An Equilibrium Model with Buy and Hold Investors

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Abstract

This paper analyzes the effects of buy and hold investors on security price dynamics in a pure-exchange, continuous-time model. Empirical studies suggest that many defined contribution plan participants follow buy and hold strategies by rarely changing asset and flow allocations due to information costs or other frictions. Similar strategies are documented for institutional investors. A buy and hold investor effectively faces an incomplete market and differs in her pricing of risk from a dynamic asset allocator. Construction of an equilibrium is achieved through a representative agent with state-dependent utility. The fraction of the stock held by the buy and hold investor emerges as an additional state variable. Characterization of equilibrium quantities is given analytically as function of the state variables. A simple calibration of our model shows that the economy with buy and hold investors can simultaneously produce a low interest rate and a high Sharpe ratio. Moreover, the model can deliver a stock return volatility more than twice that in the limited participation model while keeping interest rate volatility at reasonably low levels. Intuition for these results is also provided.

Introduction

Researchers have recently been exploring the extent to which limited stock market participation might help explain the "equity premium puzzle" of Mehra and Prescott (1985) and the "risk-free rate puzzle" of Weil (1989). For example, using the 1984 Panel Study of Income Dynamics (PSID) data, Mankiw and Zeldes (1991) document that only about a quarter of the domestic households directly invest in stocks. They also note that the aggregate consumption of stockholders is more volatile and more correlated with excess stock market returns than that of non-stockholder's. Using stockholder's consumption data in a traditional asset pricing model such as that of Breeden (1979) can help explain the size of the equity premium with a reasonable level of the relative risk aversion.

Indeed, according to the Survey of Consumer Finances (2001), the percentages of U.S. households who hold stocks through brokerage and mutual fund accounts are only 21.3% and 17.7%, respectively. However, 52.2% of the U.S. households invest in retirement accounts. Empirical studies suggest that the contributions and portfolios of many of these retirement accounts are infrequently revised. The contributors to these accounts effectively follow a buy and hold strategy. In this paper, we show that an equilibrium model with buy and hold investors can shed some new light on asset pricing puzzles.

In 2000, 2.5 trillion dollars were invested in private sector defined contribution plans[1], representing the largest pool of money invested in capital markets.[2] Of this amount, about three quarters were invested in equities.[3] (The US equity market capitalization is on the order of ten trillion.) Poterba, Venti and Wise (1999) documents that the annual contribution flow

[1]U.S. Department of Labor data.

[2]See "Hewitt Announces Launch of New 401(k) Index" at http://was.hewitt.com/hewitt /services/401k/observ/news.htm.

[3]"Perspective", Investment Company Institute, Vol. 7, No. 5, November 2001.

to 401(k) programs now exceeds a hundred billion. In addition, the study on TIAA-CREF data from 1986 to 1996 by Ameriks and Zeldes (2000) reports that: "47 percent of individuals made no change to the contribution flow during the ten year period and another 21 percent only made one change. Roughly 73 percent made no change to asset allocations over the entire 10-year period and another 14 percent made one change. A full 44 percent of the population made no changes whatsoever to either the contribution flow or asset allocation, and another 17.2 percent made one change to either stocks or flows... The finding that participants rarely change either asset or flow allocations is consistent with earlier evidence reported in Samuelson and Zeckhauser (1998)." Choi et. al. (2002) also find that defined contribution plan participants tend to choose "the path of least resistance" by doing nothing to their flow and asset allocations. These evidence demonstrates that defined contribution plan participants who rarely make changes to their investment allocations follow buy and hold [4]. In addition, buy and hold is also popular among certain institutional investors, such as pension funds, endowments and foundations, who "have static or slow-changing goals; invest large pools of assets that are difficult and expensive to move; are governed by boards with complicated decision making; and eschew market timing."[5] However, little is known about the equilibrium impact of buy and hold investors on asset price dynamics.

To address this question, we introduce buy and hold agents into a continuous-time version of the pure exchange economy of Lucas (1978). This approach offers tractability as well as easy comparisons with similar benchmark models in the dynamic asset pricing literature. There are two types of agents in this economy: dynamic asset allocators (unrestricted agents) and buy and hold investors (restricted agents). Both agents have CRRA and can dynamically

[4] Our model also covers the case where some of the 401(k) participants dynamically trade stocks through a separate brokerage account–they simply become dynamic asset allocators. However, according to the Survey of Consumer Finances (2001), a large fraction of households invest exclusively through retirement accounts.

[5] See "Institutional Investors Buy and Hold Stocks", Amarillo Business Journal, April 1, 2002.

trade a locally riskless bond. In addition, the asset allocator dynamically trades a stock, which is a claim to an exogenously specified dividend process. Due to information costs and other frictions[6], the buy and hold investor, on the other hand, can only contribute continuously to a retirement account that is invested in the stock. Her investment in the stock, in that specific sense, are constrained. The bond is in zero net supply and the stock is in positive net supply. The optimal consumption allocations, the interest rate process, and the stock price process are all determined endogenously.

The equilibrium is constructed through a representative investor assigning stochastic weights to the two types of agents. The buy and hold feature leads to path dependence for the economy. By introducing the fraction of the stock held by the buy and hold investor as another state variable, in addition to aggregate dividend and the stochastic weight, we are able to preserve the Markovian structure of the economy. Characterizations of the equilibrium quantities are then given analytically as functions of the state variables, time, the dynamic asset allocator's wealth, and the stock price. The latter two are shown to satisfy a system of coupled partial differential equations.

The complicated relationship between the stock price and state variables requires numerical analysis. The dynamic asset allocator's wealth and the equilibrium stock price are calculated by solving the coupled partial differential equations using the finite difference method. Once this is achieved, all equilibrium quantities are readily obtained.

The main results of this paper are presented by comparing the buy and hold economy with an otherwise identical complete market economy, where both agents can trade the bond and the stock dynamically, and a limited stock market participation economy where the non-stockholder can only trade the bond. A simple calibration of our model shows that unlike

[6]Like Basak and Cuoco (1998), Shapiro (2002), we do not explicitly model such frictions, but rather take them as given and focus on their asset pricing implications.

the complete market economy, the buy and hold economy can simultaneously produce a low interest rate and a high Sharpe ratio, as is the case with the limited participation economy. At the same time, the buy and hold economy can deliver a stock return volatility more than twice that in the limited participation economy while keeping interest rate volatility at reasonably low levels. Intuition for these results is also provided.

To the best of our knowledge, this is the first model to analyze the effects of buy and hold investors on equilibrium asset prices. In terms of methodology, this paper utilizes the “stochastic welfare weight” approach, first introduced in Cuoco and He (1994a,b) to characterize equilibrium with incomplete market. A partial list of recent papers that have applied this method includes Detemple and Murthy (1997) on short sale constraints, Basak and Cuoco (1998) on limited stock market participation, Basak and Gallmeyer (1999) on differential taxation, Basak (2000) on heterogenous beliefs, Shapiro (2002) on the investor recognition hypothesis, and Dumas and Maenhout (2002) on incomplete market.

The strand of literature closely related to our work is the one on the asset pricing implication of limited stock market participation. On the empirical side, Brav, Constantinides and Geczy (2002) explore the same idea of Mankiw and Zeldes (1991) using the Consumer Expenditure Survey (CEX) data, which measures consumption more accurately. (The PSID data used by Mankiw and Zeldes (1991) only contains data on food consumption.) Brav, Constantinides and Geczy (2002) find that the equity premium can be better explained and with lower levels of relative risk aversion as they look at the wealthier cohorts of asset holders. In addition, the correlation between per capita consumption growth and equity premium also rises as the definition of asset holders becomes more restricted to the wealthy families.

Vissing-Jorgensen (2002) points out that Mankiw and Zeldes (1991), as well as Brav, Constantinides and Geczy (2002), estimate relative risk aversions using the unconditional version of the Euler's equation and do not provide standard errors for the estimates. Through a boot-strap study, she finds such estimates having very large standard errors. Using the same CEX database, she estimates the relative risk aversion using the conditional Euler's equation. Her results also highlight the importance of separating the different consumption patterns of stockholders and non-stockholders in estimating relative risk aversions and in explaining the equity premium puzzle.

On the theoretical side, Saito (1996) presents a limited stock market participation model in a continuous-time production economy. In his model, non-stock holders invest all their wealth in the riskfree asset. Such higher demand (compared with complete market case) lowers the return on the riskfree asset. The equity premium is increased as a result of the reduction in the risk-free rate. Moreover, the wealth distribution between stockholders and non-stockholders determines the magnitude of the impact of limited stock market participation on asset pricing.

Basak and Cuoco (1998) demonstrate that in a pure exchange continuous time economy, limited stock market participation can produce realistic values for the risk-free rate and Sharpe ratio with reasonable relative risk aversion coefficients. Therefore, if the stock return volatility is matched to observed values, the model yields the correct size for the equity premium. They provide analytical characterization of the equilibrium when the stockholder's relative risk aversion is different from one and the non-stock holder has logarithmic preferences. In this paper, we allow both types of agents to have power utilities. Moreover, we solve for the equilibrium stock return volatility endogenously. The limited participation economy in Basak and Cuoco (1998) can be seen as a special case of the buy and hold economy in this paper by setting both the initial holding and subsequent rate of contribution to the stock to zero for the buy and hold investor.

In both Saito (1996), and Basak and Cuoco (1998), the decision to participate in the stock market is exogenously specified. Cao, Wang, and Zhang (2002), however, argue that limited participation cannot explain the equity premium puzzle in a model where agent have heterogeneous model uncertainty and stock market participation arises endogenously. In their model, the equity premium consists of two parts: the risk premium and the uncertainty premium. As participation rate decreases, the risk premium increases as the stock is held by fewer agents, but the uncertainty premium decreases since only agents with low model uncertainty participate in the stock market. These two effects act in different directions on the equity premium.

Polkovnichenko (2001) also endogenizes the participation decision by introducing a fixed cost to participate in the stock market. Moreover, in his model, agents receive labor income in addition to financial income. Agents with low labor income choose not to participate in the stock market. After calibrating the model to the Survey of Consumer Finances data, he finds that the effect of limited participation on equity premium is rather small because labor income makes the consumption growth of stockholders less volatile and imperfectly correlated with dividends.

In addition, Constantinides, Donaldson, and Mehra (2002) analysizes limited stock market participation in the context of an over-lapping generation model. Young agents in their model expect increasing future income and want to invest in the stock. However they cannot do so because of borrowing constraint. The retirees have no labor income and consume assets obtained in the prior period. The stock is therefore concentrated in the hands of the middle-aged workers who demand a high equity premium.

The rest of the paper is structured as follows. Section I introduces the economic setup. In Section II we provide the main results on the characterization of the equilibrium in a buy and hold economy when both agents have constant relative risk aversion preferences. Section III collects the results on the corresponding limited participation economy and complete market economy for comparison purposes. In Section IV, we discuss the impact of buy and hold on the equilibrium by comparing the buy and hold economy with the complete market economy and the limited participation economy, under logarithmic preferences. Section V calibrates the model under general constant relative risk aversion preferences. Section VI concludes the paper.

I. The Economy

We present a continuous-time economy on the finite time horizon $[0, T]$. Uncertainty in the economy is represented by a filtered probability space (Ω, $\mathcal{F}$, $\mathbf{F}$, $\mathcal{P}$), on which is defined a one-dimensional Brownian motion w. The common information is given by the augmented filtration $\mathbf{F} = \{\mathcal{F}_t\}$ generated by w under the probability measure $\mathcal{P}$. The sigma-field $\mathcal{F}_t$ represents information available up to time t and $\mathcal{P}$ represents agents' common beliefs. All stochastic processes are progressively measurable with respect to $\mathbf{F}$, all the equalities involving random variables are understood to hold $\mathcal{P}$-almost surely.

A. Consumption Space

There is a single perishable consumption good (the numeraire). The agents' consumption space $\mathcal{C}$ is given by the set of nonnegative consumption-rate process c with $\int_0^T |c(t)|dt < \infty$.

B. Securities Market

The investment opportunities are represented by a locally riskless bond[7] earning an instantaneous interest rate r, and one share of risky stock. The bond is in zero net supply. Its initial price is scaled to one so that the bond price satisfies

$$B(t) = \exp\left(\int_0^t r(s)ds\right). \tag{1}$$

The stock is a claim to an exogenously specified strictly positive dividend process δ, which is described by the stochastic differential equation

$$d\delta(t) = \mu_\delta(t)dt + \sigma_\delta(t)dw(t), \quad \delta(0) = \delta_0. \tag{2}$$

where μ_δ and σ_δ are stochastic processes. The stock pays dividends at the rate δ over $[0, T]$, as well as a liquidating lump sum dividend $\delta(T)$ at time T. The lump sum dividend at time T is introduced for technical reasons. In equilibrium, we will show that the stock price S follows an Itô process:

$$dS(t) = (\mu(t)S(t) - \delta(t))dt + \sigma(t)S(t)dw(t). \tag{3}$$

C. Trading Strategies

An admissible trading strategy is given by a vector process (α, θ), where $\alpha(t)$ and $\theta(t)$ represent the amounts invested in the bond and the stock at time t, respectively, with

$$\int_0^T |\alpha(t)r(t) + \theta(t)\mu(t)|dt + \int_0^T |\theta(t)\sigma(t)|^2 dt < \infty$$

and

$$\alpha(t) + \theta(t) \geq 0 \ \ \forall t \in [0, T].$$

[7] The "bond" here can be understood as a money market account.

The set of admissible trading strategies is denoted by Θ.

A trading strategy $(\alpha, \theta) \in \Theta$ is said to finance a consumption plan $c \in \mathcal{C}$ if the wealth process $W = \alpha + \theta$ satisfies the budget constraint

$$dW(t) = (\alpha(t)r(t) + \theta(t)\mu(t) - c(t))dt + \theta(t)\sigma(t)dw(t).$$

D. Agent Types and Constraints on Trading Strategies

There are two types of agents in the economy. Agent (type) 1 can invest in the stock as well as the bond without constraints. Her trading strategy at time t is denoted by the pair $(\alpha_1(t), \theta_1(t))$.

Agent (type) 2's trading strategy in the stock follows buy and hold, i.e., contributing an exogenously specified amount $x(t)dt$ per time interval dt, from her wealth (originally invested in the riskless bond) into a retirement account that is invested in the stock. Buy and hold introduces path dependence in the economy. In other words, the state of the economy depends on the historical paths of the retirement contributions and stock prices in addition to other state variables. To preserve the Markovian structure and exploit the connection to partial differential equations, we introduce η, the fraction of the stock held by agent 2, as an additional state variable. Its value is given by

$$\eta(t) = \eta(0) + \int_0^t \frac{x(s)}{S(s)} ds. \tag{4}$$

Equation (4) indicates that η has no diffusion term. Let $\theta_2(t)$ denote agent 2's time t amount in the retirement account. By definition,

$$\theta_2(t) = \eta(t)S(t). \tag{5}$$

Therefore,

$$\theta_2(t) = \left[\frac{\theta_2(0)}{S(0)} + \int_0^t \frac{x(s)}{S(s)} ds\right] S(t). \tag{6}$$

Then Equation (6) intuitively describes the value of an account with initial investment of $\theta_2(0)$ and subsequent investments at the rate $x(s)$, $\forall s, 0 \leq s \leq t$, in the stock S.

In addition, agent 2 faces no constraint in investing in the bond. We use $\alpha_2(t)$ to denote her bond position.

