2}\sigma_{\mathrm{I}}^2\tau}{\sigma_{\mathrm{I}}\sqrt{\tau}}\right) = c_{\mathrm{mkt}}$$

for each strike, K, and obtain an implied volatility, σ_{I}, as a function of strike. Since a strike has a one-to-one mapping with a moneyness, ξ, we are able to obtain an implied volatility as function of moneyness, $\sigma_{\mathrm{I}}(\xi)$, which is regarded as *implied volatility smirk*.

Table 3 shows a sample implied volatility smirk on November 4, 2003 for SPX options that mature on November 21, 2003. We list all available strikes for options with non-zero bid, choose put options for strike that is below the forward index, and choose call options for strike that is above the forward index. In other words, we pick all the out-of-the-money calls and puts with non-zero bid. We collect the information of market prices and trading volumes of these options, and compute moneyness and implied volatility. We then use a quadratic function to fit the implied volatility data by minimizing the volume weighted mean squared error

$$\frac{\sum_\xi \mathrm{Volume} \times [\sigma_{\mathrm{Imarket}} - \sigma_{\mathrm{I}}(\xi)]^2}{\sum_\xi \mathrm{Volume}}$$

and obtain

$$\sigma_{\mathrm{I}}(\xi) = 0.1447 - 0.0189\xi + 0.00595\xi^2 = 0.1447(1 - 0.1308\xi + 0.0411\xi^2).$$

When we do the fitting, we force the curve passing through the point at the money. In other words, the point $(\xi, \sigma_{\mathrm{I}}) = (0, 0.1447)$ is given by the market data. Therefore the fitted implied volatility smirk gives exact the same price as the market price for an option at the money. There is no-arbitrage between the model price and the market price for an option at the money.

The implied volatility smirk is shown graphically in Figure 2. The dots are the implied volatilities from market prices of out-of-the-money calls and puts with different strikes. The solid line is a fitted quadratic curve. One may observe that the quadratic function approximates the market implied volatility smirk very well. This is confirmed by the fitted error provided in Table 3. The root of volume weighted mean squared error is only 0.0023, which is about 1.5% of the average volatility, VIX = 0.1655, on the day.

Based on this observation and for the convenience of describing a smirk quantitatively, we introduce a few concepts:

1. *At-the-money implied volatility* (ATM-IV) is defined as

$$\sigma_{\mathrm{I}0} \equiv \sigma_{\mathrm{I}}(\xi)|_{\xi=0}. \tag{6}$$

   From the result of Figure 2, we know that on November 4, 2003, the 17 days ATM-IV of SPX option is 0.1447, or 14.47%. ATM-IV measures the level of an implied volatility smirk. The ratio between implied volatility and the ATM-IV, $\sigma_{\mathrm{I}}(\xi)/\sigma_{\mathrm{I}0}$, is regarded as a *normalized implied volatility.*

2. The *skewness of implied volatility* at-the-money is defined as the first order sensitivity of the normalized implied volatility with respect to the moneyness, i.e.,

$$\gamma_1 \equiv \frac{\partial}{\partial \xi} \left[\frac{\sigma_{\mathrm{I}}(\xi)}{\sigma_{\mathrm{I}0}} \right] \Bigg|_{\xi=0}. \tag{7}$$

   On November 4, 2003, the skewness of implied volatility for November SPX options is -0.1308. Skewness measures the slope at the money of a normalized implied volatility smirk.

3. The *smileness of implied volatility* at-the-money is defined as half of the second order sensitivity of the normalized implied volatility with respect to the moneyness, i.e.,

$$\gamma_2 \equiv \frac{1}{2} \frac{\partial^2}{\partial \xi^2} \left[\frac{\sigma_{\mathrm{I}}(\xi)}{\sigma_{\mathrm{I}0}} \right] \Bigg|_{\xi=0}. \tag{8}$$

On November 4, 2003, the smileness of implied volatility for November SPX options is 0.0411. Smileness measures the curvature at the money of a normalized implied volatility smirk.

The first quantity, the ATM-IV is well-known, but the second one and the third one are new concepts that have never been proposed before. They are referred as smirk slope and smirk curvature qualitatively in Carr and Wu (2003). With these new concepts well-defined quantitatively, an implied volatility smirk for each maturity can be fully described by using the following quadratic function

$$\sigma_{\mathrm{I}}(\xi) = \sigma_{\mathrm{I}0}(1 + \gamma_1\xi + \gamma_2\xi^2), \tag{9}$$

where the ATM-IV, $\sigma_{\mathrm{I}0}$, the skewness, γ_1, and the smileness, γ_2, depend on the time to maturity, τ. The set of three parameters, $(\sigma_{\mathrm{I}0}, \gamma_1, \gamma_2)$ is regarded as *smirkness* or *smirkness triplet* for the convenience of describing the implied volatility smirk.

One should not be confused with the two concepts: implied volatility smirk and smirkness. The former one describes the phenomenon or picture of implied volatility – moneyness curve, while the latter one is a set of three numbers, ATM-IV, skewness and smileness.

2.3 The Impact of Skewness and Smileness on Option Price

For one set of options with the same maturity, one may use three different functions of moneyness to describe implied volatility,

$$\begin{aligned}
&\text{flat} && \sigma_{\mathrm{I}} = \sigma_{\mathrm{I}0}, \\
&\text{skewed} && \sigma_{\mathrm{I}} = \sigma_{\mathrm{I}0}(1 + \gamma_1\xi), \\
&\text{smirked} && \sigma_{\mathrm{I}} = \sigma_{\mathrm{I}0}(1 + \gamma_1\xi + \gamma_2\xi^2).
\end{aligned} \tag{10}$$

The "flat" implied volatility function is a constant, which corresponds to the Black-Scholes model with a volatility being the ATM-IV. The "skewed" one is a linear function that passes through the point at the money and incorporates the skewness. The "smirked" one is a quadratic function with both the skewness and smileness. It becomes a pure smile if the skewness, γ_1, is zero. Therefore the last case is the most general one. Figure 3 shows the difference between three implied volatility functions with the market implied volatility as a reference. The market implied volatility, shown as dots in the figure, is the data on November 4, 2003 for November SPX options.

We now study the price impact of the skewness and the smileness. Again we use the market data on November 4, 2003 for options that mature on November 21, 2003 to illustrate our basic idea. Table 4 shows the option prices computed by using the Black-Scholes formula with "flat", "skewed", and "smirked" volatility functions. The root of volume weighted mean squared error is 78 cents for "flat" implied volatility function (or the Black-Scholes constant volatility case), 31 cents for "skewed" one, and 12 cents for "smirked" one. The skewness reduces the error by more than 60%, and the smileness reduces the remaining error by another 60%. With both skewness and smileness, the final error is only 12 cents, which is smaller than the smallest bid-ask spread. Within all the traded options listed in Table 1, the SPX call with strike 1125 has the smallest bid-ask spread that is 15 cents. Therefore smirked implied volatility function is good enough to capture the trend of implied volatility – moneyness curve.

Figure 4 shows the option price and price error as functions of strike for different implied volatility functions, together with the trading volumes. It is quite impressive that the price given by the Black-Scholes formula with a smirked implied volatility function is so close to the market prices of the traded options.

3 Recover risk-neutral probability distribution from "smirked" implied volatility

In this section, we present an analytical method to recover the risk-neutral probability distribution from a "smirked" implied volatility. We then identify the analytical relationships between smirkness triplet and the cumulants of risk-neutral probability of the underlying stock[4] return.

[4]The underlying stock, the underlying asset and the underlying index have the same meaning. They are used interchangeably in this paper.

Suppose the price of a European call option is given by the Black-Scholes formula in terms of a "smirked" implied volatility, $\sigma_{\rm I}$,

$$c_0 = F_0 e^{-r\tau} N(d + \sigma_{\rm I}\sqrt{\tau}) - K e^{-r\tau} N(d), \qquad d = \frac{\ln(F_0/K) - \frac{1}{2}\sigma_{\rm I}^2\tau}{\sigma_{\rm I}\sqrt{\tau}}, \tag{11}$$
$$F_0 = S_0 e^{(r-q)\tau}, \qquad \sigma_{\rm I} = \sigma_{\rm I0}(1 + \gamma_1\xi + \gamma_2\xi^2), \qquad \xi = \frac{\ln(K/F_0)}{\bar{\sigma}\sqrt{\tau}},$$

then the risk-neutral probability can be recovered from the option pricing formula.

From the risk-neutral valuation formula, the call option price can be written in an integration form

$$c_0 = e^{-r\tau}\int_K^{+\infty}(S_T - K) f(S_T, T; S_0, 0)\, dS_T, \tag{12}$$

where $f(S_T, T; S_0, 0)$ is the risk-neutral probability density function of final stock price, S_T, conditional on information up to time $t = 0$. Taking partial derivative on c_0 with respect to K, with some algebra, we have

$$1 + e^{r\tau}\frac{\partial c_0}{\partial K} = 1 - \int_K^{+\infty} f(S_T, T; S_0, 0)\, dS_T = \int_{-\infty}^{K} f(S_T, T; S_0, 0)\, dS_T \equiv F(K, T; S_0, 0), \tag{13}$$

where $F(K, T; S_0, 0)$ is the cumulative probability function evaluated at $S_T = K$. The equation can be written in another way

$$F(S_T, T; S_0, 0) = 1 + e^{r\tau}\left.\frac{\partial c_0}{\partial K}\right|_{K=S_T}. \tag{14}$$

Taking partial derivative on equation (13) with respect to K once more gives Breeden and Litzenberger's (1978) formula for the probability density function

$$e^{r\tau}\frac{\partial^2 c_0}{\partial K^2} = f(K, T; S_0, 0), \qquad \text{or} \qquad f(S_T, T; S_0, 0) = e^{r\tau}\left.\frac{\partial^2 c_0}{\partial K^2}\right|_{K=S_T}. \tag{15}$$

Applying general formulas (14, 15) to equation (11) gives following result.

Theorem 1. *The cumulative probability function recovered from the Black-Scholes option pricing formula (11) with a "smirked" implied volatility is*

$$F(S_T, T; S_0, 0) \;=\; N(-d^*) + n(d^*)\frac{\sigma_{\rm I0}}{\bar{\sigma}}(\gamma_1 + 2\gamma_2\xi^*), \tag{16}$$

where $n(\cdot)$ is the standard normal density function, given by

$$n(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}, \tag{17}$$

and

$$d^* = -\frac{\ln(S_T/F_0) + \frac{1}{2}\sigma_{\mathrm{I}}^{*2}\tau}{\sigma_{\mathrm{I}}^*\sqrt{\tau}} = -\frac{1}{2}\sigma_{\mathrm{I}}^*\sqrt{\tau} - \frac{\bar{\sigma}}{\sigma_{\mathrm{I}}^*}\xi^*, \qquad \xi^* = \frac{\ln(S_T/F_0)}{\bar{\sigma}\sqrt{\tau}}.$$

The recovered probability density function is

$$\begin{aligned} f(S_T, T; S_0, 0) \quad &= \quad \frac{n(d^*)}{S_T\sigma_{\mathrm{I}}^*\sqrt{\tau}} \left[1 + d^*\frac{\sigma_{\mathrm{I}0}}{\bar{\sigma}}(\gamma_1 + 2\gamma_2\xi^*)\right] \left[1 + \left(d^* + \sigma_{\mathrm{I}}^*\sqrt{\tau}\right)\frac{\sigma_{\mathrm{I}0}}{\bar{\sigma}}(\gamma_1 + 2\gamma_2\xi^*)\right] \\ &\quad + \frac{n(d^*)}{S_T}\frac{2\sigma_{\mathrm{I}0}}{\bar{\sigma}^2\sqrt{\tau}}\gamma_2, \end{aligned} \tag{18}$$

where $\sigma_{\mathrm{I}}^* = \sigma_{\mathrm{I}0}(1 + \gamma_1\xi^* + \gamma_2\xi^{*2})$.

Remark 1. The idea of fitting a smooth curve to an implied volatility smirk was first reported by Shimko (1993). He also recovers the risk-neutral probability from the implied volatility function by using Breeden and Litzenberger's formula. Our contribution is to simplify the implied volatility function by a quadratic function and derive explicit and intuitive formulas for risk-neutral probability.

Remark 2. The method of Rubinstein (1994) and Jackwerth and Rubinstein (1996) to recover risk-neutral probability is a numerical optimization technique. Implementing their method requires a considerable effort in nonlinear programming. In contrast, our approach here is an analytical-driven method. We only need to fit the implied volatility curve by a quadratic function, which can be done easily by standard software, such as Mathematica. Once the fitting is done, one may directly apply the analytical formulas in Theorem 1 to describe risk-neutral distribution. It seems to us that our approach is much simpler and more intuitive.

Figure 5 shows the risk-neutral probability density function recovered from a "flat", "skewed", "smiled" and "smirked" implied volatility. In general, the implied volatility skewness, γ_1, generates risk-neutral skewness, and the implied volatility smileness, γ_2, generates risk-neutral kurtosis.