E. Agents' Preferences and Endowments

Preference for agent i ($i = 1, 2$) is given by a time-additive expected utility function

$$U_i(c) = \mathrm{E}\left[\int_0^T e^{-\rho t} u_i(c(t))dt + e^{-\rho T} u_i(c(T))\right]. \tag{7}$$

In particular, we assume agents have constant relative risk aversion, i.e. $u_1(c_1) = c_1^{1-\gamma_1}/(1-\gamma_1)$ and $u_2(c_2) = c_2^{1-\gamma_2}/(1-\gamma_2)$. If γ_i equals to 1, the corresponding utility function is taken to be the logarithmic preference of $u_i(c_i) = \log(c_i)$.

At time 0, agent 1 is endowed with $1 - \eta(0)$ share of the stock, and a short position in b shares of the bond. Agent 2 is endowed with $\eta(0)$ share of the stock and b shares of the bond.

F. Equilibrium

An equilibrium for the economy is a price process (B, S) or equivalently, an interest rate and stock price process (r, S), and a set $c_i^*, (\alpha_i^*, \theta_i^*)$ of consumption and admissible trading strategies for the two agents such that

(i) (α_i^*, θ_i^*) finances c_i^* for $i = 1, 2$;

(ii) c_1^* maximizes U_1 over the set of consumptions plans $c \in \mathcal{C}$ which are financed by an admissible trading strategy $(\alpha, \theta) \in \Theta$, with $\alpha(0) + \theta(0) = (1 - \eta(0))S(0) - b$;

(iii) c_2^* maximizes U_2 over the set of consumptions plans $c \in \mathcal{C}$ which are financed by an admissible trading strategy $(\alpha, \theta) \in \Theta$, with $\alpha(0) + \theta(0) = \eta(0)S(0) + b$, and $\theta_2(t)$ given by Equation (6);

(iv) all markets clear, that is, $c_1^* + c_2^* = \delta$, $\alpha_1^* + \alpha_2^* = 0$, and $\theta_1^* + \theta_2^* = S$.

II. Equilibrium with Buy and Hold Investors

A. Portfolio Constraint and Agents' State Price Densities

Agent 1 faces a complete market. Therefore, her state price density is given by

$$\pi_1(t) = B(t)^{-1}\xi_1(t), \tag{8}$$

where $\xi_1(t)$ is defined by

$$\xi_1(t) = \exp\left(-\int_0^t \kappa(s)dw(s) - \frac{1}{2}\int_0^t |\kappa(s)|^2 ds\right). \tag{9}$$

$\kappa(t)$ is the market price of risk given by

$$\kappa(t) = \frac{\mu(t) - r(t)}{\sigma(t)}. \tag{10}$$

By Itô's lemma, ξ_1 follows

$$d\xi_1(t) = -\kappa(t)\xi_1(t)dw(t). \tag{11}$$

Agent 2 can trade the bond dynamically but her stock investment is constrained to buy and hold. He and Pearson (1991), Cvitanic and Karatzas (1992), Cuoco (1997) show that the constrained consumption and portfolio choice problem faced by agent can be transformed into that in a fictitious economy with no constraint but with a modified market price of risk and interest rate. In particular, the constraint can be written as $(\alpha_t, \theta_t) \in A_t$, where $A_t = \{(\alpha(t) \in \Re, \theta(t) = \left[\frac{\theta_2(0)}{S(0)} + \int_0^t \frac{x(s)}{S(s)} ds\right] S(t) = \theta_2(t)\}$. For $(\nu_0, \nu_-) \in \Re^2$, let

$$\Delta_t(\nu_0, \nu_-) = \sup_{(\alpha,\,\theta)\,\in\,A_t} -(\alpha\nu_0 + \theta\nu_-) \tag{12}$$

denote the support function of $-A_t$ and let

$$\tilde{A}_t = \{(\nu_0, \nu_-) \in \Re^2 : \Delta_t(\nu_0, \nu_-) < \infty\} \tag{13}$$

denote its effective domain. It is easily verified that in the case of the buy and hold constraint,

$$\tilde{A}_t = \tilde{A} = \{(0, \nu_-), \nu_- \in \Re\}. \tag{14}$$

In addition, let $\nu(t) = -\sigma(t)^{-1}\nu_-$, $\nu(t) \in \Re$,

$$\Delta_t(\nu_0, \nu_-) = -\theta_2(t)\nu_- = \theta_2(t)\sigma(t)\nu(t). \tag{15}$$

Moreover, agent 2's state price density is given by

$$\pi_2(t) = \beta_\nu(t)\xi_2(t), \tag{16}$$

where

$$\beta_\nu(t) = \exp\left(-\int_0^t (r(s) + \nu_0(s))ds\right) = \exp\left(-\int_0^t r(s)ds\right) = B(t)^{-1}, \tag{17}$$

$$\xi_2(t) = \exp\left(-\int_0^t \kappa_\nu(s)dw(s) - \frac{1}{2}\int_0^t |\kappa_\nu(s)|^2 ds\right), \tag{18}$$

and

$$\kappa_\nu(t) = \sigma(t)^{-1}(\mu(t) - r(t) + \nu_-(t)) = \kappa(t) + \sigma(t)^{-1}\nu_-(t) = \kappa(t) - \nu(t). \tag{19}$$

By Itô's lemma, ξ_2 follows

$$d\xi_2(t) = -\kappa_\nu(t)\xi_2(t)dw(t). \tag{20}$$

Since the constraint is on agent 2's stock position, and not on her bond position, only the market price of risk in the fictitious economy is affected, but not the interest rate.

B. Stochastic Weight, Interest Rate, the Market Price of Risk, and Equity Premium

Consider the utility function of the representative agent

$$U(c;\lambda) = \mathrm{E}\left[\int_0^T e^{-\rho t}u(c(t),\lambda(t))dt + e^{-\rho T}u(c(T),\lambda(T))\right], \tag{21}$$

where

$$u(c(t),\lambda(t)) = \max_{c_1+c_2=c} u_1(c_1) + \lambda(t)u_2(c_2). \tag{22}$$

Equation (22) implies the representative agent's marginal utility

$$u_c(\delta(t),\lambda(t)) = u_1'(c_1^*(t)) = \lambda(t)u_2'(c_2^*(t)). \tag{23}$$

The representative agent's marginal utility now has two sources of randomness: one comes from the aggregate dividend, the other comes from the stochastic weight. By Equation (23) and the first order conditions of both agents, the stochastic weight is given by

$$\lambda(t) = \frac{u_1'(t)}{u_2'(t)} = \frac{\psi_1\xi_1(t)}{\psi_2\xi_2(t)}, \tag{24}$$

where ψ_1 and ψ_2 are the corresponding associated Lagrange multipliers. By Equations (11), (20), (19), (24), and Itô's lemma, the stochastic weight follows the dynamics

$$d\lambda(t) = -\nu(t)(\kappa(t)-\nu(t))\lambda(t)dt - \nu(t)\lambda(t)dw(t). \tag{25}$$

Note that $\nu(t)$, the difference between the market prices of risk faced by the two agents, measures the volatility of the stochastic weight.

The first order condition of agent 1 is

$$e^{-\rho t}u_1'(c_1^*(t)) = \psi_1\pi_1(t), \tag{26}$$

where

$$d\pi_1(t) = -r(t)\pi_1(t)dt - \kappa(t)\pi_1(t)dw(t), \tag{27}$$

Equations (23), (26), and (27) together yield

$$de^{-\rho t}u_c(t) = -r(t)e^{-\rho t}u_c(t)dt - \kappa(t)e^{-\rho t}u_c(t)dw(t). \tag{28}$$

On the other hand, by Itô's lemma,

$$de^{-\rho t}u_c(t) = e^{-\rho t}[(\mathcal{D}u_c(t) - \rho u_c(t))dt + (u_{cc}(t)\sigma_\delta(t) - u_{c\lambda}(t)\nu(t)\lambda(t))dw(t)], \tag{29}$$

where $\mathcal{D}u_c(t)$ denotes the drift of $u_c(t)$. Matching the drift and diffusion terms of Equation (28) and Equation (29) respectively, we get

$$r(t) = \rho - \frac{\mathcal{D}u_c(t)}{u_c(t)} \tag{30}$$

and

$$\nu(t) = \frac{u_{cc}(t)\sigma_\delta(t) + u_c(t)\kappa(t)}{u_{c\lambda}(t)\lambda(t)}. \tag{31}$$

Let

$$\begin{aligned} A_i(t) &= -\frac{u_i''(c_i(t))}{u_i'(c_i(t))}, P_i(t) = -\frac{u_i'''(c_i(t))}{u_i''(c_i(t))}, i = 1, 2 \\ A(t) &= -\frac{u_{cc}(\delta(t), \lambda(t))}{u_c(\delta(t), \lambda(t))}, P(t) = -\frac{u_{ccc}(\delta(t), \lambda(t))}{u_{cc}(\delta(t), \lambda(t))}. \end{aligned}$$

denote the absolute risk aversion and absolute prudence coefficient at time t for agent 1, 2, and the representative agent respectively. The following lemma formalizes the above discussion.

Lemma 1 *In the **buy and hold economy**, the equilibrium interest rate and the market price of risk are given respectively by*

$$r(t) = \rho + A(t)\mu_\delta(t) - \frac{1}{2}P_1(t)A(t)A_1(t)^{-2}\kappa(t)^2 - \frac{1}{2}P_2(t)A(t)A_2(t)^{-2}\kappa_\nu(t)^2 \tag{32}$$

and

$$\kappa(t) = A(t)[\sigma_\delta(t) + A_2(t)^{-1}\nu(t)]. \tag{33}$$

Equation (33) implies the equity premium is given by

$$\mu(t) - r(t) = A(t)\sigma(t)\sigma_\delta(t) + A(t)A_2(t)^{-1}\sigma(t)\nu(t). \tag{34}$$

The optimal consumption policies c_1 and c_2 follow the processes:

$$dc_1(t) = [A_1(t)^{-1}(r(t) - \rho) + \frac{1}{2}P_1(t)A_1(t)^{-2}\kappa(t)^2]dt + A_1(t)^{-1}\kappa(t)dw(t). \tag{35}$$

$$dc_2(t) = [A_2(t)^{-1}(r(t) - \rho) + \frac{1}{2}P_2(t)A_2(t)^{-2}\kappa_\nu(t)^2]dt + A_2(t)^{-1}\kappa_\nu(t)dw(t). \tag{36}$$

Remark 1 *Equation (34) shows that the equity premium consists of two terms. The first term is the traditional Consumption CAPM term of covariation between stock return and consumption growth. In addition, there is a second term related to the covariation between stock return and the stochastic weight λ. (Recall from Equation (25) that ν measures how volatile λ is.) In a complete market, the weight λ is deterministic. The only source of randomness in the representative agent's marginal utility is from the fluctuation of aggregate consumption. With an incomplete market, λ is stochastic and its fluctuation results in an additional source of randomness in the representative agent's marginal utility (See Equation (23)). Whether the equity premium is higher or lower than that predicted by the Consumption CAPM depends critically on the sign of the shadow process ν.*

Remark 2 *Similarly, Equation (33) shows that whether the Sharpe ratio in the buy and hold economy is higher or lower than that in the complete market economy (which equals $A(t)\sigma_\delta(t)$) depends on the sign of the shadow process ν.*

C. Agent 1's Wealth and Stock Return Volatility

We look for an equilibrium in which the state variables (δ, λ, η) follow a joint Markov process. In such an equilibrium, agent 1's wealth W_1 and the stock price S must be deterministic functions of the state variables and time: $W_1(t) = J(\delta(t), \lambda(t), \eta(t), t)$ and $S(t) = F(\delta(t), \lambda(t), \eta(t), t)$ for some functions J and F. J and F are assumed to be continuously differentiable with respect to t and η, and twice continuously differentiable with respect to δ and λ.[8] Agent 1's wealth is defined by

$$J(\delta(t), \lambda(t), \eta(t), t) = \pi_1(t)^{-1}\mathrm{E}_t\left[\int_t^T \pi_1(s)c_1(s)ds + \pi_1(T)J(T)\right]. \tag{37}$$

The stock return volatility $\sigma(t)$ is then related to $J(t)$ and $F(t)$ by the following lemma:

Lemma 2 *The equilibrium stock return volatility is given by*

$$\sigma(t) = \frac{F_\delta(t)\sigma_\delta(t) - F_\lambda(t)\nu(t)\lambda(t)}{F(t)}, \tag{38}$$

where the shadow process ν satisfies

$$\nu(t) = \frac{[(1-\eta(t))F_\delta(t) - J_\delta(t)]\sigma_\delta(t)}{[(1-\eta(t))F_\lambda(t) - J_\lambda(t)]\lambda(t)}. \tag{39}$$

From now on, to keep notations at a minimum, we suppress the time subscripts whenever possible.

[8] We use F_δ and $F_{\delta\delta}$ to denote $\frac{\partial F}{\partial \delta}$ and $\frac{\partial^2 F}{\partial \delta^2}$ respectively.

D. Equilibrium Consumption Allocation

Equation (24) and the CRRA assumption on agents' preferences yield

$$\lambda = \frac{c_1^{-\gamma_1}}{c_2^{-\gamma_2}}. \tag{40}$$

Combining Equation (40) with the resource constraint

$$c_1 + c_2 = \delta, \tag{41}$$

it follows that both c_1 and c_2 are functions of δ and λ only. In particular, if we write $c_1 = h(\delta, \lambda)$, the following lemma holds:

Lemma 3 *The equilibrium consumption allocations are characterized by*

$$c_1 = h(\delta, \lambda) \textit{ and } c_2 = \delta - h(\delta, \lambda), \tag{42}$$

where

$$h : (0, +\infty) \times (0, +\infty) \to (0, \delta) \textit{ and } h(\delta, \lambda) + \lambda^{\frac{1}{\gamma_2}} h(\delta, \lambda)^{\frac{\gamma_1}{\gamma_2}} = \delta. \tag{43}$$

Remark 3 *Equation (43) can be easily solved numerically. Notice that function*

$$f(x) = x + \lambda^{\frac{1}{\gamma_2}} x^{\frac{\gamma_1}{\gamma_2}}$$

increases monotonically on $[0, \delta]$. *Moreover, since* $f(0) = 0$ *and* $f(\delta) > \delta$ *(*$\lambda > 0$*),* x *exists and is unique.*

E. Agents' Stock, Bond Investments and Total Wealth

Each agent's stock and bond investments and total wealth in equilibrium are given in the following lemma:

Lemma 4 *In equilibrium, the amounts invested in the bond by agents are given respectively by*

$$\begin{aligned}\alpha_1(t) &= J(t) - (1 - \eta(t))F(t),\\ \alpha_2(t) &= (1 - \eta(t))F(t) - J(t).\end{aligned}$$

The amounts invested in the stock by agents are given respectively by

$$\begin{aligned}\theta_1(t) &= (1 - \eta(t))F(t),\\ \theta_2(t) &= \eta(t)F(t).\end{aligned}$$

Agents' total wealth are given by

$$\begin{aligned}W_1(t) &= J(t),\\ W_2(t) &= F(t) - J(t). \qquad (44)\end{aligned}$$

F. PDE's for Agent 1's Optimal Wealth and the Stock Price

The previous four lemmas express the equilibrium interest rate, market price of risk, equity premium, stock return volatility, consumption allocations, agents' trading strategies and total wealth all in terms of the state variables, partial derivatives of agent 1's optimal wealth, and partial derivatives of the stock price. In this section, agent 1's optimal wealth and the stock price are presented as solutions of two coupled partial differential equations.