We now derive analytical relationships between the implied volatility smirkness and risk-neutral cumulants.

Suppose in a risk-neutral world, the underlying stock price at maturity, S_τ, is modelled by

$$S_\tau = S_0 e^{(r-q-\frac{1}{2}\sigma_0^2+\mu)\tau+\sigma_0\sqrt{\tau}y}, \tag{19}$$

where μ is the convexity adjustment, y is a random number with mean zero, variance 1, skewness,[5] λ_1 and kurtosis, λ_2. If y is normally distributed, then $\mu = 0$. The probability density of y is given by the Edgeworth series expansion[6]

$$f(y) = n(y) - \frac{\lambda_1}{3!}\frac{d^3 n(y)}{dy^3} + \frac{\lambda_2}{4!}\frac{d^4 n(y)}{dy^4}, \tag{20}$$

where $n(y)$ is the standard normal density function given by (17). The martingale condition in the risk-neutral probability measure requires

$$S_0 e^{(r-q)\tau} = E_0^Q[S_\tau] = E_0^Q[S_0 e^{(r-q-\frac{1}{2}\sigma_0^2+\mu)\tau+\sigma_0\sqrt{\tau}y}], \quad \text{or} \quad E_0^Q[e^{(-\frac{1}{2}\sigma_0^2+\mu)\tau+\sigma_0\sqrt{\tau}y}] = 1,$$

which determines the convexity adjustment term

$$\mu = -\frac{1}{\tau}\ln\left[1 + \frac{\lambda_1}{3!}(\sigma_0\sqrt{\tau})^3 + \frac{\lambda_2}{4!}(\sigma_0\sqrt{\tau})^4\right]. \tag{21}$$

The price of a European call option can be priced with the risk-neutral valuation formula

$$c_0^* = e^{-r\tau}E_0^Q[\max(S_\tau - K, 0)] = e^{-r\tau}\int_{-d_2}^{+\infty}(S_0 e^{(r-q-\frac{1}{2}\sigma_0^2+\mu)\tau+\sigma_0\sqrt{\tau}y} - K)f(y)dy,$$

[5]The skewness, λ_1, and kurtosis, λ_2, of a random number, x, are defined by $\lambda_1 = \kappa_3/\kappa_2^{3/2}$, $\lambda_2 = \kappa_4/\kappa_2^2$, where κ_i, $i = 1,2,3,4$ are first four cumulants, given by $\kappa_1 = E(x)$, $\kappa_2 = E(x-\kappa_1)^2 = Var(x)$, $\kappa_3 = E(x-\kappa_1)^3$, $\kappa_4 = E(x-\kappa_1)^4 - 3\kappa_2^2$.

[6]Here we use Edgeworth series expansion to expand the unknown *return* distribution near a normal distribution. Jarrow and Rudd (1982) expand the unknown *price* distribution near a lognormal distribution and result a different option-pricing formula, see e.g., Corrado and Su (1997) for an empirical test of Jarrow and Rudd's (1982) model with SPX options. It seems to us that expanding return distribution is more natural and consistent with the later advanced option-pricing models that model stock returns with a Lévy process or time-changed Lévy process, see e.g., Carr and Wu (2004).

where

$$d_2 = \frac{\ln(S_0/K) + (r - q - \frac{1}{2}\sigma_0^2 + \mu)\tau}{\sigma_0\sqrt{\tau}}.$$

With integration by parts, we have following formula[7]

$$\begin{aligned} c_0^* &= [S_0 e^{-q\tau} N(d_1) - K e^{-r\tau} N(d_2)] \left[1 + \frac{\lambda_1}{3!}(\sigma_0\sqrt{\tau})^3 + \frac{\lambda_2}{4!}(\sigma_0\sqrt{\tau})^4\right] \\ &\quad + K e^{-r\tau}\left(\frac{\lambda_1}{3!}A + \frac{\lambda_2}{4!}B\right)\sigma_0\sqrt{\tau}, \end{aligned} \tag{22}$$

where $d_1 = d_2 + \sigma_0\sqrt{\tau}$, and

$$A = -n'(-d_2) + \sigma_0\sqrt{\tau}\; n(-d_2) + \sigma_0^2\tau N(d_2) = -(d_2 - \sigma_0\sqrt{\tau})n(d_2) + \sigma_0^2\tau N(d_2), \tag{23}$$

$$\begin{aligned} B &= n''(-d_2) - \sigma_0\sqrt{\tau}\; n'(-d_2) + \sigma_0^2\tau\; n(-d_2) + (\sigma_0\sqrt{\tau})^3 N(d_2) \\ &= -(1 - d_2^2 + \sigma_0\sqrt{\tau}\; d_2 - \sigma_0^2\tau)\; n(d_2) + (\sigma_0\sqrt{\tau})^3 N(d_2). \end{aligned} \tag{24}$$

This call option pricing formula is exact under the assumption that the risk-neutral probability density is given by equation (20).

We now match the two option pricing formulas given by equations (11) and (22). Applying the three conditions at the money

$$(c_0 - c_0^*)|_{K=F_0} = 0, \qquad \left.\frac{\partial(c_0 - c_0^*)}{\partial K}\right|_{K=F_0} = 0, \qquad \left.\frac{\partial^2(c_0 - c_0^*)}{\partial K^2}\right|_{K=F_0} = 0$$

gives following result.

Theorem 2. *The implied volatility smirkness, (σ_{I0}, γ_1, γ_2), and the risk-neutral standard deviation, skewness and kurtosis, (σ_0, λ_1, λ_2) are related by following equations*

$$\begin{aligned} & 1 - 2N(d^*) = [N(d_1) - N(d_2)]\left[1 + \frac{\lambda_1}{3!}(\sigma_0\sqrt{\tau})^3 + \frac{\lambda_2}{4!}(\sigma_0\sqrt{\tau})^4\right] + \left(\frac{\lambda_1}{3!}A + \frac{\lambda_2}{4!}B\right)\sigma_0\sqrt{\tau}, \\ & N(-d^*) + n(d^*)\frac{\sigma_{I0}}{\bar{\sigma}}\gamma_1 = N(-d_2) - \left[\frac{\lambda_1}{3!}(d_2^2 - 1) - \frac{\lambda_2}{4!}(d_2^3 - 3d_2)\right] n(d_2), \\ & \frac{1}{\sigma_{I0}\sqrt{\tau}}\left(1 - d^{*2}\frac{\sigma_{I0}^2}{\bar{\sigma}^2}\gamma_1^2 + \frac{2\sigma_{I0}^2}{\bar{\sigma}^2}\gamma_2\right) n(d^*) = \frac{1}{\sigma_0\sqrt{\tau}}\left[1 - \frac{\lambda_1}{3!}(d_2^3 - 3d_2) + \frac{\lambda_2}{4!}(d_2^4 - 6d_2^2 + 3)\right] n(d_2), \end{aligned} \tag{25}$$

where

$$d^* = -\frac{1}{2}\sigma_{I0}\sqrt{\tau}, \qquad d_2 = -\frac{1}{2}\sigma_0\sqrt{\tau} + \frac{\mu\sqrt{\tau}}{\sigma_0}, \qquad d_1 = d_2 + \sigma_0\sqrt{\tau},$$

[7]The formula has been presented by Backus et al (1997) in the context of currency options.

A and B are given by (23, 24), μ is given by (21).

In the three equations in (25), the terms at the left-hand-side are only related to the smirkness, the terms at the right-hand-side are only related with the risk-neutral cumulants. Given one set of values, one may solve the three equations for another set.

Asymptotic analysis on three equations in (25) yields following approximate relationships.

Corollary 1. *The implied volatility smirkness, $(\sigma_{\mathrm{I}0}, \gamma_1, \gamma_2)$, and the risk-neutral standard deviation, skewness and kurtosis, $(\sigma_0, \lambda_1, \lambda_2)$ are related by following approximate equations*

$$\begin{aligned} \sigma_{\mathrm{I}0} &= \left(1-\frac{\lambda_2}{24}\right)\sigma_0 + \frac{\lambda_1}{4}\sigma_0^2\sqrt{\tau} + O(\sigma_0^3\tau), \\ \gamma_1 &= \frac{\lambda_1}{6\left(1-\frac{\lambda_2}{24}\right)}\frac{\bar{\sigma}}{\sigma_0} + \frac{\lambda_2}{12\left(1-\frac{\lambda_2}{24}\right)}\bar{\sigma}\sqrt{\tau} + O(\sigma_0\bar{\sigma}\sqrt{\tau}), \\ \gamma_2 &= \frac{\lambda_2}{24}\frac{\bar{\sigma}^2}{\sigma_0^2}\frac{1-\frac{\lambda_2}{16}}{\left(1-\frac{\lambda_2}{24}\right)^2} + \frac{\lambda_1\lambda_2}{96}\frac{\bar{\sigma}^2\sqrt{\tau}}{\sigma_0}\frac{1-\frac{\lambda_2}{48}}{\left(1-\frac{\lambda_2}{24}\right)^3} + O(\bar{\sigma}^2\sqrt{\tau}). \end{aligned} \tag{26}$$

Ignoring higher order terms, one may have following leading order result

$$\sigma_{\mathrm{I}0} = \sigma_0, \qquad \gamma_1 = \frac{1}{6}\lambda_1, \qquad \gamma_2 = \frac{1}{24}\lambda_2 \tag{27}$$

as a rule of thumb.

For example, on November 4, 2003, the smirkness triplet for November SPX options is $(\sigma_{\mathrm{I}0}, \gamma_1, \gamma_2) =$ (0.1447, -0.1308, 0.0411), the VIX level is $\bar{\sigma} = 0.1655$. Solving the three equations in (25) numerically gives following numerical values of standard deviation, skewness and kurtosis of the risk-neutral probability distribution of underlying index return

$$\sigma_0 = 0.1506, \qquad \lambda_1 = -0.6992, \qquad \lambda_2 = 0.8065. \tag{28}$$

Substituting these risk-neutral cumulants into three equations in (26), by taking the first terms in each equation we have the approximate values of the smirkness (0.1455, -0.1325, 0.04126). By taking the first two terms in each equation, we have the smirkness (0.1447,

-0.1300, 0.0410), which is indeed quite accurate compared with the given values (0.1447, -0.1308, 0.0411).

Comparing with the market data provided in Table 4, the volume weighted error of the call option price given by formula (22) with the parameters in equation (28) is 22.7 cents, which is somewhere between those (31 and 12 cents) given by the Black-Scholes formula with the "skewed" and the "smirked" volatility functions.

4 The Term Structure of Smirkness

Previously, we introduced a new concept of smirkness, which is a triplet of at-the-money implied volatility, σ_{I0}, skewness (slope at the money), γ_1, and smileness (curvature at the money), γ_2, of normalized implied volatility – moneyness curve. We observe that a "smirked" volatility function

$$\sigma_{\mathrm{I}} = \sigma_{\mathrm{I0}}(1 + \gamma_1\xi + \gamma_2\xi^2)$$

with both skewness and smileness is able to capture the trend of the market implied volatility. The root of volume weighted mean squared error is smaller than the smallest bid-ask spread of the traded options. The empirical exercise, demonstrated in the last section, is for the options with the shortest maturities (17 days). The same study can be done for all other maturities.

Figure 6 shows the implied volatility smirks on November 4, 2003 for options with all available maturities, such as Nov-21-03, Dec-19-03, Jan-16-04, Mar-19-04, Jun-18-04, Sep-17-04, Dec-17-04, and Jun-17-05. The times to maturity are 17, 45, 73, 136, 227, 318, 409, and 591 days respectively. For each maturity, the discount rate is determined by using the yield curve provided in Table 2 with an linear interpolation technique. The market implied volatilities, shown as dots in the figures, are computed by finding the root that equates the Black-Scholes formula with the market price. The moneyness is defined by equation (5). We fit the market implied volatilities with a quadratic function for each maturity and obtain its smirkness triplet. The fitted curves are shown as solid lines. They are presented together with the trading volumes shown as bars.

The numerical values of the term structure of the smirkness are listed in Table 5 together with the term structures of interest rate and trading volume. The first and the second nearest options are heavily traded. They occupy 80% of the total trading volume. The term structure of the interest rate is upward-sloping. The term structure of the ATM-IV is slightly upward-sloping, ranging from 14% to 18%. The skewness is quite stable with a value near -0.17. The implied volatility smirk does not flatten out as maturity increases up to almost two years. This pattern has been documented qualitatively and explained with a finite moment log stable process by Carr and Wu (2003). The term structure of smileness is oscillating between 0 and 0.04. Our approach of documenting the term structure of smirkness seems to be new in the literature.

In order to gain intuition on the evolution of the term structure of the smirkness, we have processed the market data for three different days, Oct-30-03, Nov-04-03 and Nov-12-03[8]. The final numerical values of the term structures are listed in Table 7. They are presented graphically in Figure 7. In general, the term structures are stable for longer maturity. The term structure of the ATM-IV has a more regular pattern than the other two. It indicates that investors have a better understanding about the ATM-IV than the skewness and smileness, probably because the latter two quantities have never been explicitly defined and documented in the literature.

5 The dynamics of Smirkness

In order to have some idea about the time-change dynamics of the implied volatility smirk, we need a longer time series of option prices. The market prices of SPX options from September 25, 1998 to September 3, 1999 are used. We study the options with the same maturity date, September 16, 1999. The daily option price data are provided by a commercial data supplier. The data are cleaned by using the screening procedures described in Bakshi, Cao and Chen (1997).

On each day, we determine the forward index level, F_0. For each option, we compute

[8] As explained in §2, the data of the SPX option prices on the three days are downloaded from the CBOE website.

the moneyness, ξ, by using the formula

$$\xi = \frac{\ln(F_0/K)}{\mathrm{VIX}\sqrt{\tau}}$$

with the VIX on the day. The implied volatility, $\sigma_{\mathrm{I}}(\xi)$, is determined by inverting the Black-Scholes formula. We then fit the market implied volatility with a quadratic function, $\sigma_{\mathrm{I}}(\xi) = \sigma_{\mathrm{I}0}(1 + \gamma_1\xi + \gamma_2\xi^2)$, and obtain the smirkness triplet, $(\sigma_{\mathrm{I}0}, \gamma_1, \gamma_2)$.

The time series of the smirkness for options with the same maturity date, T, is shown in Figure 8. All of the three series look quite stationary.

6 The Applications of Smirkness

Option pricing models are developed with certain number of embedded parameters, which are usually determined from market information with one of two different approaches. The first one is to estimate the parameters by using historical data of underlying stock price. The problem of this approach is that the historical data may not reflect the information of stock return in future. Practitioners often uses a second approach, that is to calibrate the parameters from the current market prices of liquidly-traded options.

Calibrating an option pricing model is a difficult task. It is commonly recognized that one should calibrate a model at least on daily basis to include the latest information of financial markets. At the end of a trading day, we have a closing price for each option with different strikes and maturities. Their trading volumes are different. Some of them, for example, the ones with the first and the second nearest maturities, are liquid. Others are not. The price of illiquid options does not tell us much information of the financial market.

One way to calibrate option pricing model is to minimize the sum of the squared errors of all available options with different strikes and maturities. The procedure, initiated by Bakshi, Cao and Chen (1997), has become a standard in the empirical study of model performance, see e.g., Carr et al (2001), Huang and Wu (2003). But this method suffers from two problems. First, the nonlinear least square is well-known to be unstable, especially for a large set of parameters, such as the ten parameters in a time-changed Lévy process in Huang and Wu (2003). The solution obtained with some standard commercial package may not be unique. For the case that the solution is not unique, the values of the parameters are not meaningful. Second, the procedure does not take into account of the liquidity effect. In principle, the option price with higher trading volume should have more weight in the target function of the numerical optimization problem.

Now we propose a new maturity-based and liquidity-based calibration. The idea is to use the distilled information of the term structure of smirkness. Since we have already considered the liquidity issue in fitting the value of smirkness, all we need to do right now is to force the term structure of smirkness implied from an option pricing model to pass through the points of the market term structures, from the first nearest term[9], to the second nearest one and so on.

For example, the Black-Scholes model

$$dS_t = (r - q)S_t dt + \sigma S_t dB_t$$

has only one parameter, i.e., volatility, σ. One may simply calibrate the model by using ATM-IV, σ_{I0}, for the volatility, σ. In fact, this is a classical way of using the Black-Scholes model.

We now present the calibration of two parsimonious models and study the term structure of smirkness implied in the calibrated option pricing models.

6.1 The Constant Elasticity of Variance (CEV) Model

In a risk-neutral world, the price of a stock, S_t, is assumed to follow a diffusion process

$$dS_t = (r - q)S_t dt + \sigma S_t^{\alpha} dB_t, \tag{29}$$

where r is risk-free rate, q is the continuous dividend yield, σ and α are constants, B_t is the standard Brownian motion. The variance of stock return is given by $v_t = \sigma^2 S_t^{2\alpha-2}$, and the elasticity of variance, defined by $\frac{S_t}{v_t}\frac{\partial v_t}{\partial S_t}$, is therefore a constant, $2\alpha - 2$.

[9]Following the convention set up by CBOE (2003) in computing the new VIX index, with 8 days left to expiration, we roll to the second contract months in order to minimize pricing anomalies that might occur close to expiration.

Cox (1975) identifies that with a power transformation, $X_t = S_t^{2(1-\alpha)}$, the CEV process can be transformed into a square root process

$$dX_t = [(1-\alpha)(1-2\alpha)\sigma^2 + 2(1-\alpha)(r-q)X_t]dt + 2(1-\alpha)\sigma\sqrt{X_t}dB_t. \tag{30}$$

Feller (1951) studies the transition probability of the square root process with a Laplace transformation approach and provides an analytical formula for the density function. Therefore the price of a European call option can be determined with risk-neutral valuation formula. We summarize some relevant results as follows.

Lemma 1. *For $0 < \alpha < 1$, the conditional risk-neutral transition probability density function $f(S_T, T; S_t, t)$ is given by (Cox 1975)*

$$\begin{aligned} f(S_T, T; S_t, t) &= 2(1-\alpha)k^{\frac{1}{2(1-\alpha)}} \left(xw^{1-4\alpha}\right)^{\frac{1}{4(1-\alpha)}} e^{-x-w} I_{-\frac{1}{2(1-\alpha)}}(2\sqrt{xw}), \\ &= 4(1-\alpha)k^{\frac{1}{2(1-\alpha)}} w^{\frac{1-2\alpha}{2(1-\alpha)}} p\left(2w; 2-\frac{1}{1-\alpha}, 2x\right), \end{aligned} \tag{31}$$

where

$$k = \frac{r-q}{(1-\alpha)\sigma^2\left[e^{2(1-\alpha)(r-q)\tau}-1\right]}, \qquad x = kS_t^{2(1-\alpha)}e^{2(1-\alpha)(r-q)\tau}, \qquad w = kS_T^{2(1-\alpha)},$$

and $I_\nu(z)$ is the modified Bessel function[10] *of the first kind of order ν, $p(z; n, \lambda)$ is the probability density function of non-central chi-square distribution*[11] *with n degrees of freedom and non-centrality parameter λ.*

[10] The modified Bessel function of the first kind of order ν is a solution of the ordinary differential equation

$$W'' + \frac{1}{z}W' - \left(1+\frac{\nu^2}{z^2}\right)W = 0.$$

The function can be written in a series form as follows

$$I_\nu = \sum_{k=0}^{+\infty} \frac{1}{k!\,\Gamma(\nu+k+1)}\left(\frac{z}{2}\right)^{2k+\nu}.$$

[11] The probability density function of non-central chi-square distribution can be written in terms of the modified Bessel function as follows (Johnson and Kotz 1970)

$$p(z; n, \lambda) = \frac{1}{2}(z/\lambda)^{\frac{1}{4}(n-2)}e^{-\frac{1}{2}(z+\lambda)}I^{\frac{n-2}{2}}(\sqrt{\lambda z}).$$

The price of a European call option is given by [12] *(Cox 1975, Cox and Ross 1976, Schroder 1989)*

$$\begin{aligned} c_t &= S_t e^{-q\tau} Q\left(2y; 2 + \frac{1}{1-\alpha}, 2x\right) - K e^{-r\tau} Q\left(2y; 2 - \frac{1}{1-\alpha}, 2x\right) \\ &= S_t e^{-q\tau} Q\left(2y; 2 + \frac{1}{1-\alpha}, 2x\right) - K e^{-r\tau}\left[1 - Q\left(2x; \frac{1}{1-\alpha}, 2y\right)\right], \end{aligned} \tag{32}$$

where $y = kK^{2(1-\alpha)}$*, and* $Q(z; n, \lambda)$ *is the complementary non-central chi-square distribution function, defined by* $Q(z; n, \lambda) = \int_z^{+\infty} p(u; n, \lambda) du$.

For $\alpha > 1$*, the transition probability density function* $f(S_T, T; S_t, t)$ *is given by*

$$f(S_T, T; S_t, t) = 2(\alpha - 1) k^{\frac{1}{2(1-\alpha)}} \left(x w^{1-4\alpha}\right)^{\frac{1}{4(1-\alpha)}} e^{-x-w} I_{\frac{1}{2(\alpha-1)}}(2\sqrt{xw}), \tag{33}$$

and the European call option pricing formula is

$$c_t = S_t e^{-q\tau} Q\left(2x; \frac{1}{\alpha-1}, 2y\right) - K e^{-r\tau}\left[1 - Q\left(2y; 2 + \frac{1}{\alpha-1}, 2x\right)\right]. \tag{34}$$

(Emanuel and MacBeth 1982)

Matching the two option pricing formulas in (32) and (11) yields smirkness implied in the CEV model as follows.

Theorem 3. *For* $0 < \alpha < 1$*, the smirkness triplet,* $(\sigma_{\mathrm{I}0}, \gamma_1, \gamma_2)$*, implied in the CEV model can be determined by*

$$1 - 2N(d^*) = Q\left(2x; 2 + \frac{1}{1-\alpha}, 2x\right) + Q\left(2x; \frac{1}{1-\alpha}, 2x\right) - 1, \tag{35}$$

$$N(-d^*) + n(d^*) \frac{\sigma_{\mathrm{I}0}}{\bar{\sigma}} \gamma_1 = Q\left(2x; \frac{1}{1-\alpha}, 2x\right), \tag{36}$$

$$\frac{1}{\sigma_{\mathrm{I}0}\sqrt{\tau}} \left(1 - d^{*2} \frac{\sigma_{\mathrm{I}0}^2}{\bar{\sigma}^2} \gamma_1^2 + \frac{2\sigma_{\mathrm{I}0}^2}{\bar{\sigma}^2} \gamma_2\right) n(d^*) = 4(1-\alpha) x \; p\left(2x; 2 - \frac{1}{1-\alpha}, 2x\right), \tag{37}$$

[12] The complementary non-central chi-square distribution function, $Q(z; n, \lambda)$, satisfies following identity

$$Q(z; n, \lambda) + Q(\lambda; 2 - n, z) = 1.$$

where $n(\cdot)$ is the standard normal density function, and

$$d^* = -\frac{1}{2}\sigma_{I0}\sqrt{\tau}, \qquad x = \frac{(r-q)S_t^{2(1-\alpha)}e^{2(1-\alpha)(r-q)\tau}}{(1-\alpha)\sigma^2\left[e^{2(1-\alpha)(r-q)\tau}-1\right]}.$$

The CEV model has two parameters, volatility, σ, and the elasticity constant, α. One should use the ATM-IV, σ_{I0}, and skewness, γ_1, of the nearest term (the most liquid ones) of the implied volatility smirk to determine these two constants.

Now we perform a calibration exercise on November 4, 2003. The S&P 500 index level is $S_t = 1053.25$ and the VIX index is $\bar{\sigma} = 0.1655$. We consider options that mature on November 21, 2003. Therefore time to maturity is $\tau = 17/365$. The corresponding risk-free rate is $r = 0.009743$. The dividend yield, computed from implied forward index level $F_t = 1052.70$, is $q = 0.02098$. The smirkness triplet for November options is $(\sigma_{I0}, \gamma_1, \gamma_2) = (0.1447, -0.1308, 0.0411)$. The task of calibrating the CEV model becomes solving equations (35) and (36) for the two parameters σ and α by using the information of at-the-money implied volatility, σ_{I0}, and skewness, γ_1.

With a given α, we can solve equation (35) for a value of σ. Substituting the values of α and σ into equation (36), we can compute γ_1. After some numerical experiments, we find that the skewness, γ_1, is positive for $\alpha > 1$, zero for $\alpha = 1$ (the Black-Scholes case) and negative for $0 \leq \alpha < 1$. The smaller the value of α between 0 and 1, the larger the absolute value of the skewness. For the extreme case of $\alpha = 0$, with $\sigma_{I0} = 0.1447$ and values of r, q, τ and S_t given above, we obtain from equation (35) $\sigma = 152.36$, from equation (36) $\gamma_1 = -0.0179$, which is much smaller than the skewness, $\gamma_1 = -0.1308$, from the market date. Therefore we have following observation.

Observation: *On November 4, 2003, We cannot find a set of values (σ, α) for the CEV model to match the at-the-money implied volatility, σ_{I0}, and skewness, γ_1, for options that mature on November 21, 2003.*

The best set of parameters would be $(\sigma, \alpha) = (152.36, 0)$. With this set of calibrated parameters, we can compute the term structure of the smirkness implied in the CEV model from theorem 3. The results are shown in Table 7 numerically and in Figure 9 graphically together with that from the market data.

6.2 The Finite Moment Log Stable (FMLS) process

Recently Carr and Wu (2003) propose a Finite Moment Log $\alpha-$Stable (FMLS) process for stock price

$$dS_t = (r-q)S_t dt + \sigma S_t dL_t^{\alpha,-1}, \tag{38}$$

where $dL_t^{\alpha,-1}$ has an $\alpha-$stable distribution with zero drift, dispersion of $dt^{1/\alpha}$, and a skew parameter -1: $L_\alpha(0, dt^{1/\alpha}, -1)$. The $\alpha-$stable process is a pure jump Lévy process when $0 < \alpha < 2$. When $\alpha = 2$, the $\alpha-$stable process is degenerated to a standard Brownian motion multiplied by $\sqrt{2}$, i.e., $L_t^{2,\beta} = \sqrt{2}\ B_t$.