Theorem 1 *Agent 1's optimal wealth J solves the following PDE,*

$$\mathcal{D}J - rJ - \kappa\sigma(1 - \eta)F + c_1 = 0, \qquad (45)$$

with terminal condition

$$J(\delta, \lambda, \eta, T) = h(\delta(T), \lambda(T)),$$

where

$$\mathcal{D}J = J_\delta \mu_\delta - J_\lambda(\kappa - \nu)\nu\lambda + J_\eta x F^{-1} + J_t + \frac{1}{2}[J_{\delta\delta}\sigma_\delta^2 + J_{\lambda\lambda}\nu^2\lambda^2] - J_{\delta\lambda}\sigma_\delta\nu\lambda,$$

κ is given by Equation (33) and ν is given by Equation (39).

Similarly, the stock price is also given by a PDE:

Theorem 2 *The stock price F solves the following PDE,*

$$\mathcal{D}F - rF - \kappa\sigma F + \delta = 0, \tag{46}$$

with terminal condition

$$F(\delta, \lambda, \eta, T) = \delta(T),$$

where

$$\mathcal{D}F = F_\delta \mu_\delta - F_\lambda(\kappa - \nu)\nu\lambda + F_\eta x F^{-1} + F_t + \frac{1}{2}[F_{\delta\delta}\sigma_\delta^2 + F_{\lambda\lambda}\nu^2\lambda^2] - F_{\delta\lambda}\sigma_\delta\nu\lambda.$$

By now, we have expressed all terms in the two PDE's in terms of only the state variables — (δ, λ, η), t, J, F, J and F's first-order partial derivatives. The plan is to solve for J and F numerically. All the other variables in the model, e.g. κ, ν, μ, σ, r, can be expressed in terms of only the state variables — (δ, λ, η), time t, J, F, J's and F's first-order partial derivatives. Therefore, once J and F are known, the rest can be easily computed. The system of coupled second order PDE's on J and F is solved using the numerical methodology described in the appendix.

III. The Limited Participation Economy and the Complete Market Economy

For comparison purposes, this section collects the results on the equilibrium characterizations of limited participation economy and complete market economy in two corollaries. The third corollary compares the interest rate in a complete market economy with that in a buy and hold economy.

Corollary 1 *In the* ***limited participation economy***[9] *, the equilibrium is characterized by*

$$r(t) = \rho + A(t)\mu_\delta(t) - \frac{1}{2}A(t)P_1(t)\sigma_\delta(t)^2, \tag{47}$$

$$\kappa(t) = A_1(t)\sigma_\delta(t), \tag{48}$$

$$\nu(t) = \kappa(t) = A_1(t)\sigma_\delta(t), \tag{49}$$

$$\sigma(t) = \frac{F_\delta(t)\sigma_\delta(t) - F_\lambda(t)\nu(t)\lambda(t)}{F(t)},$$

$$\mu(t) - r(t) = A_1(t)\sigma(t)\sigma_\delta(t). \tag{50}$$

The stock price $F_{LP}(\delta, \lambda, t)$ satisfies the following PDE:

$$\mathcal{D}F - rF - \kappa\sigma F + \delta = 0, \tag{51}$$

[9] In the limited participation economy of Basak and Cuoco (1998), the non-stock holder is restricted to log preference to obtain closed form solution. The limited participation model considered here allows both agents to have arbitrary CRRA coefficients. In addition, we numerically solve for the stock price and investigate the equilibrium volatility.

with terminal condition

$$F(\delta, \lambda, T) = \delta(T), \tag{52}$$

where

$$\mathcal{D}F \quad = \quad F_\delta \mu_\delta + F_t + \frac{1}{2}[F_{\delta\delta}\sigma_\delta^2 + F_{\lambda\lambda}\nu^2\lambda^2] - F_{\delta\lambda}\sigma_\delta\nu\lambda.$$

Agents' optimal consumption processes are

$$dc_1(t) = A(t)[A_1(t)^{-1}\mu_\delta(t) + \frac{1}{2}P_1(t)A_2(t)^{-1}\sigma_\delta(t)^2]dt + \sigma_\delta(t)dw(t),$$

and

$$dc_2(t) = A(t)A_2(t)^{-1}[\mu_\delta(t) - \frac{1}{2}P_1(t)\sigma_\delta(t)^2]dt.$$

Corollary 2 *In the* ***complete market economy***[10] *, the equilibrium is characterized by*

$$r(t) = \rho + A(t)\mu_\delta(t) - \frac{1}{2}A(t)P(t)\sigma_\delta(t)^2, \tag{53}$$

$$\kappa(t) = A(t)\sigma_\delta(t),$$

$$\nu(t) = 0, \tag{54}$$

$$\sigma(t) = \frac{F_\delta(t)\sigma_\delta(t)}{F(t)},$$

$$\mu(t) - r(t) = A(t)\sigma(t)\sigma_\delta(t). \tag{55}$$

The stock price $F_{CM}(\delta, \lambda, t)$ satisfies the following PDE:

$$\mathcal{D}F - rF - \kappa\sigma F + \delta = 0, \tag{56}$$

[10]The complete market economy is identical to that of Basak and Cuoco (1998). Again, we numerically solve for the stock price and the equilibrium volatility.

with terminal condition

$$F(\delta, \lambda, T) = \delta(T), \tag{57}$$

where

$$\mathcal{D}F \quad = \quad F_\delta \mu_\delta + F_t + \frac{1}{2} F_{\delta\delta} \sigma_\delta^2.$$

Agents' optimal consumption processes are

$$\begin{aligned} dc_i(t) \quad &= \quad [A(t)A_i(t)^{-1}\mu_\delta(t) - \frac{1}{2}A(t)A_i(t)^{-1}P(t)\sigma_\delta(t)^2 + \frac{1}{2}A(t)^2A_i(t)^{-2}P_i(t)\sigma_\delta(t)^2]dt \\ &\quad +A(t)A_i(t)^{-1}\sigma_\delta(t)dw(t), \quad i = 1, 2. \end{aligned}$$

The next corollary compares the interest rate in the complete market economy with that in the buy and hold economy.

Corollary 3 *If agents have homogenous constant relative risk aversion, for a given pair of (δ, λ), the interest rate in the buy and hold economy is bounded from above by the interest rate in the complete market economy. The two are equal if and only if $\nu(\delta, \lambda, \eta, t) = 0$.*

IV. Effects of Buy and Hold Investors on the Equilibrium (the Case of Logarithmic Preferences)

To study the impact of buy and hold investors, in this section we specialize our economy to both agents having logarithmic preferences and the dividend process following geometric Brownian motion (GBM). (We will allow both agents having relative risk aversion different from one in the calibration section). In particular, we compare the equilibrium in the buy and hold (BH) economy with that in the complete market (CM) economy and the one in the limited participation (LP) economy. The latter two are studied in detail by Basak and Cuoco (1998), so that we directly invoke their results. The BH model is also calibrated using their parameters, which are taken to match the data in Mehra and Prescott (1985). The mean and volatility for consumption growth rate are set at $\bar{\mu} = 0.0183$ and $\bar{\sigma} = 0.0357$, respectively. Time preference parameter $\rho = 0.001$. In addition, we assume the buy and hold investor's contribution rate into the stock $x(t) = \bar{x}\delta(t)$, where $\bar{x}$ is a constant. Table 1 indicates that the ratio between average contribution given participation and per capita consumption for the whole population seems quite stable over the years. The constant $\bar{x}$ is set to 12%.[11]

A. Equilibrium in the Log Case with GBM Dividend

Specializing the results in Section III, we obtain the following characterization of asset price dynamics and optimal consumption allocations.

Corollary 4 *When both agents in the* ***buy and hold economy*** *have* ***logarithmic*** *preferences and aggregate dividend follows GBM:*

$$d\delta(t) = \bar{\mu}\delta(t)dt + \bar{\sigma}\delta(t)dw(t), \tag{58}$$

the equilibrium consumption allocations are given by

$$c_1(t) = \frac{\delta(t)}{1+\lambda(t)}, c_2(t) = \frac{\lambda(t)\delta(t)}{1+\lambda(t)}. \tag{59}$$

[11]Department of Labor data indicates that retirement contribution is typically 20% as large as per capita consumption (See Table 1). In addition, Exhibit 8 of Ameriks and Zeldes (2000) shows that on average, about 60% of contribution flow goes to stocks. Therefore, $\bar{x} \approx 20\% \times 60\% = 12\%$. The purpose of the proportional assumption is to get an idea of the relative size of the contribution rate. The model can easily accommodate more sophisticated functional forms for $x(t)$. In fact, most of the results are driven by the "holding" rather than the "buying" of the stock by the restricted agent as the last section on intuitions will demonstrate.

The weighting process λ follows

$$d\lambda(t) = \frac{(1+\lambda(t))(\bar{\sigma}-\kappa(t))[(1+\lambda(t))\bar{\sigma}-\kappa(t)]}{\lambda(t)}dt + (1+\lambda(t))(\bar{\sigma}-\kappa(t))dw(t). \tag{60}$$

The interest rate and market price of risk are characterized respectively by

$$r(t) = \rho + \bar{\mu} - \bar{\sigma}^2 - \frac{(\bar{\sigma}-\kappa(t))^2}{\lambda(t)} \tag{61}$$

and

$$k(t) = \bar{\sigma} + \frac{\nu(t)\lambda(t)}{1+\lambda(t)}. \tag{62}$$

The equity premium is given by

$$\mu(t) - r(t) = \sigma(t)\bar{\sigma} + \sigma(t)\frac{\nu(t)\lambda(t)}{1+\lambda(t)} = cov_t(\frac{dS}{S}, \frac{d\delta}{\delta}) - \frac{1}{1+\lambda(t)}cov_t(\frac{dS}{S}, d\lambda). \tag{63}$$

Agents' optimal consumptions follow

$$dc_i^*(t) = \mu_{c_i^*}(t)c_i^*(t)dt + \sigma_{c_i^*}(t)c_i^*(t)dw(t),\ i = 1, 2$$

where

$$\begin{aligned}
\mu_{c_1^*}(t) &= \bar{\mu} + (\bar{\sigma}-\kappa)(-2\kappa+\nu), \\
\mu_{c_2^*}(t) &= \bar{\mu} - \frac{(\bar{\sigma}-\kappa)(-2\kappa+\nu)}{\lambda}, \\
\sigma_{c_1^*}(t) &= \kappa, \\
\sigma_{c_2^*}(t) &= \bar{\sigma} + \frac{(\bar{\sigma}-\kappa)}{\lambda}.
\end{aligned}$$

To facilitate comparisons across economies, we also reiterate the corresponding results for the complete market economy and limited participation economy. In the ***complete market economy****,*

$$\begin{aligned}
\mu_{c_1^*}(t) &= \mu_{c_2^*}(t) = \bar{\mu}, \\
\sigma_{c_1^*}(t) &= \sigma_{c_2^*}(t) = \bar{\sigma}.
\end{aligned}$$

*While in the **limited participation economy**,*

$$\mu_{c_1^*}(t) = \bar{\mu} + (\lambda + \lambda^2)\bar{\sigma}^2,$$

$$\mu_{c_2^*}(t) = \bar{\mu} - (1 + \lambda)\bar{\sigma}^2,$$

$$\sigma_{c_1^*}(t) = (1 + \lambda)\bar{\sigma},$$

$$\sigma_{c_2^*}(t) = 0.$$

B. Effects on Equilibrium Asset Price Dynamics

Basak and Cuoco (1998) show that with logarithmic preferences, in both the complete market economy and limited participation economy, the expected return of the stock $\mu_S^{CM/LP} = \rho + \bar{\mu}$, and the volatility of the stock $\sigma_S^{CM/LP} = \bar{\sigma}$. In other words, restricting agent 2 from investing in the stock has no impact on either the expected return or the volatility of the stock.

Figure 1 shows that stock return volatility in the buy and hold economy can be both higher or lower than that in the complete market economy or the limited participation economy. When the buy and hold investor's stock holding η is held constant at small values, increasing her consumption share λ leads to higher stock return volatility. But with log utility, the effect is small.

The effect on of buy and hold on expected stock return is similar to that on stock return volatility, as shown in Figure 2. Expected stock return in the buy and hold economy can be both higher or lower than that in the complete market economy or the limited participation economy. Increasing buy and hold investor's consumption share λ leads to higher expected stock return. Again, the change is small with log utility, .

Figure 3 compares the Sharpe ratio in the buy and hold economy with those in the complete market and the limited participation economy. Note that with log utility, agent

1's fraction of wealth invested in the stock is $\sigma^{-1}\kappa$. As η increases from 0 to 1, agent 1 optimally chooses to hold less stock. Since the change in σ is small, the Sharpe ratio, given by κ, must decrease to induce her to invest less in stock. As η reaches 1, the Sharpe ratio declines to zero and agent 1 optimally choose to hold zero amount in the stock. In addition, holding η constant, the Sharpe ratio increases as the buy and hold investor's consumption share increases. The same intuition also applies to the equity premium, which is obtained just by multiplying the Sharpe ratio by the stock return volatility. Figure 4 plots a similar picture for the equity premium as that for the Sharpe ratio.

Figure 5 shows that the interest rate in the buy and hold economy r^{BH} is uniformly lower than the interest rate in the complete market economy, given by $r^{CM} = \rho + \bar{\mu} - \bar{\sigma}^2$. This is obviously seen from Equation (61). In fact, Corollary 3 states that r^{BH} is bounded from above by r^{CM} as long as agents have the same relative risk aversion coefficient. In the complete market economy with homogenous agents, there is no borrowing or lending between two agents. In the corresponding economy with buy and hold investors, as η decreases, Sharpe ratio increases, inducing agent 1 to invest more in stock by borrowing, the interest rate decreases. We know r^{BH} is quadratic in ν. Moreover, r^{BH} attains its maximum at $\nu = 0$. The maximum value is equal to the interest rate in the corresponding complete market economy. Figure 6 indicates that ν is approximately linear in η given a particular consumption share λ. Therefore r appears close to quadratic in η. In addition, when η, the fraction of the stock held by the buy and hold investor is zero, r^{BH} is very close to $r^{LP} = \rho + \bar{\mu} - (1 + \lambda(t))\bar{\sigma}^2$, the interest rate in the corresponding limited participation economy. Moreover, holding η constant, the higher the consumption share for the buy and hold investor, the lower the interest rate can potentially become.

C. Effects on Expected Consumption Growth and Volatility

Figures 7 and 8 show that in the economy with buy and hold investors, the higher the consumption share claimed by the buy and hold investor, the higher the unrestricted agent's expected consumption growth rate and volatility become. Moreover, the unrestricted agent's expected consumption growth rate and volatility will exceed those in the complete market case if she holds more stock than what she would hold in the corresponding complete market economy. When $\eta = 0$, the unrestricted agent holds all of the stock and her expected consumption growth rate and volatility are close to that in the limited participation economy.

Figures 9 and 10 show that the buy and hold investor's expected consumption growth rate and volatility start off close to the non-stock holder case in the limited participation economy at $\eta = 0$ and gradually increase as she holds more shares. Moreover, her expected consumption growth rate and volatility will exceed those in the complete market case if she holds more stock than what she would hold had she faced a complete market.

V. The Effects of Buy and Hold Investors on the Equilibrium (the CRRA Model)

In this section we allow both agents to have general constant relative risk aversion preferences. We demonstrate that the qualitative features of the model discussed in the previous section carries over to the more general model. A simple calibration with $\gamma_1 = 1$ and $\gamma_2 = 3$ suggests that the model with buy and hold investors can match the data better than both the complete market model and the limited participation model.