The stock price at a future time, T, is then written explicitly as

$$S_T = S_t e^{(r-q+\mu)\tau + \sigma L_\tau^{\alpha,-1}}, \qquad \tau = T - t,$$

or

$$\ln(S_T/S_t) \sim L_\alpha((r-q+\mu)\tau, \sigma\tau^{1/\alpha}, -1) \tag{39}$$

where the convexity adjustment term is given by $\mu = \sigma^\alpha \sec\frac{\pi\alpha}{2}$ to ensure the martingale condition, $E_t^Q[S_T] = S_t e^{(r-q)\tau}$. The characteristic function of $\ln(S_T/S_t)$ is given by (Carr and Wu 2003)

$$f(\tau;\phi) = E_t^Q\left[e^{i\phi\ln(S_T/S_t)}\right] = e^{i\phi(r-q)\tau + (i\phi - (i\phi)^\alpha)\mu\tau}. \tag{40}$$

We recall following general option pricing formula.

Lemma 2. *If the characteristic function of* $\ln(S_T/S_t)$ *is denoted as* $f(\tau;\phi) = E_t^Q\left[e^{i\phi\ln(S_T/S_t)}\right]$, *then the price of a European call option can be written as (Bakshi and Madan 2000)*

$$c_t = S_t e^{-q\tau}\Pi_1(t,\tau) - Ke^{-r\tau}\Pi_2(t,\tau),$$

where

$$\begin{aligned}\Pi_1(t,\tau) &= \frac{1}{2}+\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\phi\ln(K/S_t)}\times f(\tau;\phi-i)}{i\phi\ f(\tau;-i)}\right]d\phi,\\ \Pi_2(t,\tau) &= \frac{1}{2}+\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\phi\ln(K/S_t)}\times f(\tau;\phi)}{i\phi}\right]d\phi.\end{aligned}$$

Applying the general formula to the FMLS process yields following result.

Lemma 3. *If the underlying stock price is modelled by the FMLS process in equation (38), then the price of a European call option is given by*

$$c_t = S_t e^{-q\tau}\Pi_1(t,\tau) - K e^{-r\tau}\Pi_2(t,\tau), \tag{41}$$

where

$$\begin{aligned}\Pi_1(t,\tau) &= \frac{1}{2}+\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\phi\ln(K/F_t)+(i\phi+1-(i\phi+1)^\alpha)\mu\tau}}{i\phi}\right]d\phi,\\ \Pi_2(t,\tau) &= \frac{1}{2}+\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\phi\ln(K/F_t)+(i\phi-(i\phi)^\alpha)\mu\tau}}{i\phi}\right]d\phi,\end{aligned}$$

where $F_t = S_t e^{(r-q)\tau}$ *is the current froward price, and* $\mu = \sigma^\alpha \sec\frac{\pi\alpha}{2}$ *is the convexity adjustment.*

Matching the two option pricing formulas in (41) and (11) yields smirkness implied in the FMLS process option pricing model as follows.

Theorem 4. *For* $1<\alpha<2$*, the smirkness triplet,* $(\sigma_{\mathrm{I0}},\ \gamma_1,\ \gamma_2)$*, implied in the FMLS process option pricing model can be determined by*

$$1-2N(d^*) = \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{(i\phi+1-(i\phi+1)^\alpha)\mu\tau}-e^{(i\phi-(i\phi)^\alpha)\mu\tau}}{i\phi}\right]d\phi, \tag{42}$$

$$N(-d^*)+n(d^*)\frac{\sigma_{\mathrm{I0}}}{\bar\sigma}\gamma_1 = \frac{1}{2}-\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{(i\phi-(i\phi)^\alpha)\mu\tau}}{i\phi}\right]d\phi, \tag{43}$$

$$\frac{1}{\sigma_{\mathrm{I0}}\sqrt{\tau}}\left(1-d^{*2}\frac{\sigma_{\mathrm{I0}}^2}{\bar\sigma^2}\gamma_1^2+\frac{2\sigma_{\mathrm{I0}}^2}{\bar\sigma^2}\gamma_2\right)n(d^*) = \frac{1}{\pi}\int_0^\infty \mathrm{Re}[e^{(i\phi-(i\phi)^\alpha)\mu\tau}]d\phi, \tag{44}$$

where $n(\cdot)$ *is the standard normal density function, and* $d^* = -\frac{1}{2}\sigma_{\mathrm{I0}}\sqrt{\tau}$.

The model has only two parameters, (σ, α), as well. Similar to the CEV model, one should use the ATM-IV, σ_{I0}, and skewness, γ_1, of the implied volatility smirk of the nearest term to calibrate the model.

On November 4, 2003, VIX index level is $\bar{\sigma} = 0.1655$. The ATM-IV is $\sigma_{\mathrm{I0}} = 0.1447$. The skewness is $\gamma_1 = -0.1308$. The time to maturity of the nearest term is $\tau = 17/365$. Then $d^* = -\frac{1}{2}\sigma_{\mathrm{I0}}\sqrt{\tau} = -0.0156141$. The task of calibrating the FMLS process option pricing model becomes solving following two equations

$$\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{(i\phi+1-(i\phi+1)^\alpha)\tau\sigma^\alpha \sec\frac{\pi\alpha}{2}} - e^{(i\phi-(i\phi)^\alpha)\tau\sigma^\alpha \sec\frac{\pi\alpha}{2}}}{i\phi}\right] d\phi = 1 - 2N(d^*) = 0.0124577,$$

$$\frac{1}{2} - \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{(i\phi-(i\phi)^\alpha)\tau\sigma^\alpha \sec\frac{\pi\alpha}{2}}}{i\phi}\right] d\phi = N(-d^*) + n(d^*)\frac{\sigma_{\mathrm{I0}}}{\bar{\sigma}}\gamma_1 = 0.460611$$

for two unknowns, volatility parameter, σ, and the tail index, α. This task can be accomplished within a few seconds by using three lines of MATHEMATICA code. The result is $\sigma = 0.1086$, $\alpha = 1.8141$.

With this set of calibrated parameters, we can compute the term structure of the smirkness implied in the FMLS process option pricing model from theorem 4. The results are shown in Table 7 numerically and in Figure 9 graphically together with that from the CEV model and the market data. A direct comparison shows that the FMLS model is much better than the CEV model in terms of capturing the skewness of the implied volatility smirk. This observation, which agrees with Carr and Wu (2003), indicates that the underlying process indeed has jumps rather than a simple pure diffusion.

Other models, such as the jump-diffusion model of Merton (1976), the stochastic volatility model of Heston (1993), the variance Gamma process of Madan, Carr and Chang (1998), the affine jump diffusion model of Duffie, Pan and Singleton (2000), the time-changed Lévy process of Carr and Wu (2003), have more parameters. More points from the term structure of smirkness are needed in calibration. The details of implementing our calibration procedure will be presented in a subsequent research.

7 Conclusions

In this paper, we propose a new concept of smirkness, which is defined as a triplet of at-the-money implied volatility, skewness (slope at the money) and smileness (curvature at the money) of implied volatility – moneyness curve. The moneyness is the logarithm of the strike price over the forward price, normalized by the standard deviation of expected return on maturity.

Empirical evidence from S&P 500 index options shows that a quadratic function with both skewness and smileness fits the market implied volatility smirk very well. The volume weighted error can be smaller than the smallest bid-ask spread of traded options. Theoretical analysis shows that the smirkness triplet is related to the cumulants of the risk-neutral probability of the underlying asset return. With these new quantities well-defined, we are able to study the term structure, its evolution and time-change dynamics of an implied volatility smirk.

Our research suggests that the three term structures of the at-the-money implied volatility, skewness and smileness should be constructed on a daily basis. These term structures distilled from the current prices of options with different strikes and maturities provide a lot of information about investor's fair expectation on the index return distribution over different terms in the future. Therefore they should be used to calibrate option pricing models.

References

[1] Ait-Sahalia, Y., Y. Wang, and F. Yared, 2001, Do options markets correctly assess the probabilities of movement in the underlying asset? *Journal of Econometrics* 102, 67-110.

[2] Backus, David, Silverio Foresi, Kai Li, and Liuren Wu, 1997, Accounting for bias in Black-Scholes, Working paper, New York University.

[3] Bakshi, Gurdip, Charles Cao, and Zhiwu Chen, 1997, Empirical performance of alternative option pricing models, *Journal of Finance* 52, 2003-2049.

[4] Bakshi, Gurdip, and Dilip Madan, 2000, Spanning and derivative-security valuation, *Journal of Financial Economics* 55, 205-238.

[5] Bollen, Nicolas P. B., and Robert E. Whaley, 2004, Does net buying pressure affect the shape of implied volatility functions? *Journal of Finance* 59, 711-753.

[6] Breeden, Douglas, and Robert Litzenberger, 1978, Prices of state-contingent claims implicit in option prices, *Journal of Business* 51, 621-651.

[7] Carr, Peter, and Liuren Wu, 2003, The finite moment log stable process and option pricing, *Journal of Finance* 58, 753-777.

[8] Carr, Peter, and Liuren Wu, 2004, Time-changed Lévy processes and option pricing, *Journal of Financial Economics* 17(1), 113-141.

[9] Carr, Peter, Hélyette Geman, Dilip Madan, and Marc Yor, 2001, Stochastic volatility for Lévy processes, *Mathematical Finance* 13, 345-382.

[10] Chicago Board Options Exchange, 2003, VIX —- CBOE Volatility Index, http://www.cboe.com/micro/vix/vixwhite.pdf.

[11] Corrado, Charles J., and Tie Su, 1997, Implied volatility skews and stock index skewness and kurtosis in S&P 500 index option prices, *Journal of Derivatives* 4, 8-19.

[12] Cox, John, 1975, Notes on option pricing I: Constant elasticity of variance diffusions, Working paper, Stanford University.

[13] Cox, John, and Stephen A. Ross, 1976, The valuation of options for alternative stochastic processes, *Journal of Financial Economics* 3, 145-166.

[14] Dennis, Patrick, and Stewart Mayhew, 2002, Risk-neutral skewness: Evidence from stock options, *Journal of Financial and Quantitative Analysis* 37, 471-493.

[15] Duffie, Darrell, Jun Pan, and Kenneth Singleton, 2000, Transformation analysis and asset pricing for affine jump diffusions, *Econometrica* 68, 1343-1376.

[16] Emanuel, David C., and James D. MacBeth, 1982, Further results on the constant elasticity of variance call option pricing model, *Journal of Financial and Quantitative Analysis* 17, 533-554.

[17] Eraker, Bjorn, 2004, Do stock prices and volatility jump? Reconciling evidence from spot and option prices, *Journal of Finance* (forthcoming).

[18] Feller, William, 1951, Two singular diffusion problems, *Annals of Mathematics* 54, 173-182.

[19] Foresi, Silverio, and Liuren Wu, 2002, Crash-o-phobia: A domestic fear or a worldwide concern, Working paper, Zicklin School of Business, City University of New York.

[20] Heston, Stephen, 1993, Closed-form solution for options with stochastic volatility, with application to bond and currency options, *Review of Financial Studies* 6, 327-343.

[21] Huang, Jing-zhi, and Liuren Wu, 2004, Specification analysis of option pricing models based on time-changed Lévy processes, *Journal of Finance* 59, 1405-1439.

[22] Jackwerth, Jens Carsten, and Mark Rubinstein, 1996, Recovering probability distributions from option prices, *Journal of Finance* 51, 1611-1631.

[23] Jarrow, Robert, and A. Rudd, 1982, Approximate option valuation for arbitrary stochastic processes, *Journal of Financial Economics* 10, 347-369.

[24] Johnson, Norman L., and Samuel Kotz, 1970, *Distributions in Statistics: Continuous Univariate Distribution — 2.* New York, Wiley.

[25] Madan, Dilip, Peter Carr, and Eric Chang, 1998, The variance Gamma process and option pricing, *European Finance Review* 2, 79-105.

[26] Merton, Robert C., 1976, Option pricing when underlying stock returns are discontinuous, *Journal of Financial Economics* 3, 125-144.

[27] Rubinstein, Mark, 1994, Implied binomial trees, *Journal of Finance* 49, 771-818.

[28] Schroder, Mark, 1989, Computing the constant elasticity of variance option pricing formula, *Journal of Finance* 44, 211-219.

[29] Shimko, David, 1993, Bounds probability, *RISK* 6, 33-37.

Table 1: The market data of SPX options on November 4, 2003. The maturity date is November 21 2003, therefore time to maturity is $\tau = 17$ days, or 17/365 year. The S&P 500 index level is $S_0 = 1053.25$. The risk-free rate is $r = 0.9743\%$. The implied forward index level, computed by using formula $F_0 = K + e^{r\tau}(c_0(K) - p_0(K))$ with $K = 1055$ to minimize $|c_0(K) - p_0(K)|$, is $F_0 = 1052.70$. The fact of $F_0 < S_0$ implies that expected dividend yield, q, is larger than the risk-free rate, r.

	Call					Put				
Strike	Last Sale	Bid	Ask	Mid-value	Vol	Last Sale	Bid	Ask	Mid-value	Vol
850	204.6	201.6	203.6	202.6	0	0.1	0.05	0.15	0.1	0
875	160.4	176.7	178.7	177.7	0	0.15	0.1	0.3	0.2	0
900	150	151.8	153.8	152.8	0	0.2	0.15	0.25	0.2	0
925	135	126.9	128.9	127.9	0	0.3	0.3	0.5	0.4	307
935	124	116.9	118.9	117.9	0	0.55	0.25	0.55	0.4	26
945	0	107	109	108	0	1.3	0.35	0.65	0.5	0
950	102.5	102.1	104.1	103.1	10	0.55	0.45	0.6	0.525	832
960	0	92.2	94.2	93.2	0	0.65	0.35	0.8	0.575	0
970	0	82.4	84.4	83.4	0	0	0.5	1	0.75	0
975	82	77.6	79.6	78.6	0	1	0.8	1.1	0.95	1362
980	72.5	72.7	74.7	73.7	1	1.05	0.8	1.3	1.05	34
985	63	67.9	69.9	68.9	0	1.3	1	1.45	1.225	347
990	70.5	63.1	65.1	64.1	0	1.4	1.2	1.65	1.425	188
995	58.6	58.3	60.3	59.3	13	1.7	1.6	1.9	1.75	1109
1005	49	49	51	50	45	2.3	2.25	2.6	2.425	493
1010	44.6	44.5	46.5	45.5	146	2.9	2.4	3.2	2.8	190
1015	40	40	42	41	29	3.7	2.9	3.7	3.3	965
1020	39.7	35.7	37.7	36.7	6	3.6	3.6	4.3	3.95	1118
1025	31.9	31.5	33.5	32.5	1952	4.6	4.5	5.1	4.8	2005
1030	28.5	27.4	29.4	28.4	9	5.5	5.1	6	5.55	373
1035	23.7	24	25.6	24.8	187	7.2	6.3	7.5	6.9	2525
1040	20.3	20	22	21	107	8.2	7.7	8.5	8.1	1190
1045	17	17	18.5	17.75	14	10.1	9.4	10.6	10	94
1050	14.5	14.5	15.4	14.95	3033	11.9	11.6	12.2	11.9	3478
1055	11.5	11.1	12.7	11.9	1823	14.5	13.4	15	14.2	627
1060	9.8	9.5	10	9.75	2603	17	16	17.2	16.6	188
1065	7.1	7.2	8.1	7.65	243	20	19	20.6	19.8	24
1070	5.5	5.4	6.4	5.9	2623	23.7	22.1	24.1	23.1	204
1075	4.1	4.1	4.7	4.4	1759	26.8	25.7	27.2	26.45	1704
1080	3.2	3.1	3.7	3.4	303	30	29.6	31.6	30.6	6
1085	2.4	2	2.8	2.4	79	0	33.7	35.7	34.7	0
1090	1.9	1.5	2	1.75	362	40	38	40	39	2
1095	1.1	1	1.5	1.25	10	0	42.5	44.5	43.5	0
1100	0.95	0.85	1.1	0.975	149	49	47.2	49.2	48.2	42
1115	0.45	0.3	0.5	0.4	32	0	61.6	63.6	62.6	0
1125	0.25	0.15	0.3	0.225	39	72	71.4	73.4	72.4	13

Table 2: The U.S. daily treasury yield curve rates between October 30, 2003 and November 12, 2003.

Maturity / Date	1 mo	3 mo	6 mo	1 yr	2 yr	3 yr	5 yr	7 yr	10 yr	20 yr
10/30/03	0.97	0.96	1.04	1.32	1.86	2.39	3.29	3.83	4.36	5.25
10/31/03	0.96	0.96	1.04	1.31	1.85	2.36	3.27	3.80	4.33	5.20
11/03/03	0.97	0.96	1.05	1.33	1.90	2.44	3.34	3.88	4.40	5.25
11/04/03	0.97	0.95	1.03	1.31	1.86	2.40	3.28	3.81	4.33	5.19
11/05/03	0.96	0.96	1.05	1.35	1.94	2.46	3.35	3.88	4.38	5.24
11/06/03	0.95	0.96	1.06	1.38	2.01	2.55	3.43	3.96	4.45	5.32
11/07/03	0.95	0.96	1.07	1.40	2.04	2.60	3.47	3.99	4.48	5.33
11/10/03	0.92	0.97	1.07	1.39	2.06	2.63	3.49	4.00	4.49	5.34
11/12/03	0.92	0.95	1.06	1.39	2.05	2.59	3.45	3.95	4.44	5.29

Table 3: The price and implied volatility on November 4, 2003 for out-of-the-money SPX options that mature on November 21, 2003. Puts are chosen for strikes below the forward index level, $F_0 = 1052.70$. Calls are chosen for strike above the forward index level. The implied volatility is the root that equates the Black-Scholes formula with the market price, and the moneyness is defined as $\ln(K/F_0)/(\sigma\sqrt{\tau})$, where $\tau = 17/365$ year, and $\sigma = 16.55\%$ is the VIX closing index on the day. It is taken to be a proxy of the average volatility. The fitted implied volatility, given by $\sigma_{\mathrm{I}} = 0.1447(1 - 0.1308\xi + 0.0411\xi^2)$, is obtained by fitting the market implied volatility with a quadratic function that passes through the point at the money and minimizes the volume weighted mean squared error. RMSE, root of mean squared error. RVWMSE, root of volume weighted mean squared error.

	Strike	Market	Volume	Moneyness	ImV.	Fitted ImV.	FitEr
Put	850	0.1	0	-5.9881	0.3760	0.4713	0.0953
	875	0.2	0	-5.1765	0.3568	0.4020	0.0452
	900	0.2	0	-4.3878	0.3081	0.3422	0.0341
	925	0.4	307	-3.6206	0.2861	0.2911	0.0050
	935	0.4	26	-3.3196	0.2655	0.2730	0.0075
	945	0.5	0	-3.0217	0.2537	0.2561	0.0025
	950	0.525	832	-2.8740	0.2451	0.2482	0.0031
	960	0.575	0	-2.5808	0.2273	0.2331	0.0058
	970	0.75	0	-2.2907	0.2159	0.2192	0.0033
	975	0.95	1362	-2.1467	0.2144	0.2127	-0.0022
	980	1.05	34	-2.0035	0.2070	0.2064	-0.0006
	985	1.225	347	-1.8610	0.2019	0.2005	-0.0014
	990	1.425	188	-1.7193	0.1965	0.1948	-0.0017
	995	1.75	1109	-1.5782	0.1936	0.1893	-0.0043
	1005	2.425	493	-1.2983	0.1835	0.1792	-0.0043
	1010	2.8	190	-1.1593	0.1773	0.1746	-0.0027
	1015	3.3	965	-1.0210	0.1719	0.1702	-0.0018
	1020	3.95	1118	-0.8835	0.1675	0.1660	-0.0015
	1025	4.8	2005	-0.7466	0.1641	0.1621	-0.0020
	1030	5.55	373	-0.6103	0.1571	0.1584	0.0013
	1035	6.9	2525	-0.4747	0.1559	0.1550	-0.0009
	1040	8.1	1190	-0.3398	0.150	0.1518	0.0017
	1045	10	94	-0.2055	0.1498	0.1488	-0.0008
	1050	11.9	3478	-0.0719	0.1460	0.1460	0.0001
ATM	1052.70			-0.0001	0.1447	0.1447	0.0000
Call	1055	11.9	1823	0.0611	0.1435	0.1435	-0.00003
	1060	9.75	2603	0.1935	0.1439	0.1412	-0.0026
	1065	7.65	243	0.3253	0.1413	0.1391	-0.0022
	1070	5.9	2623	0.4564	0.1393	0.1373	-0.0020
	1075	4.4	1759	0.5869	0.1367	0.1356	-0.0011
	1080	3.4	303	0.7169	0.1375	0.1342	-0.0034
	1085	2.4	79	0.8462	0.1348	0.1329	-0.0018
	1090	1.75	362	0.9749	0.1346	0.1319	-0.0028
	1095	1.25	10	1.1030	0.1344	0.1310	-0.0034
	1100	0.975	149	1.2306	0.1375	0.1304	-0.0071
	1115	0.4	32	1.6098	0.1426	0.1296	-0.0130
	1125	0.225	39	1.8598	0.1468	0.1301	-0.0168
RMSE							0.0190
RVWMSE							0.0023

Table 4: The prices and price errors computed by the Black-Scholes formula with flat, skewed and smirked volatility functions for out-of-the-money SPX options. The set of options used is the same as that in Table 3. The flat volatility function, $\sigma_{\mathrm{I}} = 0.1447$, is a constant given by at-the-money implied volatility. The skewed one, $\sigma_{\mathrm{I}} = 0.1447(1 - 0.1308\xi)$, is a linear function passing through the point at the money. The smirked one, $\sigma_{\mathrm{I}} = 0.1447(1 - 0.1308\xi + 0.0411\xi^2)$, is a quadratic function obtained by fitting the market implied volatility. The errors are the difference between the computed prices and the market prices. RMSE, root of mean squared error. RVWMSE, root of volume weighted mean squared error.

Strike	Market	Volume	Flat	FlatEr	Skew	SkewEr	Smirk	SmirkEr
850	0.1	0	0	-0.1	0.0008	-0.0993	0.6163	0.5163
875	0.2	0	0	-0.2	0.0026	-0.1974	0.4933	0.2933
900	0.2	0	0	-0.2	0.0092	-0.1908	0.4368	0.2368
925	0.4	307	0.0001	-0.3999	0.0332	-0.3668	0.4493	0.0493
935	0.4	26	0.0005	-0.3995	0.0556	-0.3444	0.4797	0.0797
945	0.5	0	0.0022	-0.4979	0.0931	-0.4069	0.5307	0.0307
950	0.525	832	0.0042	-0.5209	0.1203	-0.4047	0.5664	0.0414
960	0.575	0	0.0142	-0.5609	0.2005	-0.3745	0.6646	0.0896
970	0.75	0	0.0432	-0.7068	0.3325	-0.4175	0.8127	0.0627
975	0.95	1362	0.0725	-0.8775	0.4272	-0.5228	0.9127	-0.0373
980	1.05	34	0.1187	-0.9313	0.5477	-0.5023	1.0355	-0.0145
985	1.225	347	0.1895	-1.0355	0.7006	-0.5244	1.1866	-0.0384
990	1.425	188	0.2953	-1.1298	0.8939	-0.5312	1.3727	-0.0523
995	1.75	1109	0.4494	-1.3007	1.1371	-0.6129	1.6024	-0.1476
1005	2.425	493	0.9723	-1.4527	1.8217	-0.6033	2.2365	-0.1886
1010	2.8	190	1.3844	-1.4156	2.2925	-0.5075	2.6693	-0.1307
1015	3.3	965	1.9306	-1.3694	2.8723	-0.4277	3.2030	-0.0970
1020	3.95	1118	2.6388	-1.3112	3.5816	-0.3684	3.8595	-0.0906
1025	4.8	2005	3.5377	-1.2623	4.4431	-0.3569	4.6634	-0.1366
1030	5.55	373	4.6557	-0.8943	5.4810	-0.0690	5.6425	0.0925
1035	6.9	2525	6.0190	-0.8811	6.7208	-0.1792	6.8266	-0.0734
1040	8.1	1190	7.6503	-0.4497	8.1885	0.0885	8.2462	0.1462
1045	10	94	9.5675	-0.4325	9.9090	-0.0910	9.9311	-0.0689
1050	11.9	3478	11.7824	-0.1178	11.9050	0.0050	11.9078	0.0078
1052.70				0.0000		0.0000		0.0000
1055	11.9	1823	12.0003	0.1103	11.8957	-0.0043	11.8977	-0.0023
1060	9.75	2603	9.8214	0.0714	9.4970	-0.2530	9.5167	-0.2333
1065	7.65	243	7.9350	0.2849	7.4135	-0.2365	7.4667	-0.1833
1070	5.9	2623	6.3262	0.4262	5.6442	-0.2558	5.7414	-0.1586
1075	4.4	1759	4.9754	0.5754	4.1794	-0.2207	4.3239	-0.0761
1080	3.4	303	3.8590	0.4590	3.0005	-0.3995	3.1885	-0.2115
1085	2.4	79	2.9510	0.5510	2.0815	-0.3186	2.3023	-0.0978
1090	1.75	362	2.2244	0.4744	1.3900	-0.3600	1.6285	-0.1215
1095	1.25	10	1.6525	0.4025	0.8898	-0.3602	1.1294	-0.1206
1100	0.975	149	1.2097	0.2347	0.5436	-0.4314	0.7689	-0.2061
1115	0.4	32	0.4341	0.0341	0.0887	-0.3113	0.2215	-0.1785
1125	0.225	39	0.2033	-0.0217	0.0188	-0.2062	0.0918	-0.1332
RMSE				0.7504		0.3591		0.1566
RVWMSE				0.7758		0.3127		0.1229

Table 5: The term structure of smirkness on November 4, 2003. The trading volume is the number of the out-of-the-money options with different strikes for the same maturity. The discount rate is computed by using linear interpolation with the yield curve on November 4, 2003 provided in Table 2. The smirkness triplet $(\sigma_{I0}, \gamma_1, \gamma_2)$ is obtained by fitting the implied volatility smirk with a quadratic function, $\sigma(\xi) = \sigma_{I0}(1 + \gamma_1\xi + \gamma_2\xi^2)$, where ξ is the moneyness.