First we look at the complete market case. If we calibrate the complete market model such that the interest rate is reasonable at close to 3% as shown in Figure 11, but then the

Sharpe ratio is too low (Figure 12). If we increase agents' relative risk aversion to increase the Sharpe ratio, the interest rate becomes too high. The model cannot generate a low interest rate and a high Sharpe ratio at the same time. This is the well known equity premium/risk free rate puzzle. The reason behind this puzzle is obvious from Corollary 2. Both the interest rate and the Sharpe ratio respond positively to an increase to the representative agent's risk aversion. Increasing the Sharpe ratio inevitably increases the interest rate.

Next, we examine the limited participation model. Figure 13 and 14 show that the limited participation model can produce a low interest rate and a high Sharpe ratio with λ around 4. [12]As shown in Corollary 1, in the limited participation model, the Sharpe ratio responds only to agent 1's risk aversion. Therefore, we can use agent 1's risk aversion to match the Sharpe ratio and agent 2's risk aversion to match the risk free rate. However, as indicated by Figure 15 and 16, with λ around 4, the stock return volatility σ is too low while the volatility for the risk free rate is much too high compared with those in the data. Consequently, the expected return is also low for the stock (Figure 17).

Finally, Figures 18 and 19 show that the model with buy and hold investors with λ around 4 and η around 0.3 can generate both a low interest rate and a high Sharpe ratio.[13] (Note that the stock here refers to unlevered equity.) In addition, it can produce stock return volatility more than twice that in the limited participation model and a higher expected return (Figures 20 and 21). We decompose the stock return volatility into its two components:

$$\sigma = \frac{\partial \log F}{\partial \log \delta}\bar{\sigma} - \frac{\partial \log F}{\partial \log \lambda}\nu.$$

The first component is caused by the dividend (consumption) risk. The second component is related to the risk of random shifting wealth represented by the weighting process. The

[12] $\lambda = 4$ corresponds to a consumption share of 85% claimed by the non-stockholder. This value is close to that observed in data.

[13] $\lambda = 4$ here corresponds to a consumption share of 85% claimed by the buy and hold investor. η around 0.3 seems also realistic given the fraction of stocks held by defined contribution plans.

higher stock return volatility is mainly a result of higher sensitivity of the stock price with respect to the dividend risk ($\frac{\partial \log F}{\partial \log \delta}$) in the buy and hold economy (Figure 22) than that in the corresponding limited participation economy (Figure 23). To keep things simple, we consider the case where the contribution rate is zero, i.e., $x = 0$. If there is a negative shock to the dividend, in the absence of buy and hold constraint, the less risk averse agent 1 would sell shares to the more risk averse agent 2, and the stock price drops to S_1 where agent 1 no longer wants to sell. With the buy and hold constraint, selling is impossible. Therefore, to reach equilibrium, the stock price has to drop further than S_1 such that agent 1 does not want to sell at such a low price. In the case of a positive shock to the dividend, the less risk averse agent 1 would buy shares from the more risk averse agent 2, and the stock price rises to S_2 where agent 1 no longer wants to buy. With the buy and hold constraint, buying is impossible. Therefore, to reach equilibrium, the stock price has to rise further than S_2 such that agent 1 does not want to buy at such a high price. The pressure to trade after a dividend shock becomes more severe as the buy and hold investor holds more shares of the stock, leading to higher sensitivity of the stock price with respect to the dividend risk.

Moreover, as shown in Figure 24, interest rate volatility in the buy and hold economy with λ around 4 and η around 0.3 is kept at 3%, which is within the reasonable range. The interest rate sensitivities to both dividend risk and the risk to random shifting wealth represented by the weighting process are comparable in both types of economies. Similarly we decompose the interest rate volatility into its two components:

$$\sigma_r = \frac{\partial \log r}{\partial \log \delta}\bar{\sigma} - \frac{\partial \log r}{\partial \log \lambda}\nu.$$

The lower interest rate volatility in the buy and hold economy compared to the limited participation economy is mostly due to the fact that ν, the volatility of the weighting process λ is higher in the corresponding limited participation economy (Figures 25 and 26). This is because in the limited participation economy, a positive (negative) dividend shock increases (decreases) only the value of the stock owned by the stockholder (the non-stockholder owns no stock by definition), while in the buy and hold economy, such shock increases (decreases) the values of stock holdings of *both* agents, therefore having a smaller effect on the weighting process in the latter economy.

VI. Conclusion

This paper presents a continuous-time pure-exchange economy populated by dynamic asset allocators and buy and hold investors. The buy and hold investor faces portfolio constraint and therefore has state price density different from that of the dynamic asset allocator. The construction of equilibrium is achieved through a representative investor with stochastic weights assigned to the two types of agents. In equilibrium, the fraction of the stock held by the buy and hold investor emerges as an additional state variable to capture the information of the historical dividend and stock price. We characterize all equilibrium quantities as functions of the state variables. The main results of this paper are then presented by comparing the buy and hold economy with an otherwise identical complete market economy and a limited participation economy. A simple calibration of our model shows that unlike the complete market economy, the buy and hold economy can simultaneously produce a low interest rate and a high Sharpe ratio, as is the case with limited participation economy. Moreover, the buy and hold economy can deliver stock return volatility more than twice that in the limited participation economy while keeping interest rate volatility at reasonably low levels. Intuition for these results is also provided.

VII. Appendices

A. Proofs

A.1. Proof of Lemma 1.

Define $f_1(\cdot)$ and $f_2(\cdot)$ to be the inverse function of $u_1'(\cdot)$ and $u_2'(\cdot)$ respectively. Then the identity

$$c_1 = f_1(u_1'(c_1)) \tag{64}$$

holds. Taking partial derivatives of Equation (64) with respect to c_1 yields

$$f_1'(u_1'(c_1)) = u_1''(c_1)^{-1} = -A_1^{-1}u_c(\delta,\lambda)^{-1}, \tag{65}$$

Taking partial derivatives of Equation (65) with respect to c_1 leads to

$$f_1''(u_1'(c_1)) = -u_1'''(c_1)u_1''(c_1)^{-3} = P_1A_1^{-2}u_c(\delta,\lambda)^{-2}. \tag{66}$$

Substituting Equation (23) into Equation (64) yields

$$c_1 = f_1(u_c(\delta,\lambda)). \tag{67}$$

Applying Itô's lemma to Equation (67) and using Equations (28), (65), (66) lead to Equation (35).

The derivation of c_2's dynamics is essentially the same, except that

$$c_2 = f_2(\lambda^{-1}u_c(\delta,\lambda)) \tag{68}$$

and Equation (25) is used.

Adding the drift terms of c_1 and c_2 then equating the sum to μ_δ give the expression for interest rate in Equation (32).

By definition,

$$f_1(u_c(\delta,\lambda)) + (f_2(\lambda^{-1}u_c(\delta,\lambda)) = \delta. \tag{69}$$

Differentiating both sides of Equation (69) with respect to δ and using Equation (65) and

$$f_2'(u_2'(c_2)) = u_2''(c_2)^{-1} \tag{70}$$

yield

$$u_2''(c_2)^{-1} = \lambda(u_1(c_1)'')^{-1}u_{cc}(\delta,\lambda)^{-1} \tag{71}$$

Differentiating both sides of Equation (69) with respect to λ and using Equation (71) lead to

$$u_{c\lambda}(\delta,\lambda) = \lambda^{-1}AA_2^{-1}u_c(\delta,\lambda). \tag{72}$$

Substituting Equation (72) into Equation (31) and re-arranging terms yield Equation (33). Equation (34) follows from Equation (33).

A.2. Proof of Lemma 2.

It follows from Equation (37) and the martingale representation theorem that there exists a process φ_1 with $\int_0^T |\varphi_1(t)|^2 dt < \infty$ a.s. such that

$$\pi_1(t)J(t) + \int_0^t \pi_1(s)c_1(s)ds = J(0) + \int_0^t \varphi_1(s)dw(s) \tag{73}$$

is a martingale. Matching the diffusion terms on both sides of Equation (73) we get

$$\sigma_J = \pi_1^{-1}\varphi_1 + \kappa J, \tag{74}$$

where σ_J denotes the diffusion of J. On the other hand, standard argument such as that in Cuoco (1997) suggests that agent 1's optimal investment in the stock is given by

$$\theta_1 = \sigma^{-1}(\pi_1^{-1}\varphi_1 + \kappa J). \tag{75}$$

Comparing Equation (74) and Equation (75) yields

$$\sigma_J = \sigma\theta_1 = \sigma(1-\eta)F. \tag{76}$$

Alternatively, σ_J can be written as

$$\sigma_J = J_\delta\sigma_\delta - J_\lambda\nu\lambda. \tag{77}$$

Putting equations (76) and (77) together:

$$\sigma = \frac{J_\delta\sigma_\delta - J_\lambda\nu\lambda}{(1-\eta)F}. \tag{78}$$

Similarly, the diffusion of the stock price F, is given by

$$\sigma F = F_\delta\sigma_\delta - F_\lambda\nu\lambda. \tag{79}$$

Solving for σ from Equation (79) and equating the result to the expression for σ in Equation (78) leads to the following equation

$$J_\delta\sigma_\delta - J_\lambda\nu\lambda = (1-\eta)(F_\delta\sigma_\delta - F_\lambda\nu\lambda). \tag{80}$$

Solving ν from Equation (80), we obtain Equation (39).

A.3. Proof of Theorem 1.

The PDE for J is obtained by using the fact that the martingale in Equation (73) must have a zero drift.

A.4. Proof of Theorem 2.

Define W_2, the optimal wealth process[14] for agent 2 by

$$W_2(t) = \pi_2(t)^{-1}\mathrm{E}_t\left[\int_t^T \pi_2(s)[c_2(s) - \Delta_s(\nu(s))]ds + \pi_2(T)[c_2(T) - \Delta_T(\nu(T))]\right], \tag{81}$$

[14] Cuoco (1997) proves the optimal wealth of constrained agent are financed by associated trading strategies that satisfy agent's portfolio constraint.

where the function $\Delta_t(\nu) = \theta_2(t)\sigma(t)\nu(t)$ is given in Equation (15). It then follows from Equations (44) and (81) that the martingale

$$\pi_2(t)\left[F(t) - J(t)\right] + \int_0^t \pi_2(s)\left[c_2(s) - \theta_2(s)\sigma(s)\nu(s)\right] ds \tag{82}$$

must have a zero drift term, i.e.,

$$\mathcal{D}F - \mathcal{D}J - r(F - J) - (\kappa - \nu)\sigma\theta_2 + c_2 - \theta_2\sigma\nu = 0. \tag{83}$$

Substituting Equation (45) into Equation (83), we obtain the PDE for stock price F.

A.5. Proof of Corollary 1 and 2.

See Basak and Cuoco (1997).

A.6. Proof of Corollary 3.

Substituting Equation (33) into Equation (32) yields an expression for r^{BH} as a quadratic polynomial in ν with a negative coefficient for the quadratic term. The zero order term of the polynomial is equal to r^{CM}. It is easy to see that the first order term vanishes if $\gamma_1 = \gamma_2$ and the polynomial achieves its minimum at $\nu = 0$.

A.7. Proof of Corollary 4.

Solving Lemma 3 with $\gamma_1 = \gamma_2 = 1$ directly yields Equation (59). The rest follows from substituting Equation (59) into Lemma 1 and Lemma 2.

B. Numerical Method

The two coupled PDE's in Equation (45) and Equation (46) are solved using the explicit finite difference method. This is a terminal value problem. Other finite difference schemes, such as the implicit method and the Crank-Nicolson method cannot be applied here because implicit methods require solving a system of linear difference equations. The PDE's we need to solve transform into a system of non-linear difference equations after discretization.

First, Equation (46) can be rewritten as

$$F_t = -\mathcal{L}F(\delta, \lambda, \eta, t), \tag{84}$$

where $\mathcal{L}$ is an operator on F that satisfies

$$\begin{aligned}&\mathcal{L}F(\delta, \lambda, \eta, t) = \\ &F_\delta \mu_\delta - F_\lambda(\kappa - \nu)\nu\lambda + F_\eta \bar{x}\delta F^{-1} + \frac{1}{2}[F_{\delta\delta}\sigma_\delta^2 + F_{\lambda\lambda}\nu^2\lambda^2] - F_{\delta\lambda}\sigma_\delta\nu\lambda + \delta - rF + \kappa\sigma F.\end{aligned}$$

Then we represent function $F(\delta, \lambda, \eta, t)$ by its values at the discrete set of points:

$$\begin{aligned}&\delta(k) = k\Delta\delta, k = 0, 1, ..., K \\ &\lambda(l) = l\Delta\lambda, l = 0, 1, ..., L \\ &\eta(m) = m\Delta\eta, m = 0, 1, ..., M \\ &t(n) = n\Delta t, n = 0, 1, ..., N\end{aligned}$$

where the Δ's are the grid spacings along each dimension. We discretize Equation (84) by using forward differencing along the time dimension and central differencing along the spatial dimension:

$$
\begin{aligned}
F_t(\delta,\lambda,\eta,t) &= \frac{F(\delta,\lambda,\eta,t+\Delta t)-F(\delta,\lambda,\eta,t)}{\Delta t},\\
F_\delta(\delta,\lambda,\eta,t) &= \frac{F(\delta+\Delta\delta,\lambda,\eta,t)-F(\delta-\Delta\delta,\lambda,\eta,t)}{2\Delta\delta},\\
F_{\delta\delta}(\delta,\lambda,\eta,t) &= \frac{F(\delta+\Delta\delta,\lambda,\eta,t)-2F(\delta,\lambda,\eta,t)+F(\delta-\Delta\delta,\lambda,\eta,t)}{(\Delta\delta)^2},\\
F_{\delta\lambda}(\delta,\lambda,\eta,t) &= \frac{F_\delta(\delta,\lambda+\Delta\lambda,\eta,t)-F_\delta(\delta,\lambda-\Delta\lambda,\eta,t)}{2\Delta\lambda},\\
&\ldots
\end{aligned}
$$

This way, we obtain a method of calculating F through iterating backward, starting at time T:

$$
\begin{aligned}
&F(\delta,\lambda,\eta,T) = \delta(T),\\
&F(\delta,\lambda,\eta,t-\Delta t) = F(\delta,\lambda,\eta,t) + \Delta t \mathcal{L} F(\delta,\lambda,\eta,t),\ t = T, T-\Delta t, ..., \Delta t.
\end{aligned}
$$

To insure stability of the numerical solution, the time-spacing needs to be taken much smaller relative to the space-spacings. Heuristically, if you start with a stable solution, then increasing the number of points along the space dimensions by a factor of n requires increasing the number of points along the time dimension by a factor of n^2 to retain stability. Detailed discussion of stability conditions for explicit finite difference can be found in Press et. al. (1993).

Once the value for F is known, its first- and second-order partial derivatives can be computed as shown above. The same calculation is carried out simultaneously for J. Since we have expressed all the equilibrium quantities, e.g. κ, ν, μ, σ, r, in terms of only the state variables — (δ,λ,η), t, J, F, J and F's first- and second-order partial derivatives, they are also readily obtained. Although we have only demonstrated the computation strategy for the buy and hold economy, similar methodology can be applied to solving equilibriums in the limited participation economy and complete market economy.

References

[1] Abel, A., 2002, The Effects of a Baby Boom on Stock Prices and Capital Accumulation in the Presence of Social Security, *Econometrica*, forthcoming.

[2] Allen, F. and D. Gale, 1994, Limited Market Participation and Volatility of Asset Prices, *American Economic Review*, 84, 933-955.

[3] Ameriks, J. and S. Zeldes, 2000, How Do Household Portfolio Shares Vary with Age? working paper, Columbia University.

[4] Bansal, R. and A. Yaron, 2003, Risks For the Long Run: A Potential Resolution of Asset Pricing Puzzles, *Journal of Finance*, forthcoming.