Maturity τ (days)	Trading volume	Discount rate r (%)	ATM-IV σ_{I0}	Skewness γ_1	Smileness γ_2
17	26,661	0.9743	0.1447	-0.1308	0.0411
45	37,983	0.9651	0.1473	-0.1894	0.0158
73	4,066	0.9559	0.1509	-0.2141	0.0103
136	6,825	0.9896	0.1608	-0.2063	0.0049
227	4,848	1.0989	0.1683	-0.1623	0.0230
318	4,050	1.2381	0.1727	-0.1905	0.0425
409	3,987	1.3763	0.1759	-0.1688	0.0086
591	1,401	1.6506	0.1786	-0.1574	0.0309

Table 6: The evolution of the term structure of smirkness from Oct-30-03 to Nov-04-03 and then to Nov-12-03. The discount rate is computed by using linear interpolation with the yield curve on the corresponding dates provided in Table 2. The smirkness triplet $(\sigma_{\mathrm{I0}}, \gamma_1, \gamma_2)$ is obtained by fitting the implied volatility smirk with a quadratic function, $\sigma(\xi) = \sigma_{\mathrm{I0}}(1 + \gamma_1\xi + \gamma_2\xi^2)$, where ξ is the moneyness.

Observed on Oct-30-03

Maturity (days)	22	50	78	141	232	323	414	596
Discount rate	0.9713	0.9667	0.9621	1.0040	1.1165	1.2557	1.3925	1.6618
ATM-IV σ_{I0}	0.1474	0.1528	0.1568	0.1661	0.1729	0.1776	0.1805	0.1833
Skewness γ_1	-0.1599	-0.1820	-0.1648	-0.1606	-0.1765	-0.1856	-0.1988	-0.2096
Smileness γ_2	0.0220	0.0103	0.0190	0.0250	0.0109	0.0108	-0.0055	-0.0103

Observed on Nov-04-03

Maturity (days)	17	45	73	136	227	318	409	591
Discount rate	0.9743	0.9651	0.9559	0.9896	1.0989	1.2381	1.3763	1.6506
ATM-IV σ_{I0}	0.1447	0.1473	0.1509	0.1608	0.1683	0.1727	0.1759	0.1786
Skewness γ_1	-0.1308	-0.1894	-0.2141	-0.2063	-0.1623	-0.1905	-0.1688	-0.1574
Smileness γ_2	0.0411	0.0158	0.0103	0.0049	0.0230	0.0425	0.0086	0.0309

Observed on Nov-12-03

Maturity (days)	9	37	65	128	219	310	401	583
Discount rate	0.9097	0.9234	0.9372	0.9947	1.1267	1.2908	1.4551	1.7842
ATM-IV σ_{I0}	0.1632	0.1553	0.1575	0.1658	0.1712	0.1743	0.1776	0.1798
Skewness γ_1	-0.1357	-0.1723	-0.1719	-0.2062	-0.2117	-0.2052	-0.1867	-0.1838
Smileness γ_2	0.0256	0.0168	0.0291	0.0063	0.0049	0.0031	-0.0006	0.0188

Table 7: The term structure of smirkness observed from the market data on Nov-04-03, and implied in calibrated option pricing models: the Constant Elasticity of Variance (CEV) model and the Finite Moment Log Stable (FMLS) process.

The term structure of smirkness observed from the market data on Nov-04-03

Maturity (days)	17	45	73	136	227	318	409	591
Discount rate (%)	0.9743	0.9651	0.9559	0.9896	1.0989	1.2381	1.3763	1.6506
Dividend (%)	2.098	1.656	1.548	1.704	1.565	1.615	1.609	1.474
Forward price	1052.70	1052.35	1052.00	1050.45	1050.20	1049.80	1050.51	1056.27
ATM-IV σ_{I0}	0.1447	0.1473	0.1509	0.1608	0.1683	0.1727	0.1759	0.1786
Skewness γ_1	-0.1308	-0.1894	-0.2141	-0.2063	-0.1623	-0.1905	-0.1688	-0.1574
Smileness γ_2	0.0411	0.0158	0.0103	0.0049	0.0230	0.0425	0.0086	0.0309

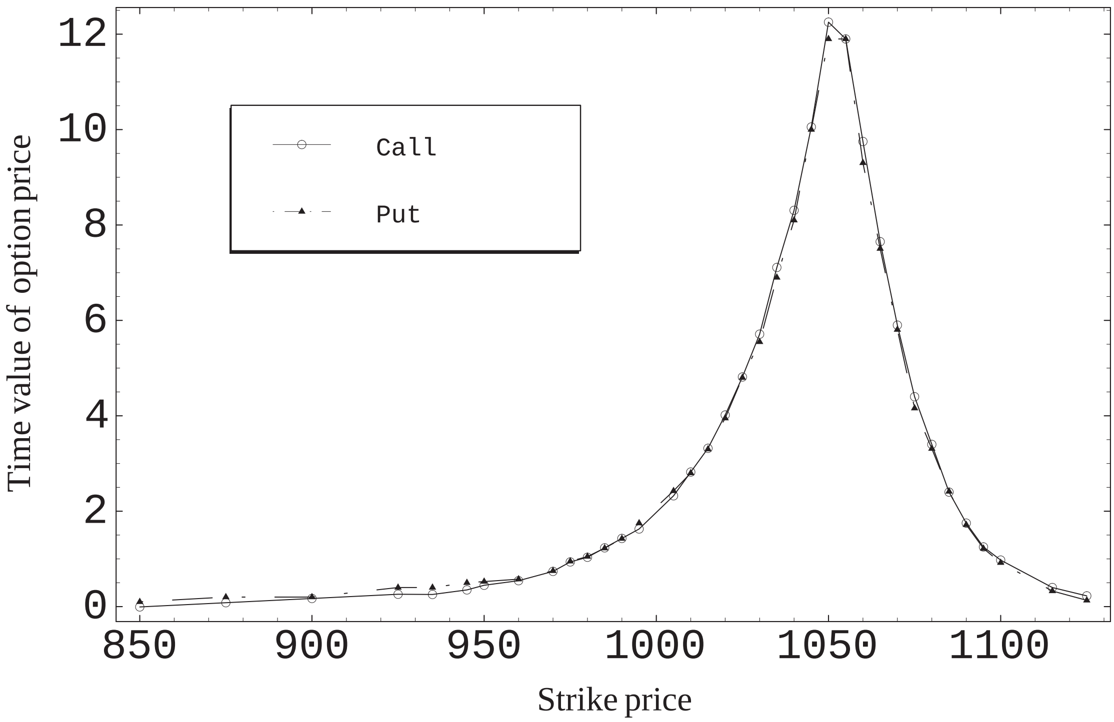
The term structure of smirkness implied in the CEV model, $dS_t = (r-q)S_t dt + \sigma S_t^{\alpha} dB_t$, with calibrated parameters: $(\sigma, \alpha) = (152.36, 0)$

Maturity (days)	17	45	73	136	227	318	409	591
ATM-IV σ_{I0}	0.1447	0.1447	0.1448	0.1449	0.1449	0.1450	0.1450	0.1447
Skewness γ_1	-0.0179	-0.0291	-0.0370	-0.0506	-0.0653	-0.0774	-0.0878	-0.1056
Smileness γ_2	0.00011	0.00028	0.00046	0.00085	0.00143	0.00201	0.00259	0.00375

The term structure of smirkness implied in the FMLS process, $dS_t = (r-q)S_t dt + \sigma S_t dL_t^{\alpha,-1}$, with calibrated parameters: $(\sigma, \alpha) = (0.1086, 1.8141)$

Maturity (days)	17	45	73	136	227	318	409	591
ATM-IV σ_{I0}	0.1447	0.1508	0.1537	0.1574	0.1603	0.1621	0.1634	0.1652
Skewness γ_1	-0.1308	-0.1195	-0.1137	-0.1059	-0.0992	-0.0947	-0.0913	-0.0863
Smileness γ_2	0.0486	0.0416	0.0383	0.0341	0.0307	0.0285	0.0269	0.0246

Figure 1: The time value of SPX options as a function of strike price, K, on November 4, 2003 for both calls and puts that mature on November 21, 2003.



The time to maturity is $\tau = 17$ days $= 17/365$ year. The discount rate over the period, derived from the U.S. treasury yield curve on November 4, 2003, is $r = 0.9743\%$. The implied forward price, derived from put-call parity for the strike that minimizes the absolute difference between a call and a put prices, is $F_0 = 1052.70$. The time value of an option is defined as the difference between option price and its intrinsic value. Therefore the time value of a call is

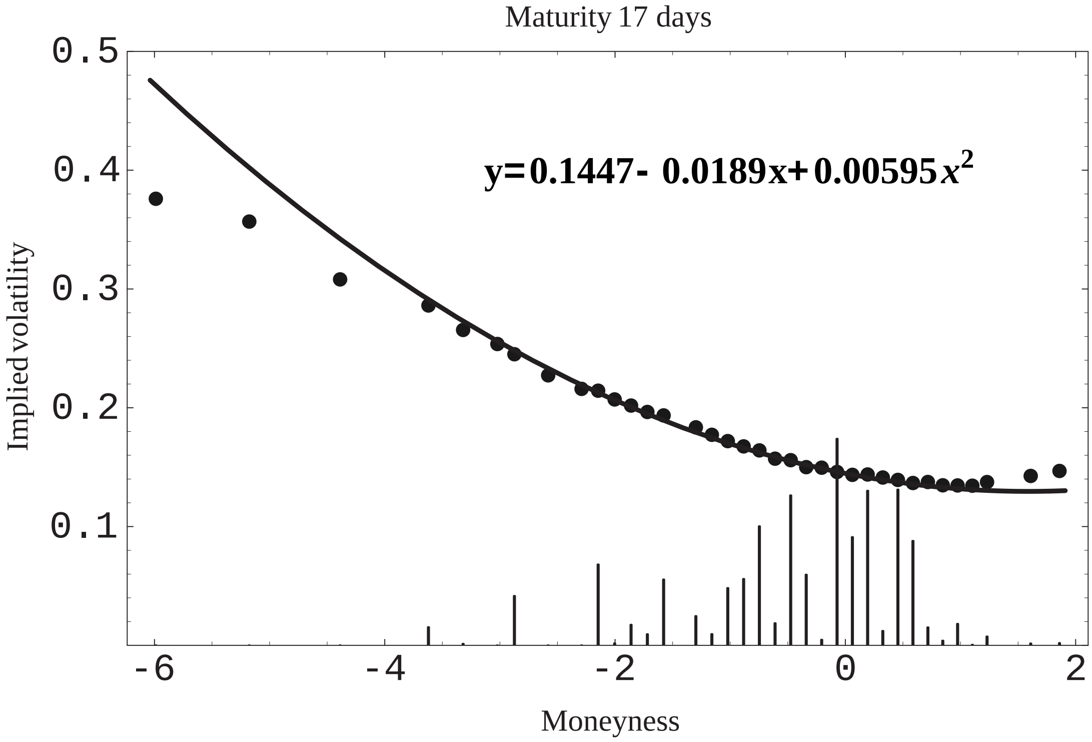
$$c_{tv}(K) = c_0(K) - e^{-r\tau} \max(F_0 - K, 0),$$

and the time value of a put is

$$p_{tv}(K) = p_0(K) - e^{-r\tau} \max(K - F_0, 0).$$

The fact that the two functions, $c_{tv}(K)$ and $p_{tv}(K)$, almost collapse each other indicates that the put-call parity holds for any strike price, K.

Figure 2: The implied volatility smirk on November 4, 2003 for out-of-the-money SPX options that mature on November 21, 2003. The implied volatility, σ_{I}, as a function of moneyness, ξ, is regarded as an implied volatility smirk.



The implied volatility, σ_{I}, is defined as the root that equates the Black-Scholes formula with the market price,

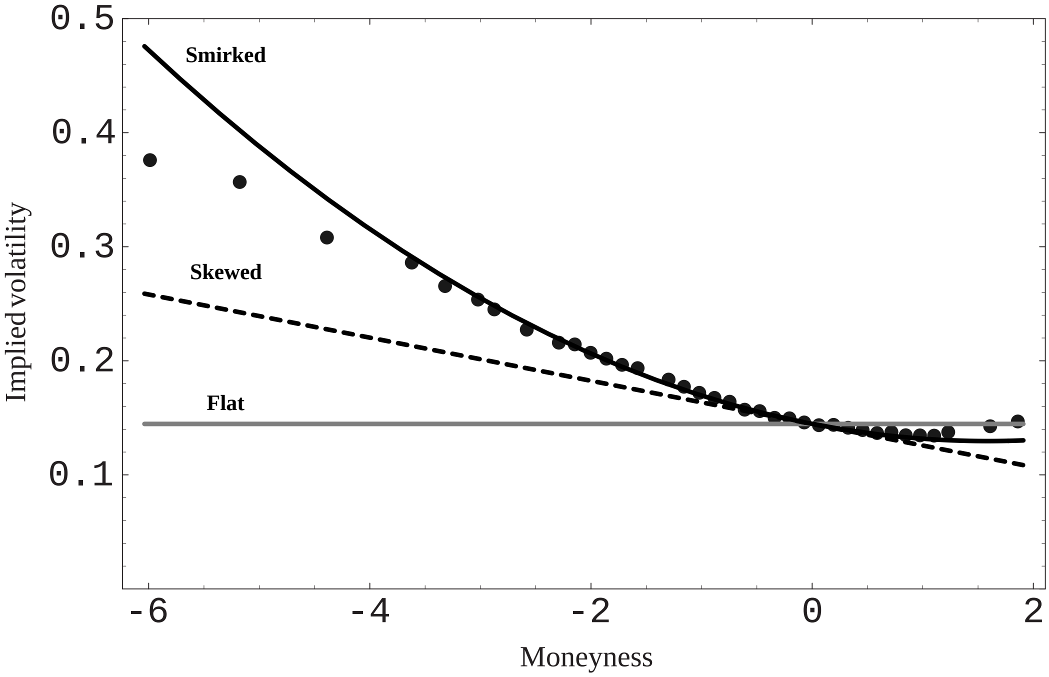
$$F_0 e^{-r\tau} N\left(\frac{\ln(F_0/K) + \frac{1}{2}\sigma_{\mathrm{I}}^2\tau}{\sigma_{\mathrm{I}}\sqrt{\tau}}\right) - Ke^{-r\tau} N\left(\frac{\ln(F_0/K) - \frac{1}{2}\sigma_{\mathrm{I}}^2\tau}{\sigma_{\mathrm{I}}\sqrt{\tau}}\right) = c_{\mathrm{mkt}},$$

and the moneyness, ξ, is defined by

$$\xi \equiv \frac{\ln(K/F_0)}{\bar{\sigma}\sqrt{\tau}}.$$

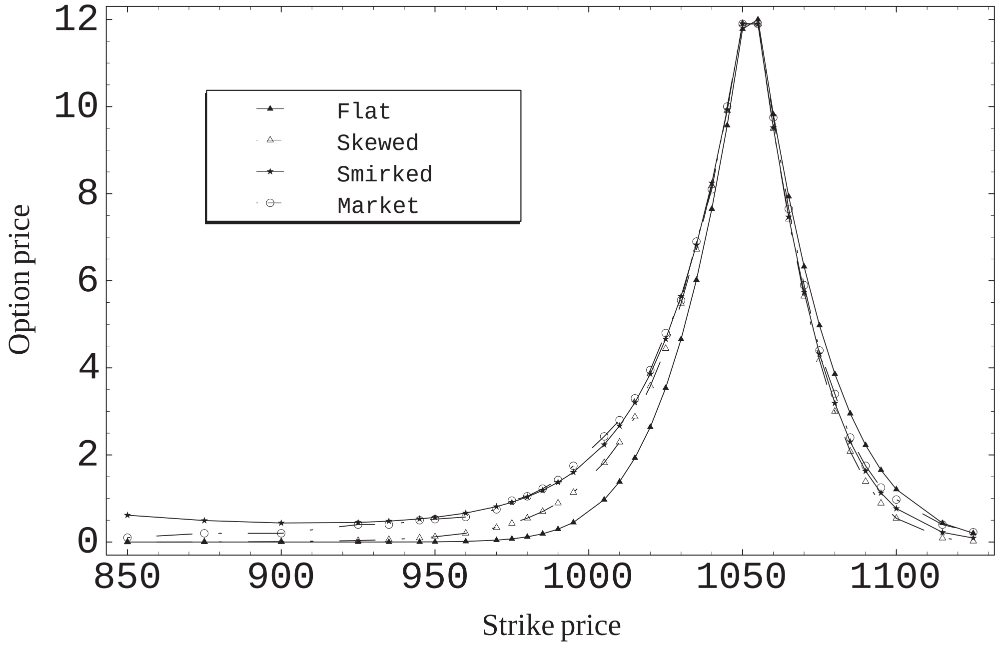
Here the time to maturity is $\tau = 17$ days $= 17/365$ year. The discount rate over the period, derived from the U.S. treasury yield curve on November 4, 2003, is $r = 0.9743\%$. The implied forward price, derived from put-call parity for the strike that minimizes the absolute difference between a call and a put prices, is $F_0 = 1052.70$. The benchmark volatility, $\bar{\sigma}$, is taken to be the VIX index reported by the CBOE on the day, $\bar{\sigma} = 16.55\%$. In the diagram, the dots are computed from the market prices of the out-of-the-money calls and puts. The solid line is generated by fitting the market implied volatility with a quadratic function that passes through the point at the money and minimizes the volume weighted mean squared errors of the implied volatility. The bar chart is the trading volume normalized by 20,000 contracts for the corresponding options traded on November 4, 2003.

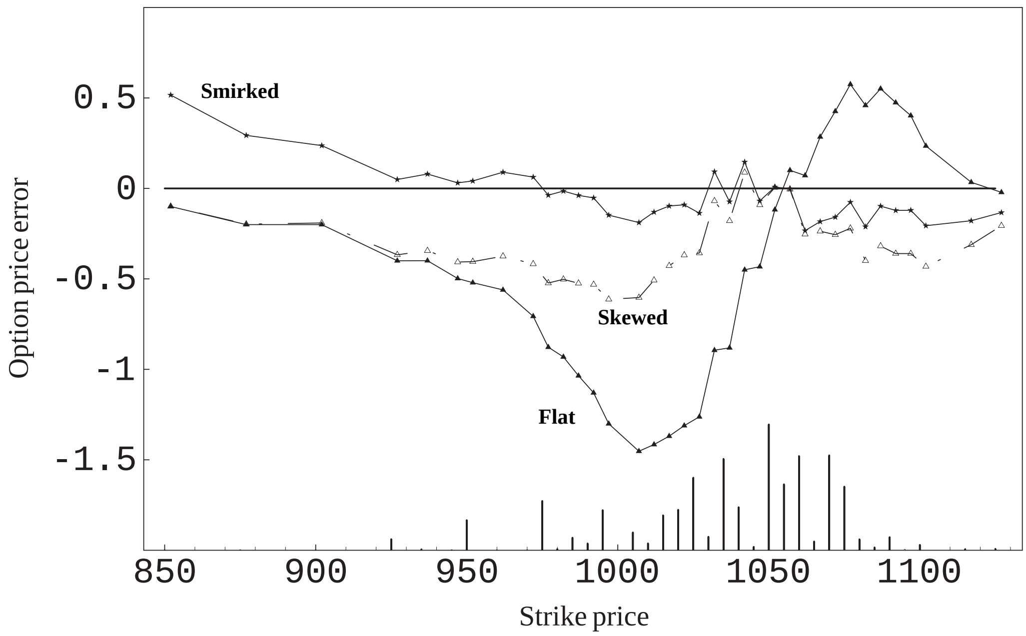
Figure 3: The flat, skewed and smirked implied volatility functions together with market implied volatilities (shown as dots) on November 4, 2003 for SPX options that mature on November 21, 2003.



The "flat" volatility function is $\sigma_{\mathrm{I}} = 0.1447$, which is the ATM-IV. The "skewed" one is a linear volatility function, $\sigma_{\mathrm{I}} = 0.1447(1 - 0.1308\xi)$. The "smirked" one is a quadratic function, $\sigma_{\mathrm{I}} = 0.1447(1 - 0.1308\xi + 0.0411\xi^2)$.

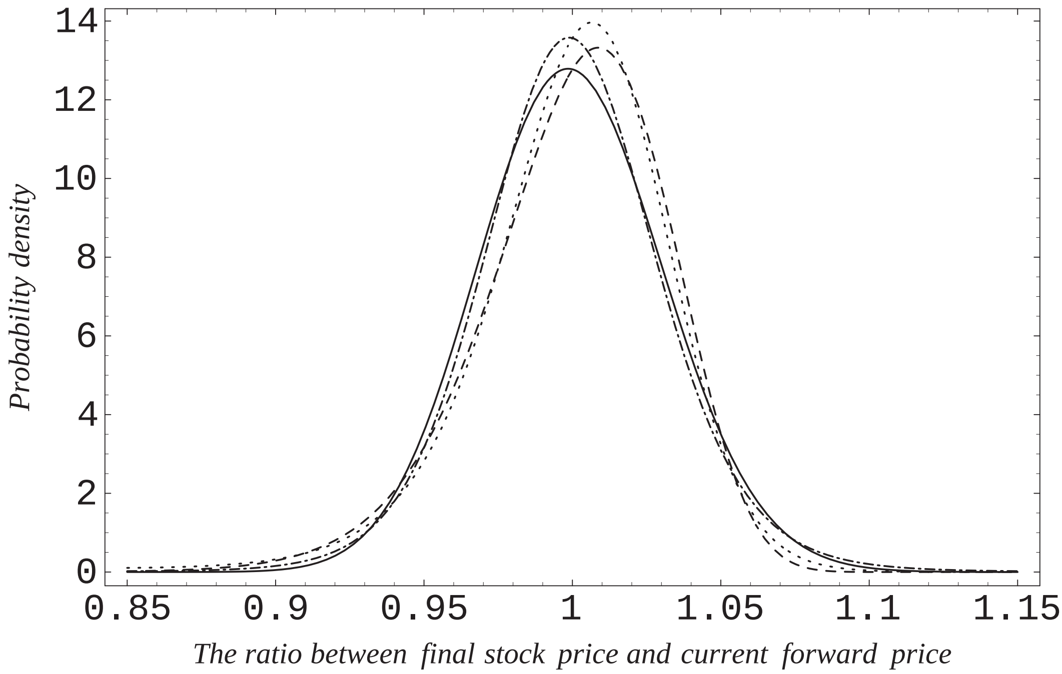
Figure 4: The computed option price and its error as functions of strike on November 4, 2003 for out-of-the-money SPX options that mature on November 21, 2003.





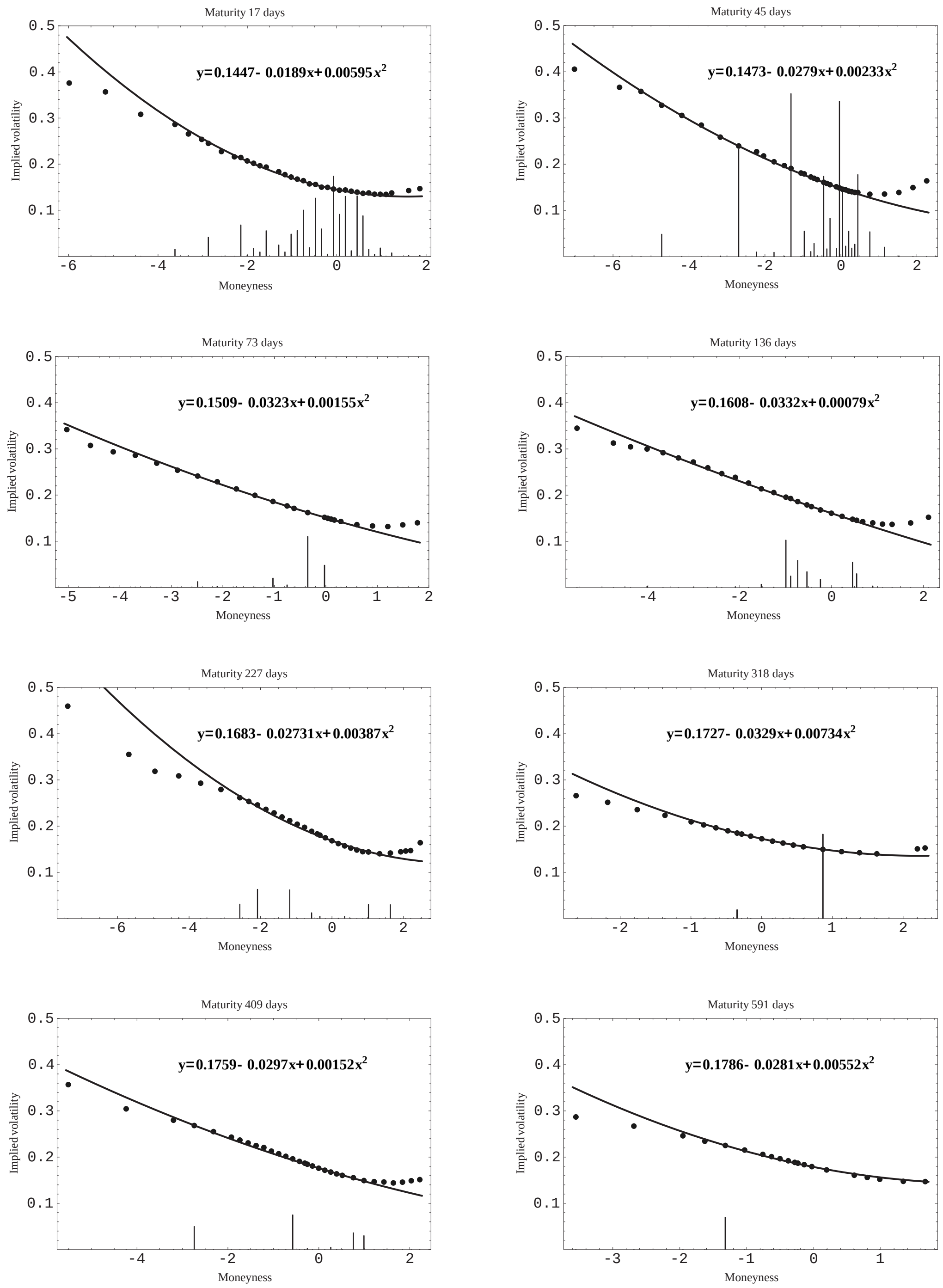
The computed option price is determined by using the Black-Scholes formula with flat, skewed and smirked volatility functions. The "flat" one is $\sigma_{\mathrm{I}} = 0.1447$, which is the ATM-IV. The "skewed" one is a linear volatility function, $\sigma_{\mathrm{I}} = 0.1447(1 - 0.1308\xi)$. And the "smirked" one is a quadratic function, $\sigma_{\mathrm{I}} = 0.1447(1 - 0.1308\xi + 0.0411\xi^2)$. The price error is the difference between the computed price and the market price. At the money, $K = F_0 = 1052.70$, three computed option prices are the same as the market price, therefore errors are zero. The option price error is shown together with the trading volume normalized by 5,000 contracts.