[5] Basak, S. 1995, A General Equilibrium Model of Portfolio Insurance, *Review of Financial Studies*, 8, 1059-1090.

[6] Basak, S., 2000, A Model of Dynamic Equilibrium Asset Pricing with Heterogeneous Beliefs and Extraneous Risk, *Journal of Economic Dynamics and Control*, 24, 63-95.

[7] Basak, S. and D. Cuoco, 1998, An Equilibrium Model with Restricted Stock Market Participation, *Review of Financial Studies*, 11, 309-341.

[8] Basak, S. and M. Gallmeyer, 2001, Capital Market Equilibrium with Differential Taxation, *European Finance Review*, forthcoming.

[9] Brav, A., G. Constantinides, and C. Geczy, 2002, Asset Pricing with Heterogeneous Consumers and Limited Participation: Empirical Evidence, *Journal of Political Economy*, 110, 793-824.

[10] Campbell, J., Consumption-Based Asset Pricing, NBER, working paper.

[11] Cao, H., T. Wang, and H. Zhang, 2002, Model Uncertainty, Limited Market Participation, and Asset Prices, working paper, University of North Carolina.

[12] Choi, J., D. Laibson, B. Madrian, and A. Metrick, 2002, Defined Contribution Pensions: Plan Rules, Participant Choices, and the Path of Least Resistance, *Tax Policy and the Economy, MIT Press.*

[13] Cochrane, J., F. Longstaff, P., Santa-Clara, Two Trees: Asset Price Dynamics Induced by Market Clearing, working paper, UCLA.

[14] Constantinides, G., J. Donaldson, and R. Mehra, 2002 Junior Can't Borrow: A New Perspective on the Equity Premium Puzzle, *Quarterly Journal of Economics*, 117, 269-296.

[15] Constantinides, G., and D., Duffie, 1996, Asset Pricing with Heterogeneous Consumers, *Journal of Political Economy*, 104, 219-240.

[16] Cox, J. and C.-f. Huang, 1989, Optimal Consumption and Portfolio Policies When Asset Prices Follow a Diffusion Process, *Journal of Economic Theory*, 49, 33-83.

[17] Cuoco, D., 1997, Optimal Consumption and Equilibrium Prices with Portfolio Constraints and Stochastic Income, *Journal of Economic Theory*, 72, 33-73.

[18] Cuoco, D. and H. He, 1994a, Dynamic Aggregation and Computation of Equilibria in Finite Dimensional Economies with Incomplete Financial Markets, working paper, University of Pennsylvania.

[19] Cuoco, D. and H. He, 1994b, Dynamic Equilibrium in Infinite Dimensional Economies with Incomplete Financial Markets, working paper, University of Pennsylvania.

[20] Cvitanic, J. and I. Karatzas, 1992, Convex Duality in Constrained Portfolio Optimization, *The Annals of Applied Probability*, 2, 767-818.

[21] Dammon, R, C. Spatt, and H. Zhang, 2001, Optimal Consumption and Investment with Capital Gains Taxes , *Review of Financial Studies*, 14, 583-616.

[22] Dammon, R, C. Spatt, and H. Zhang, 2004, Optimal Asset Location and Allocation with Taxable and Tax-Deferred Investing, *Journal of Finance*, forthcoming.

[23] Detemple, J. and S. Murthy, 1997, Equilibrium Asset Prices and No-Arbitrage with Portfolio Constraints, *Review of Financial Studies*, 10, 1133-1174.

[24] Dumas, B., 1989, Two-Person Dynamic Equilibrium in the Capital Market, *Review of Financial Studies*, 2, 157-188.

[25] Dumas, B. and P. Maenhout, 2002, A Central-Planning Approach to Dynamic Incomplete-Market Equilibrium, working paper, INSEAD.

[26] He, H., and N. Pearson, 1991, Consumption and Portfolio Policies with Incomplete Market and Short- Sale Constraint: The Infinite Dimensional Case, *Journal of Economic Theory*, 54, 259-304.

[27] Huberman, G., S. Iyengar, and W. Jiang, 2003, Defined Contribution Pension Plans: Determinants of Participation and Contribution Rates, working paper, Columbia University.

[28] Huberman, G., P. Sengmuller, 2002, Company Stock in 401 (k) Plans, working paper, Columbia University.

[29] Hull, J., and A. White, 1990, Valuing Derivative Securities Using the Explicit Finite Difference Method, *Journal of Financial and Quantitative Analysis*, 25, 87-100.

[30] Liu, J., and E. Neis, 2002, Endogenous Retirement and Portfolio Choice, working paper, UCLA.

[31] Lucas R., 1978, Asset Prices in an Exchange Economy, *Econometrica*, 46, 1429-1445.

[32] Mankiw, N. and S. Zeldes, 1991, The Consumption of Stockholders and Non-Stockholders, *Journal of Financial Economics*, 29, 97-112.

[33] Mehra, R. and E. Prescott, 1985, The Equity Premium: A Puzzle, *Journal of Monetary Economics*, 15, 145-161.

[34] Merton, R. 1971, Optimal Consumption and Portfolio Rules in a Continuous Time Model, *Journal of Economic Theory*, 3, 373-413.

[35] O'Brien, T., 2002, Time Diversification, Dollar Cost Averaging, and Other Dynamic Asset Allocation Issues, working paper, University of Connecticut.

[36] Polkovnichenko, V., 2001, Limited Stock Market Participation and the Equity Premium Puzzle, working paper, University of Minnesota.

[37] Poterba, J., S. Venti and D. Wise, 1999, Pre-retirement Cashouts and Foregone Retirement Saving: Implications for 401(k) Asset Accumulation, NBER working paper No. 7314.

[38] Press, W., B. Flannery, S. Teukolsky, and W. Vetterling. 1993, *Numerical Recipes in C*, (2nd ed.) Cambridge: Cambridge University Press.

[39] Saito, M., 1996, Limited Participation and Asset Pricing, working paper, University of British Columbia.

[40] Samuelson, P., 1994, The Long-Term Case for Equities, *Journal of Portfolio Management*, 21, 15-24.

[41] Samuelson, W. and R. Zeckhauser, 1998, Status Quo Bias in Decision Making, *The Journal of Risk and Uncertainty*, 1(1), 7-59.

[42] Schwert, G., 1989, Why Does Stock Market Volatility Change Over Time, *Journal of Finance*, 44, 1115-1153.

[43] Shapiro, A., 2002, The Investor Recognition Hypothesis in a Dynamic General Equilibrium: Theory and Evidence, *Review of Financial Studies*, 15(1), 97-141.

[44] Shiller, R., 1981, Do Stock Prices Move Too Much to Be Justified by Subsequent Changes in Dividends? *American Economic Review*, 71, 421-436.

[45] Storesletten, K., C. Telmer, A. Yaron, Asset Pricing with Idiosyncrtic Risk and Overlapping Generations, working paper, University of Pennsylvania.

[46] Sundaresan, S., 2000, Continuous-Time Methods in Finance: A Review and an Assessment, *Journal of Finance*, 55, 1569-1622.

[47] Vissing-Jorgensen, A., 2002, Limited Asset Market Participation and the Elasticity of Intertemporal Substitution, *Journal of Political Economy*, 110, 825-853.

[48] Weil, P., 1989, The Equity Premium Puzzle and the Risk free Rate Puzzle, *Journal of Monetary Economics*, 24, 401-421.

Table 1: **401(k) Contribution as a Percentage of Personal Consumption (1989-1998)**

The table displays the relationship between the average 401(k) plan contribution and the corresponding personal consumption data for that year. Consumption data is from NIPA (National Income and Product Accounts Tables), CPI is obtained from CRSP, and 401(k) contribution data is from the Department of Labor. The first column is real personal consumption of non-durable goods and services measured in 1998 dollars, the second column is the average annual personal contribution into 401(k) accounts given participation, also measured in 1998 dollars. The last column indicates the ratio between contribution and consumption.

Year	Consumption	Contribution	Contribution/Consumption
1989	16,439	3,455	0.21
1990	16,484	3,070	0.19
1991	16,553	3,202	0.19
1992	16,824	3,317	0.20
1993	17,038	3,368	0.20
1994	17,284	3,296	0.19
1995	17,555	3,326	0.19
1996	17,721	3,484	0.20
1997	18,205	3,471	0.19
1998	18,709	3,628	0.19

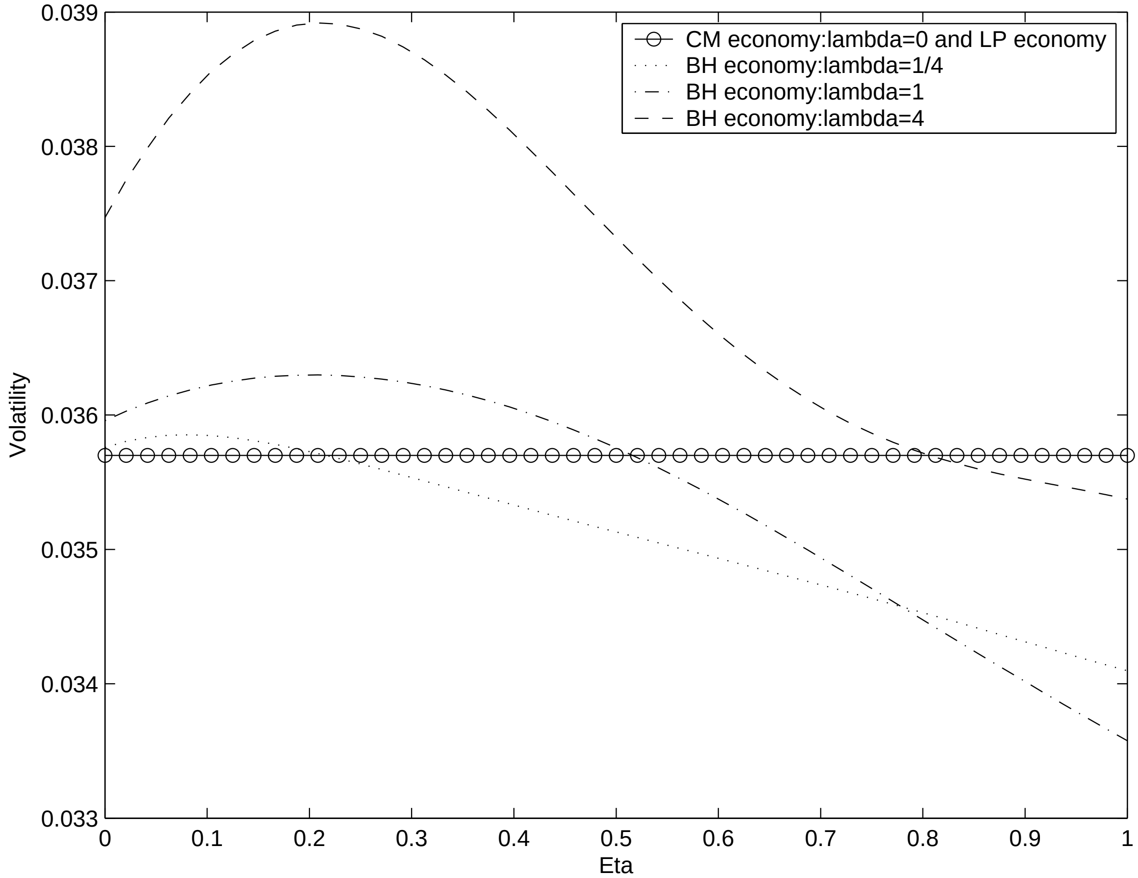

Figure 1: **Stock return volatility in the buy and hold economy vs that in the complete market/limited participation economy.**
The x-axis corresponds to η, the fraction of the stock held by the buy and hold investor. The circled line is the case for complete market economy and limited participation economy. The dashed line is for buy and hold economy with $\lambda = 4$, where λ is the ratio between the consumption of buy and hold investor and the consumption of the dynamic asset allocator. The dash-dotted line is for buy and hold economy with $\lambda = 1$. The dotted line is for buy and hold economy with $\lambda = 1/4$. Parameter values of the economy: agent's relative risk aversion γ_1=γ_2=1, impatience parameter ρ=0.001, consumption growth volatility $\bar{\sigma}$=0.0357, expected consumption growth rate $\bar{\mu}$=0.0183, time horizon T=50, the ratio between contribution and aggregate consumption $\bar{x}$=0.12, current aggregate consumption δ=1.

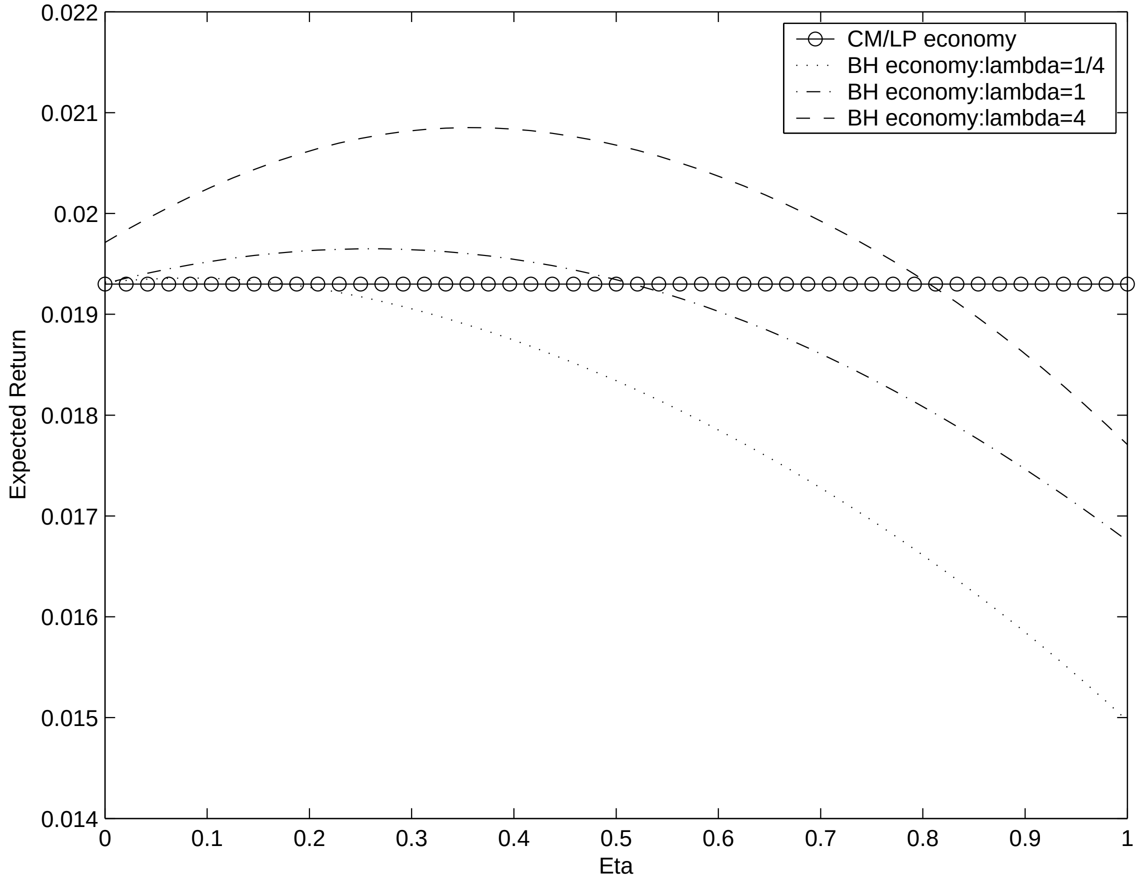

Figure 2: **Expected return in the buy and hold economy vs that in the complete market/limited participation economy.**
The x-axis corresponds to η, the fraction of the stock held by the buy and hold investor. The circled line is the case for complete market economy and limited participation economy. The dashed line is for buy and hold economy with $\lambda = 4$, where λ is the ratio between the consumption of buy and hold investor and the consumption of the dynamic asset allocator. The dash-dotted line is for buy and hold economy with $\lambda = 1$. The dotted line is for buy and hold economy with $\lambda = 1/4$. Parameter values of the economy: agent's relative risk aversion γ_1=γ_2=1, impatience parameter ρ=0.001, consumption growth volatility $\bar{\sigma}$=0.0357, expected consumption growth rate $\bar{\mu}$=0.0183, time horizon T=50, the ratio between contribution and aggregate consumption $\bar{x}$=0.12, current aggregate consumption δ=1.