Figure 5: The risk-neutral probability density functions recovered from "flat", "skewed", "smiled" and "smirked" implied volatility functions on November 4, 2003 for options that mature on November 21, 2003.



The solid line is the probability density function recovered from a "flat" implied volatility, $\sigma_{\mathrm{I}} = \sigma_{\mathrm{I0}} = 0.1447$. It is a lognormal distribution density function. The dashed line is recovered from a "skewed" implied volatility, $\sigma_{\mathrm{I}} = \sigma_{\mathrm{I0}}(1 + \gamma_1\xi) = 0.1447(1 - 0.1308\xi)$. The dash-doted line is recovered from a "smiled" implied volatility, $\sigma_{\mathrm{I}} = \sigma_{\mathrm{I0}}(1 + \gamma_2\xi^2) = 0.1447(1 + 0.0411\xi^2)$. The doted line is recovered from a "smirked" implied volatility, $\sigma_{\mathrm{I}} = \sigma_{\mathrm{I0}}(1 + \gamma_1\xi + \gamma_2\xi^2) = 0.1447(1 - 0.1308\xi + 0.0411\xi^2)$.

Figure 6: The implied volatility smirks on November 4, 2003 for options with all available maturities, including Nov-21-03, Dec-19-03, Jan-16-04, Mar-19-04, Jun-18-04, Sep-17-04, Dec-17-04, and Jun-17-05. The times to maturity are 17, 45, 73, 136, 227, 318, 409, and 591 days respectively.



The dots are computed from the market prices of the out-of-the-money calls and puts. The solid line is generated by fitting the market implied volatility with a quadratic function that passes through the point at the money and minimizes the volume weighted mean squared errors of the implied volatility. The bar chart is the trading volume normalized by 20,000 contracts for the corresponding options traded on November 4, 2003.

Figure 7: The evolution of the term structure of smirkness from Oct-30-03 to Nov-04-03 and then to Nov-12-03. The smirkness triplet $(\sigma_{I0}, \gamma_1, \gamma_2)$ is obtained by fitting the implied volatility smirks with a quadratic function, $\sigma(\xi) = \sigma_{I0}(1 + \gamma_1\xi + \gamma_2\xi^2)$, where σ_{I0} is the ATM-IV, γ_1 is the skewness, γ_2 is the smileness, and ξ is the moneyness.

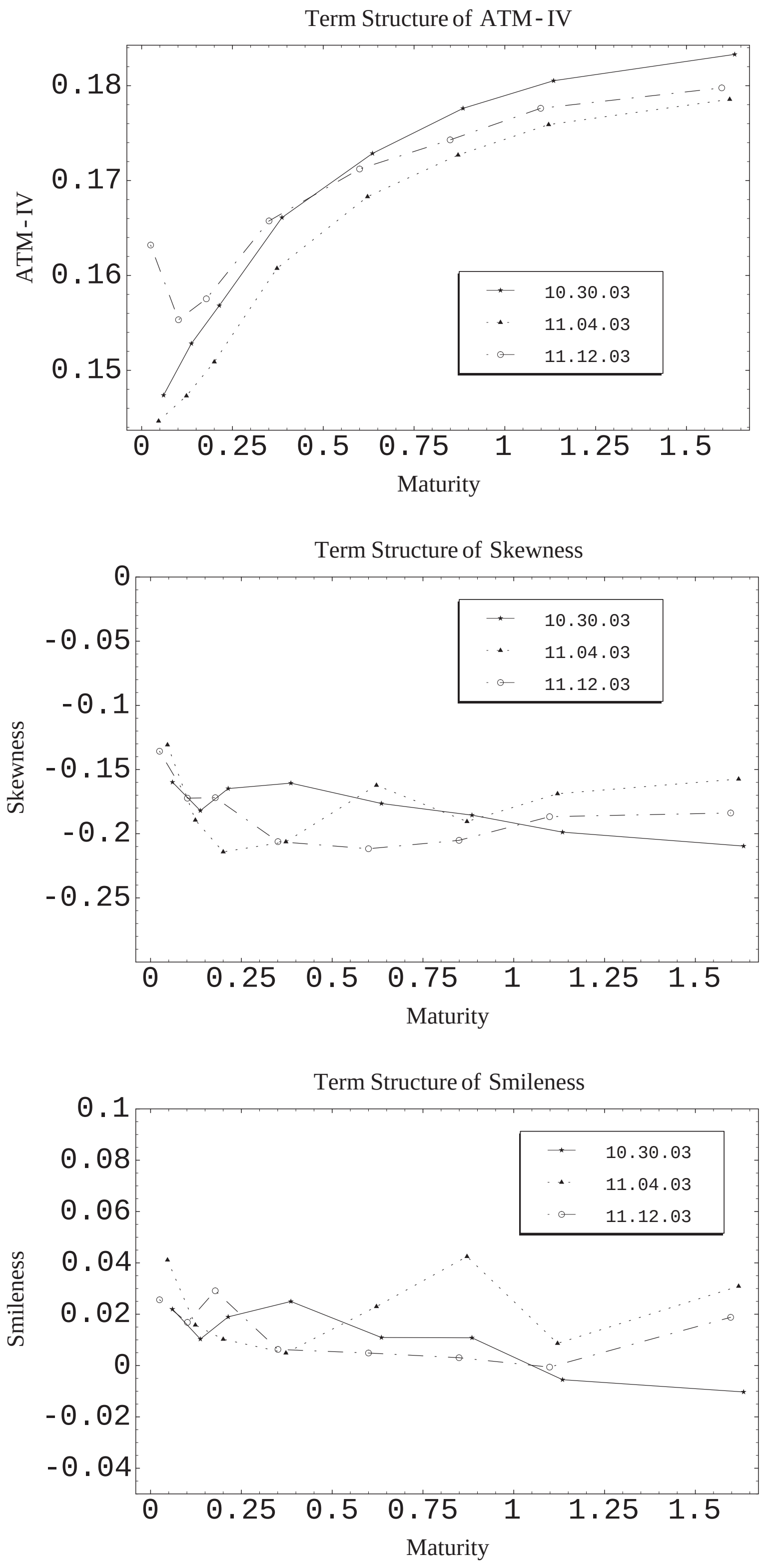


Figure 8: The time-change dynamics of the smirkness triplet, $(\sigma_{I0}, \gamma_1, \gamma_2)$ for options with the maturity date, September 16, 1999. The time series is from September 25, 1998 to September 3, 1999. The horizontal axis is calendar time t (year). The maturity date is at $t = 1$.

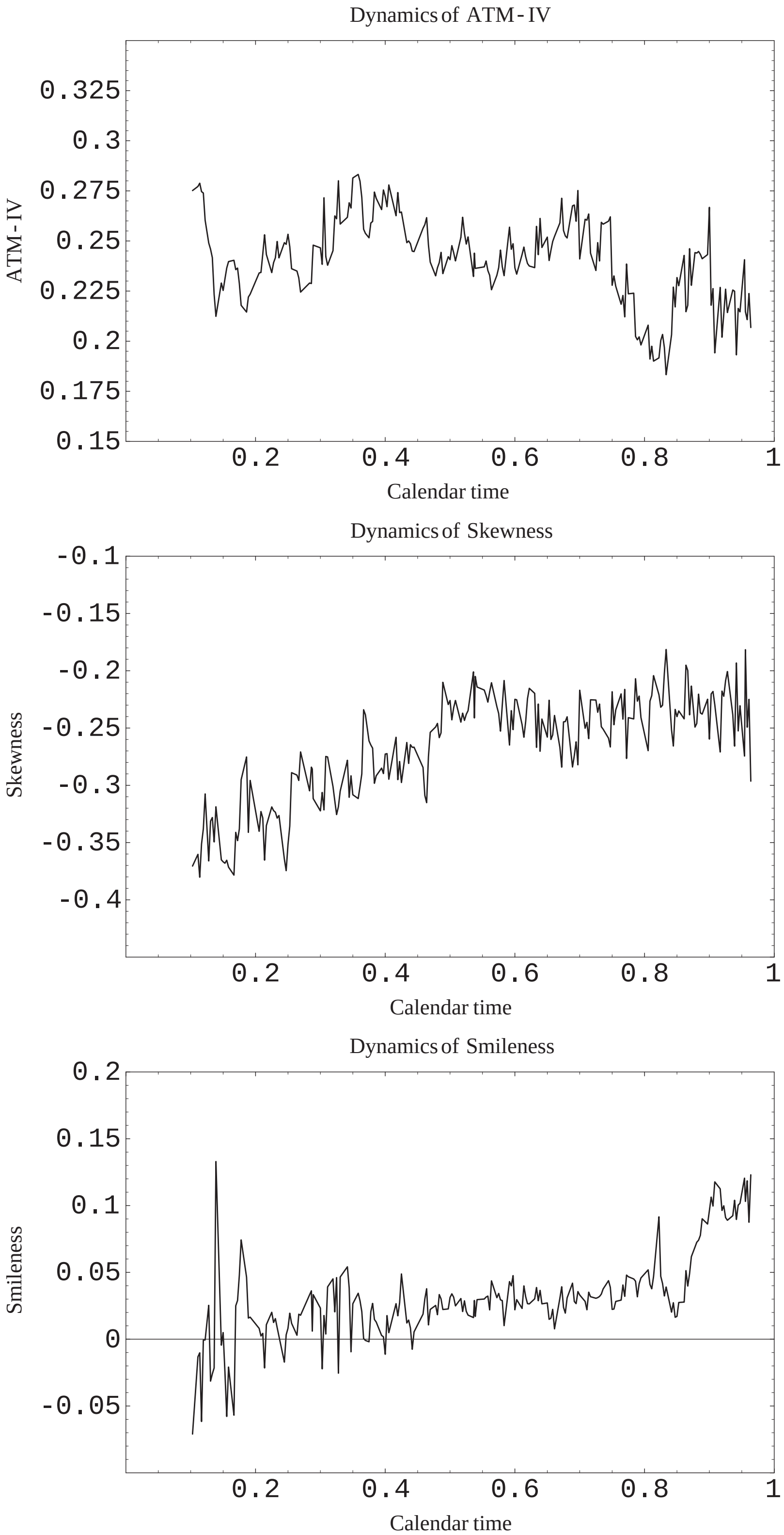


Figure 9: The term structure of smirkness implied in two calibrated option pricing models, and observed from the market data on November 4, 2003. The two models are the Constant Elasticity of Variance (CEV) model, $dS_t = (r-q)S_t dt + \sigma S_t^{\alpha} dB_t$, with calibrated parameters: $(\sigma, \alpha) = (152.36, 0)$, and the Finite Moment Log α-Stable (FMLS) process, $dS_t = (r-q)S_t dt + \sigma S_t dL_t^{\alpha,-1}$, with calibrated parameters: $(\sigma, \alpha) = (0.1086, 1.8141)$.