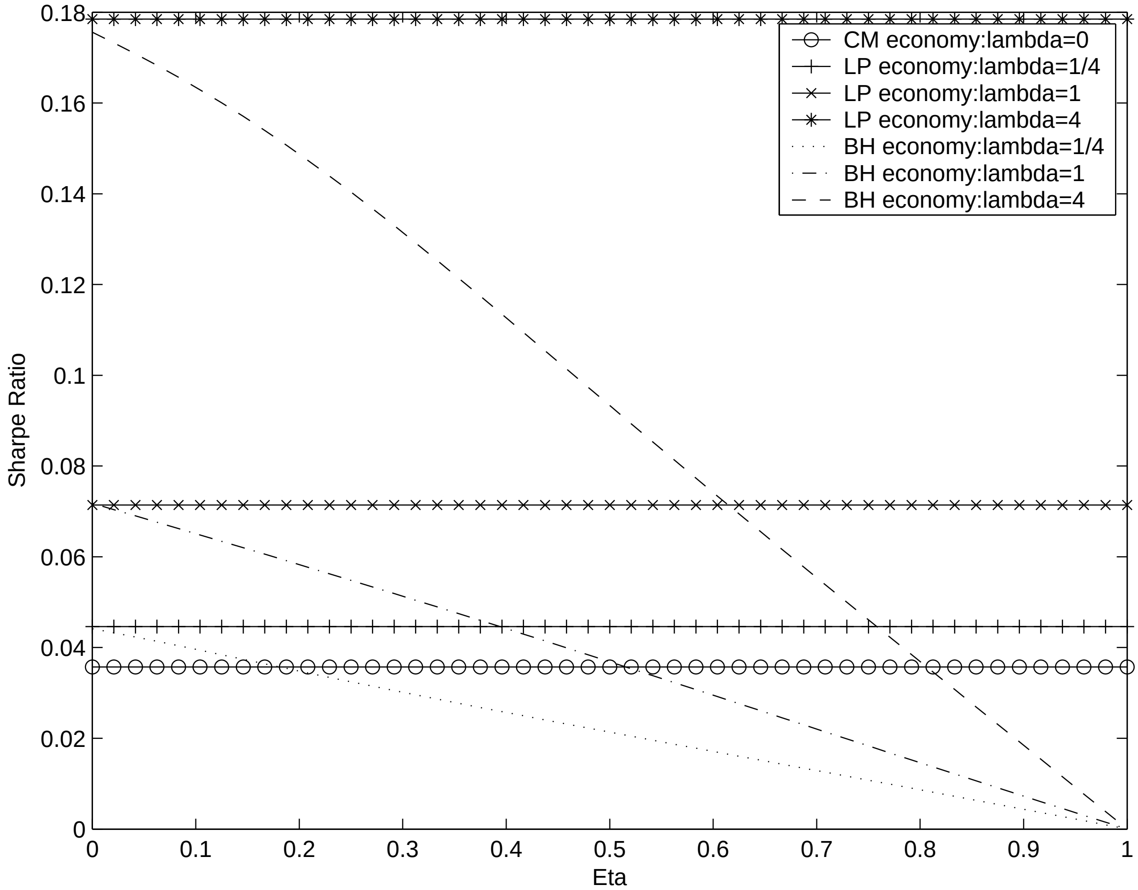

Figure 3: **Sharpe ratio in the buy and hold economy vs that in the complete market/limited participation economy.**

The x-axis corresponds to η, the fraction of the stock held by the buy and hold investor. The circled line is the case for complete market economy. The line with '$*$' is for limited participation economy with $\lambda = 4$, where λ is the ratio between the consumption of the non-stockholder and the consumption of the stockholder. The line with '$\times$' is for limited participation economy with $\lambda = 1$. The line with '$|$' is for limited participation economy with $\lambda = 1/4$. The dashed line is for buy and hold economy with $\lambda = 4$, where λ is the ratio between the consumption of the buy and hold investor and the consumption of the dynamic asset allocator. The dash-dotted line is for buy and hold economy with $\lambda = 1$. The dotted line is for buy and hold economy with $\lambda = 1/4$. Parameter values of the economy: agent's relative risk aversion γ_1=γ_2=1, impatience parameter ρ=0.001, consumption growth volatility $\bar{\sigma}$=0.0357, expected consumption growth rate $\bar{\mu}$=0.0183, time horizon T=50, the ratio between contribution and aggregate consumption $\bar{x}$=0.12, current aggregate consumption δ=1.

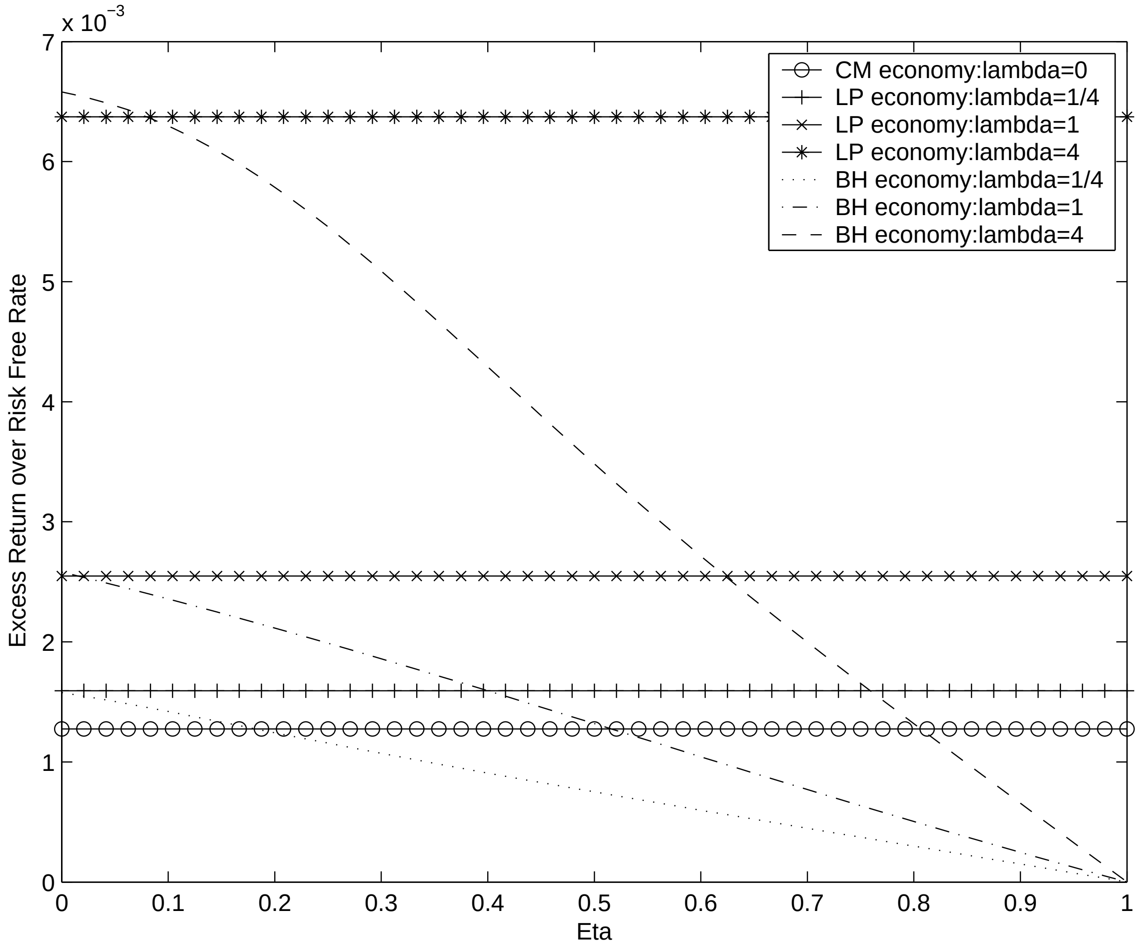

Figure 4: **Equity premium in the buy and hold economy vs that in the complete market/limited participation economy.**
The x-axis corresponds to η, the fraction of the stock held by the buy and hold investor. The circled line is for complete market economy. The line with '$*$' is for limited participation economy with $\lambda = 4$, where λ is the ratio between the consumption of the non-stockholder and the consumption of the stockholder. The line with '$\times$' is for limited participation economy with $\lambda = 1$. The line with '$|$' is for limited participation economy with $\lambda = 1/4$. The dashed line is for buy and hold economy with $\lambda = 4$, where λ is the ratio between the consumption of the buy and hold investor and the consumption of the dynamic asset allocator. The dash-dotted line is for buy and hold economy with $\lambda = 1$. The dotted line is for buy and hold economy with $\lambda = 1/4$. Parameter values of the economy: agent's relative risk aversion γ_1=γ_2=1, impatience parameter ρ=0.001, consumption growth volatility $\bar{\sigma}$=0.0357, expected consumption growth rate $\bar{\mu}$=0.0183, time horizon T=50, the ratio between contribution and aggregate consumption $\bar{x}$=0.12, current aggregate consumption δ=1.

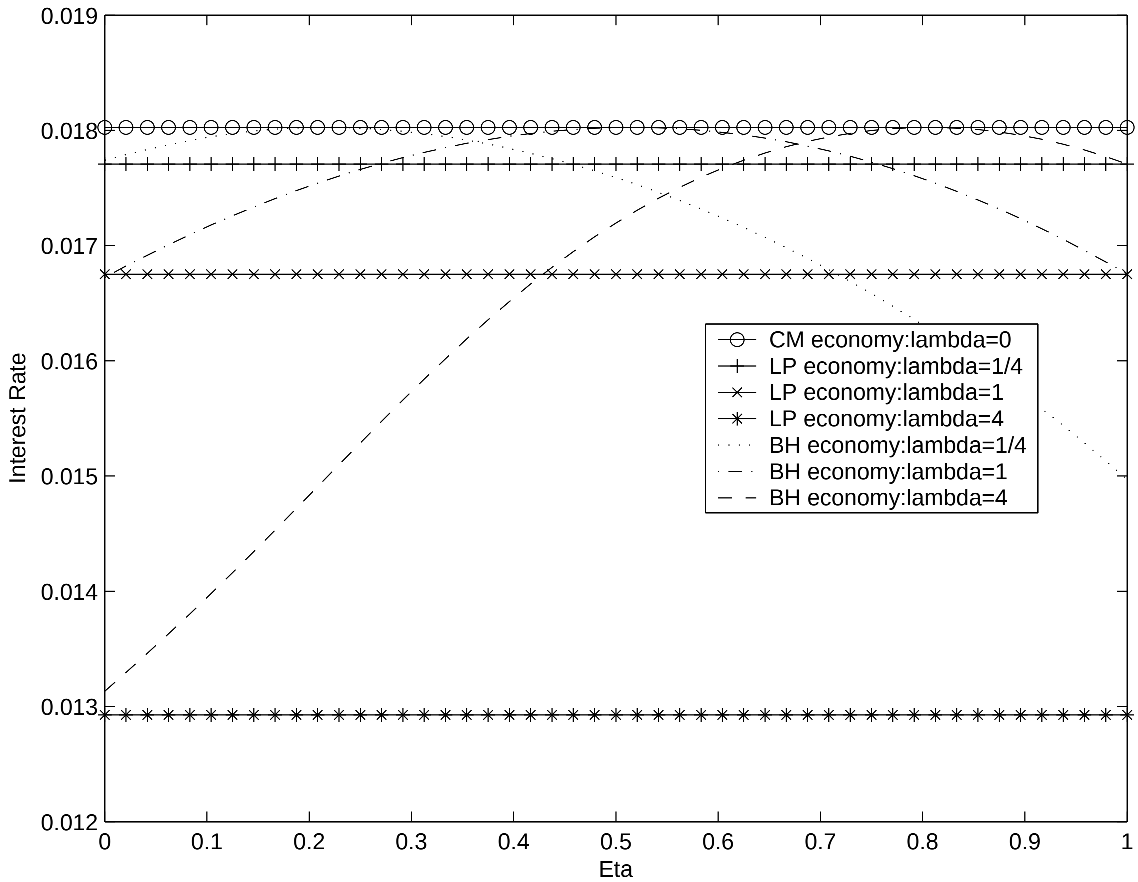

Figure 5: **Interest rate in the buy and hold economy vs that in the complete market/limited participation economy.**
The x-axis corresponds to η, the fraction of the stock held by the buy and hold investor. The circled line is the case for complete market economy. The line with '$*$' is for limited participation economy with $\lambda = 4$, where λ is the ratio between the consumption of the non-stockholder and the consumption of the stockholder. The line with '$\times$' is for limited participation economy with $\lambda = 1$. The line with '$|$' is for limited participation economy with $\lambda = 1/4$. The dashed line is for buy and hold economy with $\lambda = 4$, where λ is the ratio between the consumption of the buy and hold investor and the consumption of the dynamic asset allocator. The dash-dotted line is for buy and hold economy with $\lambda = 1$. The dotted line is for buy and hold economy with $\lambda = 1/4$. Parameter values of the economy: agent's relative risk aversion γ_1=γ_2=1, impatience parameter ρ=0.001, consumption growth volatility $\bar{\sigma}$=0.0357, expected consumption growth rate $\bar{\mu}$=0.0183, time horizon T=50, the ratio between contribution and aggregate consumption $\bar{x}$=0.12, current aggregate consumption δ=1.

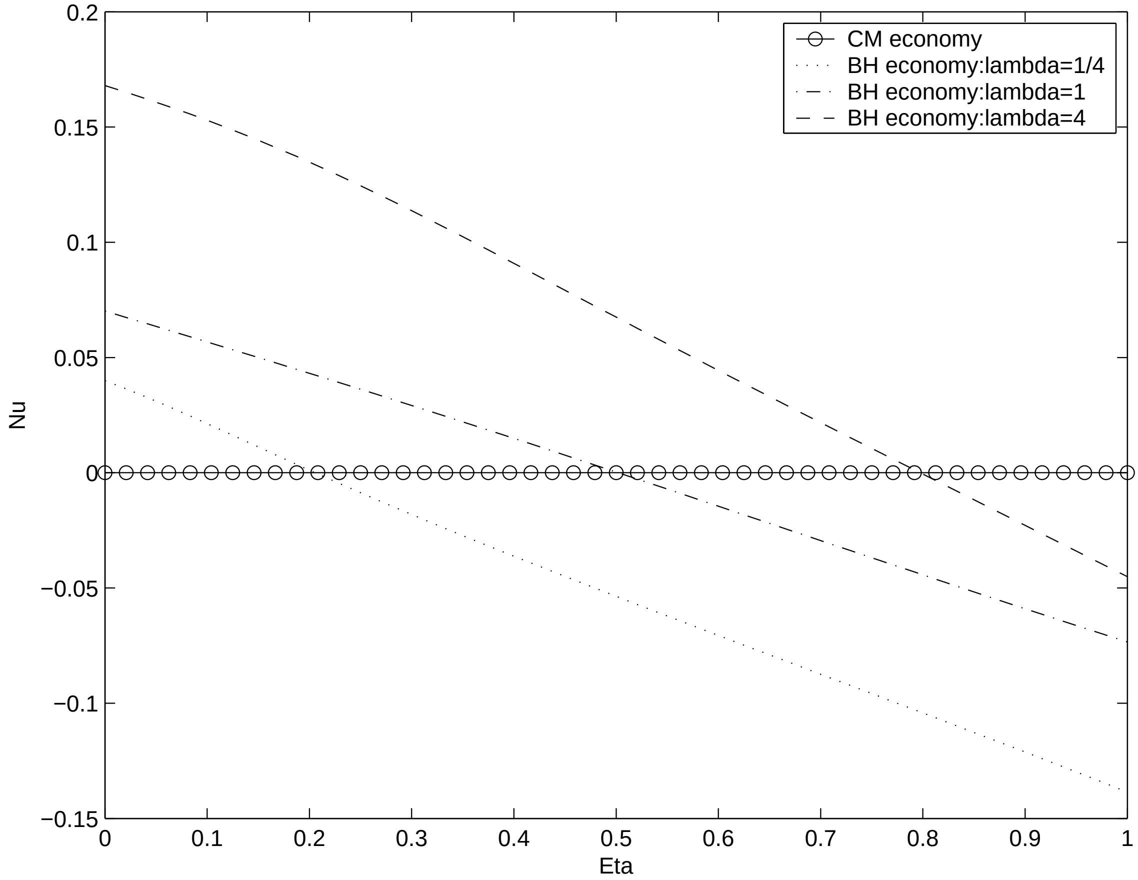

Figure 6: **Shadow process ν in the buy and hold economy vs that in the complete market economy.**
The x-axis corresponds to η, the fraction of the stock held by the buy and hold investor. The circled line is the case for complete market economy and limited participation economy. The dashed line is for buy and hold economy with $\lambda = 4$, where λ is the ratio between the consumption of buy and hold investor and the consumption of the dynamic asset allocator. The dash-dotted line is for buy and hold economy with $\lambda = 1$. The dotted line is for buy and hold economy with $\lambda = 1/4$. Parameter values of the economy: agent's relative risk aversion γ_1=γ_2=1, impatience parameter ρ=0.001, consumption growth volatility $\bar{\sigma}$=0.0357, expected consumption growth rate $\bar{\mu}$=0.0183, time horizon T=50, the ratio between contribution and aggregate consumption $\bar{x}$=0.12, current aggregate consumption δ=1.

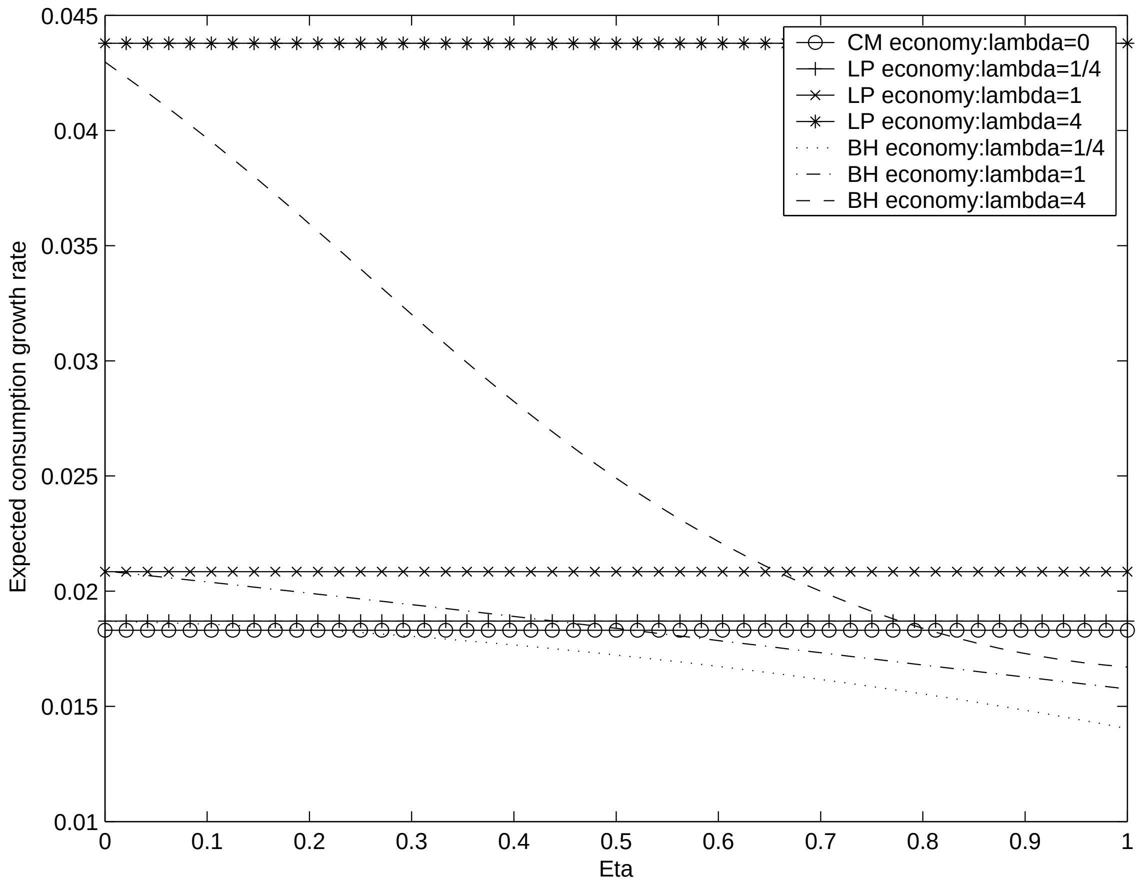

Figure 7: **Unrestricted agent's expected consumption growth rate in the buy and hold economy vs that in the complete market/limited participation economy.** The x-axis corresponds to η, the fraction of the stock held by the buy and hold investor. The circled line is the case for complete market economy. The line with '$*$' is for limited participation economy with $\lambda = 4$, where λ is the ratio between the consumption of the non-stockholder and the consumption of the stockholder. The line with '$\times$' is for limited participation economy with $\lambda = 1$. The line with '$|$' is for limited participation economy with $\lambda = 1/4$. The dashed line is for buy and hold economy with $\lambda = 4$, where λ is the ratio between the consumption of buy and hold investor and the consumption of the dynamic asset allocator. The dash-dotted line is for buy and hold economy with $\lambda = 1$. The dotted line is for buy and hold economy with $\lambda = 1/4$. Parameter values of the economy: agent's relative risk aversion γ_1=γ_2=1, impatience parameter ρ=0.001, consumption growth volatility $\bar{\sigma}$=0.0357, expected consumption growth rate $\bar{\mu}$=0.0183, time horizon T=50, the ratio between contribution and aggregate consumption $\bar{x}$=0.12, current aggregate consumption δ=1.

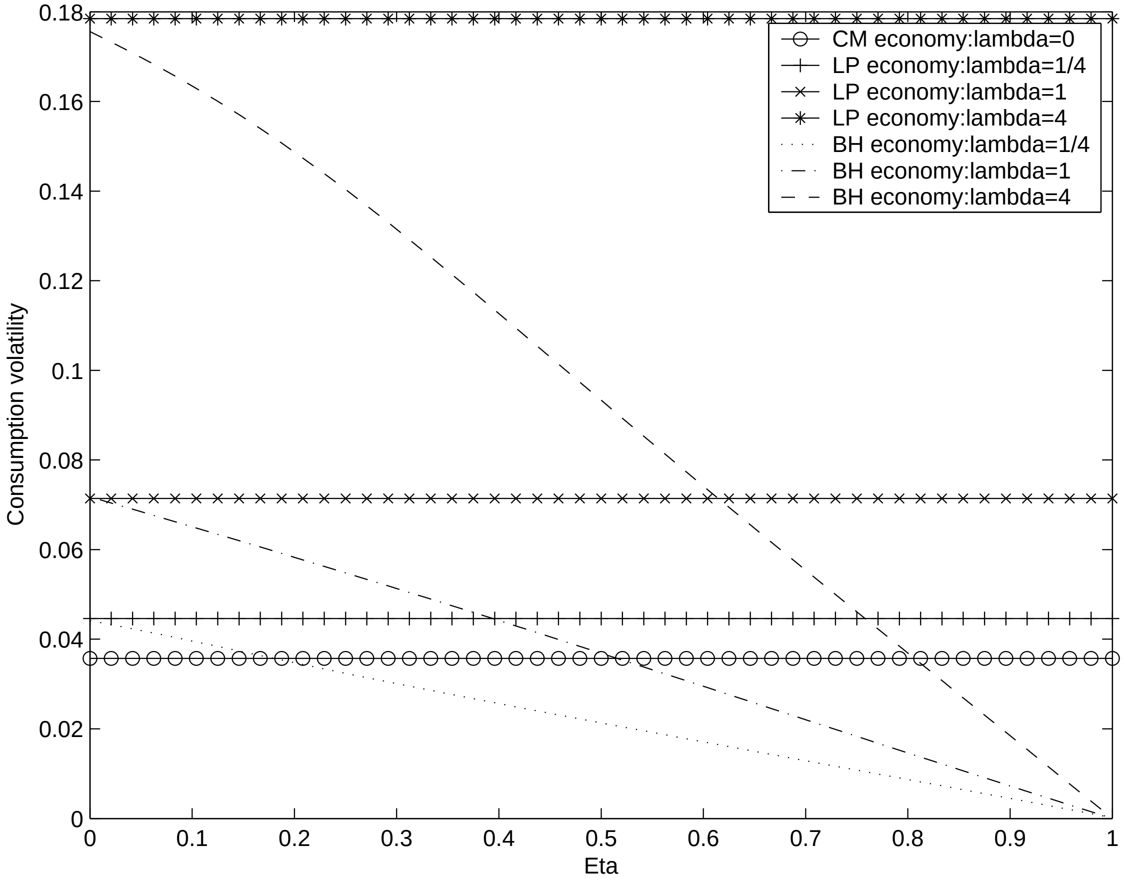

Figure 8: **Unrestricted agent's consumption volatility in the buy and hold economy vs that in the complete market/limited participation economy.**
The x-axis corresponds to η, the fraction of the stock held by the buy and hold investor. The circled line is the case for complete market economy. The line with '$*$' is for limited participation economy with $\lambda = 4$, where λ is the ratio between the consumption of the non-stockholder and the consumption of the stockholder. The line with '$\times$' is for limited participation economy with $\lambda = 1$. The line with '$|$' is for limited participation economy with $\lambda = 1/4$. The dashed line is for buy and hold economy with $\lambda = 4$, where λ is the ratio between the consumption of buy and hold investor and the consumption of the dynamic asset allocator. The dash-dotted line is for buy and hold economy with $\lambda = 1$. The dotted line is for buy and hold economy with $\lambda = 1/4$. Parameter values of the economy: agent's relative risk aversion γ_1=γ_2=1, impatience parameter ρ=0.001, consumption growth volatility $\bar{\sigma}$=0.0357, expected consumption growth rate $\bar{\mu}$=0.0183, time horizon T=50, the ratio between contribution and aggregate consumption $\bar{x}$=0.12, current aggregate consumption δ=1.

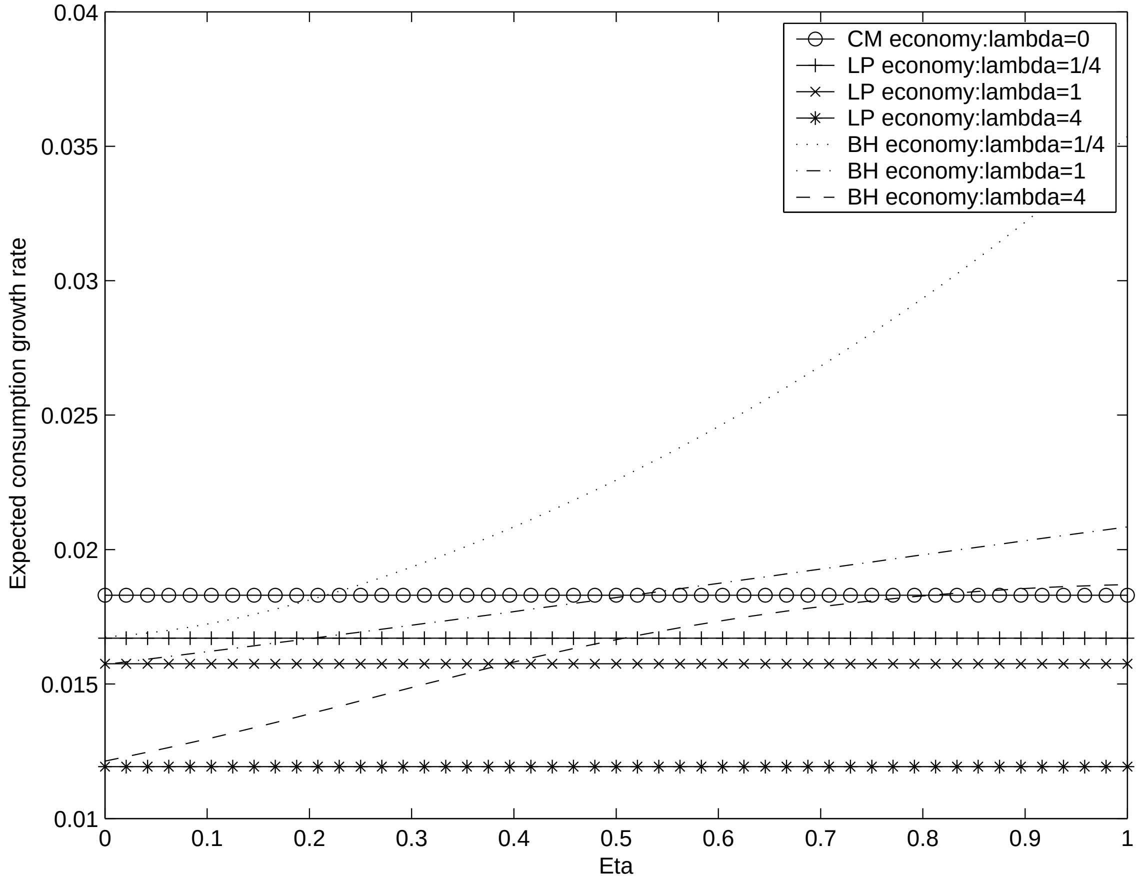

Figure 9: **Restricted agent's expected consumption growth rate in the buy and hold economy vs that in the complete market/limited participation economy.** The x-axis corresponds to η, the fraction of the stock held by the buy and hold investor. The circled line is the case for complete market economy. The line with '$*$' is for limited participation economy with $\lambda = 4$, where λ is the ratio between the consumption of the non-stockholder and the consumption of the stockholder. The line with '$\times$' is for limited participation economy with $\lambda = 1$. The line with '$|$' is for limited participation economy with $\lambda = 1/4$. The dashed line is for buy and hold economy with $\lambda = 4$, where λ is the ratio between the consumption of buy and hold investor and the consumption of the dynamic asset allocator. The dash-dotted line is for buy and hold economy with $\lambda = 1$. The dotted line is for buy and hold economy with $\lambda = 1/4$. Parameter values of the economy: agent's relative risk aversion γ_1=γ_2=1, impatience parameter ρ=0.001, consumption growth volatility $\bar{\sigma}$=0.0357, expected consumption growth rate $\bar{\mu}$=0.0183, time horizon T=50, the ratio between contribution and aggregate consumption $\bar{x}$=0.12, current aggregate consumption δ=1.

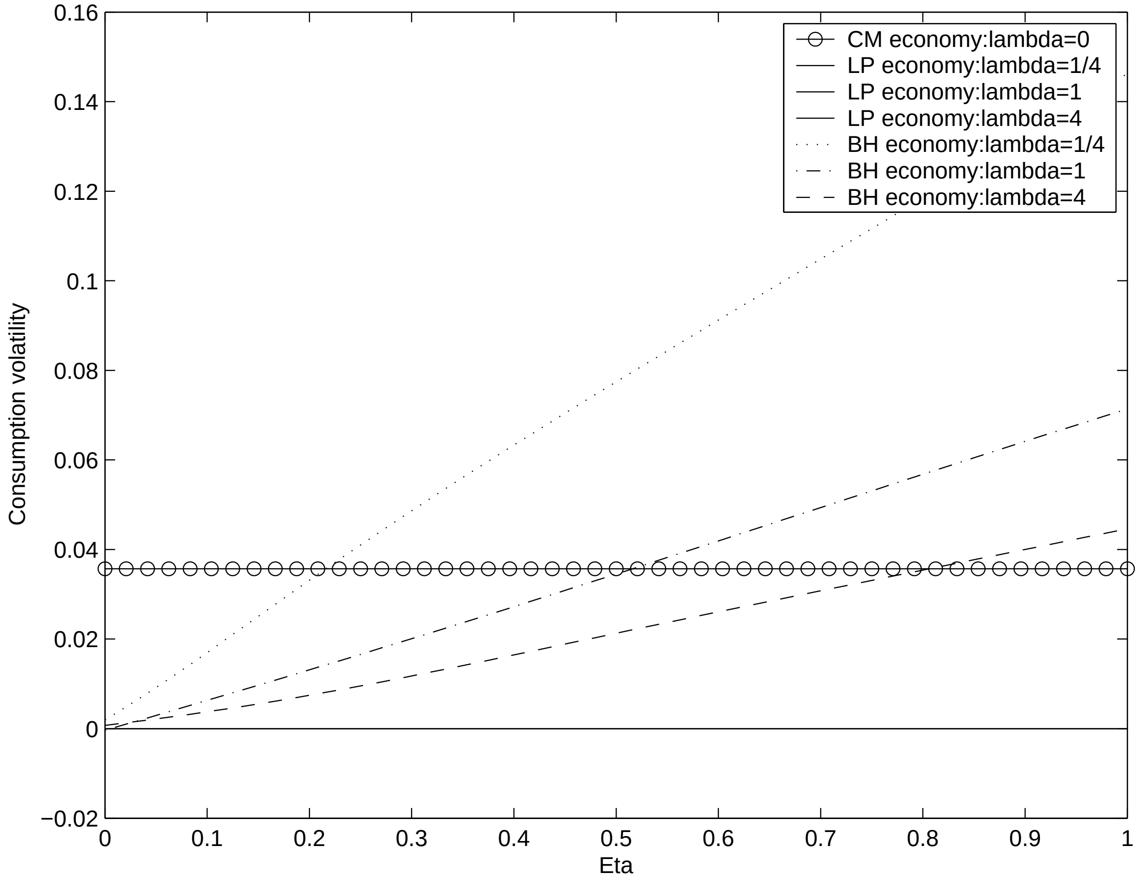

Figure 10: **Restricted agent's consumption volatility in the buy and hold economy vs that in the complete market/limited participation economy.**
The x-axis corresponds to η, the fraction of the stock held by the buy and hold investor. The circled line is the case for complete market economy. The solid line represents limited participation economy. The dashed line is for buy and hold economy with $\lambda = 4$, where λ is the ratio between the consumption of buy and hold investor and the consumption of the dynamic asset allocator. The dash-dotted line is for buy and hold economy with $\lambda = 1$. The dotted line is for buy and hold economy with $\lambda = 1/4$. Parameter values of the economy: agent's relative risk aversion γ_1=γ_2=1, impatience parameter ρ=0.001, consumption growth volatility $\bar{\sigma}$=0.0357, expected consumption growth rate $\bar{\mu}$=0.0183, time horizon T=50, the ratio between contribution and aggregate consumption $\bar{x}$=0.12, current aggregate consumption δ=1.

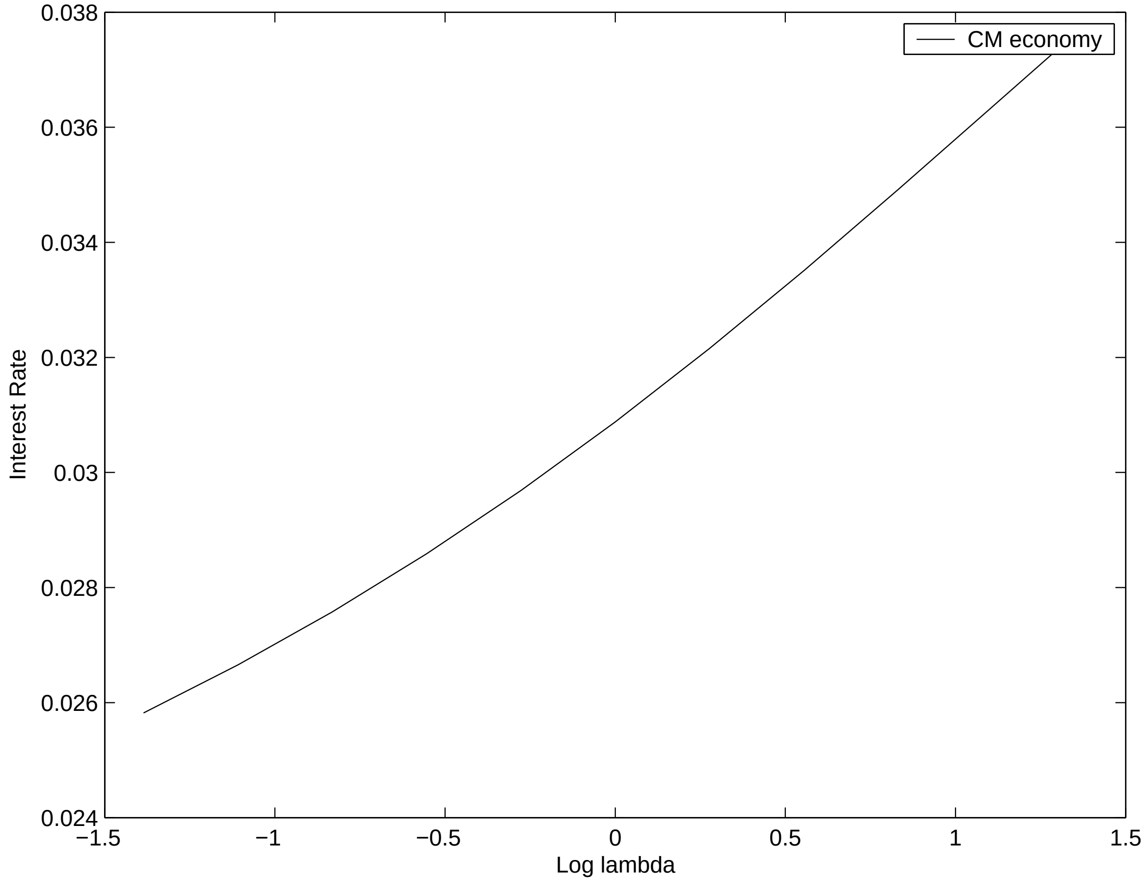

Figure 11: **Interest rate in the complete market economy.**
The x-axis corresponds to log λ. The corresponding range for λ is [1/4,4], which translates into [0.5, 0.85] for the consumption share claimed by agent 2. Parameter values of the economy: agent's relative risk aversion $\gamma_1 = 1$, $\gamma_2 = 3$, impatience parameter $\rho = 0.001$, consumption growth volatility $\bar{\sigma}$=0.0357, expected consumption growth rate $\bar{\mu}$=0.0183, time horizon T=50, the ratio between contribution and aggregate consumption $\bar{x}$=0.12, current aggregate consumption δ=1.

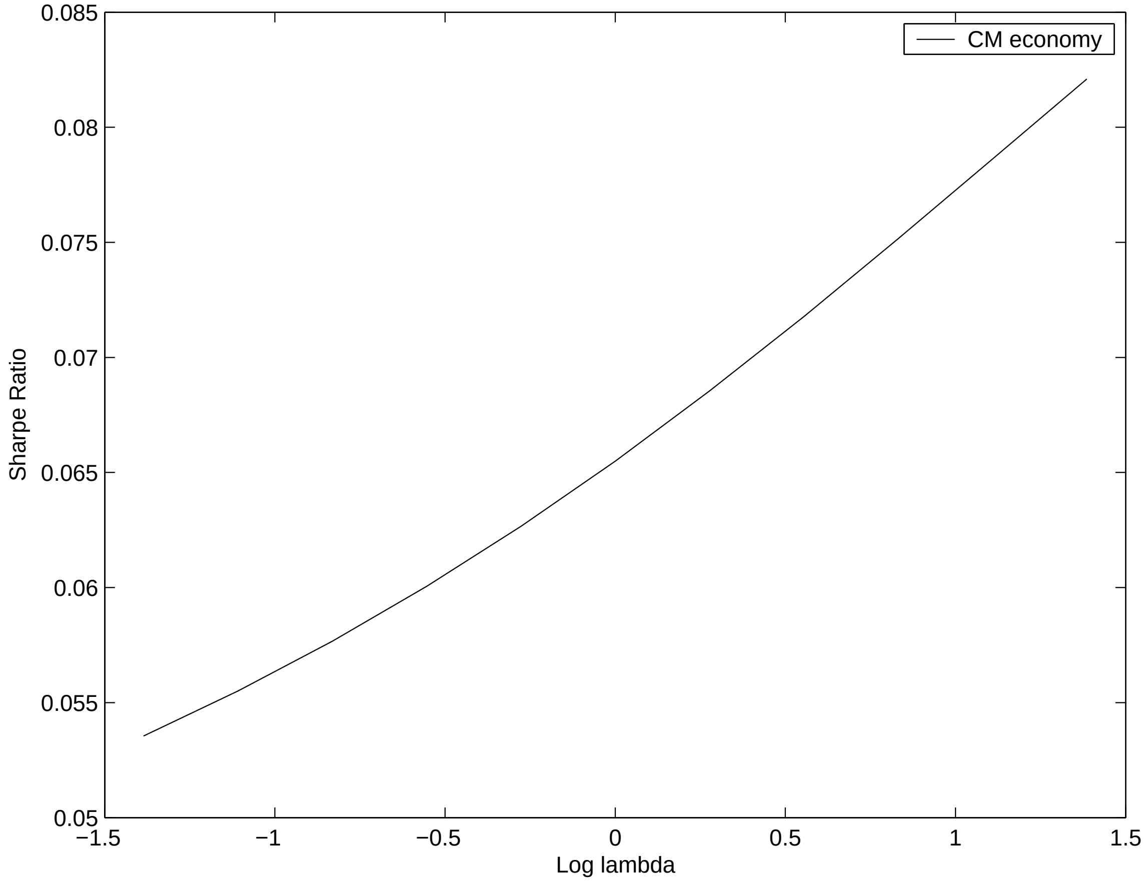

Figure 12: **Sharpe ratio in the complete market economy.**
The x-axis corresponds to log λ. The corresponding range for λ is [1/4,4], which translates into [0.5, 0.85] for the consumption share claimed by agent 2. Parameter values of the economy: agent's relative risk aversion $\gamma_1 = 1$, $\gamma_2 = 3$, impatience parameter $\rho = 0.001$, consumption growth volatility $\bar{\sigma}$=0.0357, expected consumption growth rate $\bar{\mu}$=0.0183, time horizon T=50, the ratio between contribution and aggregate consumption $\bar{x}$=0.12, current aggregate consumption δ=1.

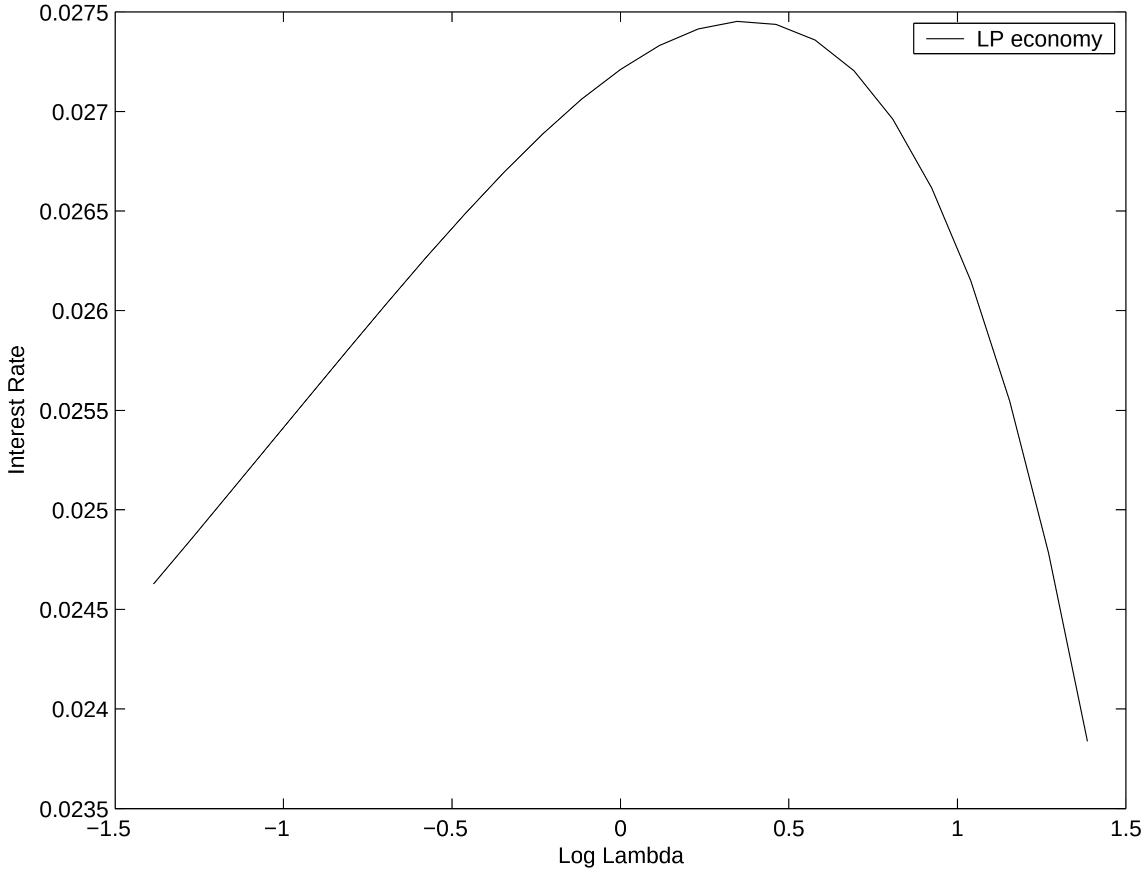

Figure 13: **Interest rate in the limited participation economy.**
The x-axis corresponds to log λ. The corresponding range for λ is [1/4,4], which translates into [0.5, 0.85] for the consumption share claimed by agent 2. Parameter values of the economy: agent's relative risk aversion $\gamma_1 = 1$, $\gamma_2 = 3$, impatience parameter $\rho = 0.001$, consumption growth volatility $\bar{\sigma}$=0.0357, expected consumption growth rate $\bar{\mu}$=0.0183, time horizon T=50, the ratio between contribution and aggregate consumption $\bar{x}$=0.12, current aggregate consumption δ=1.

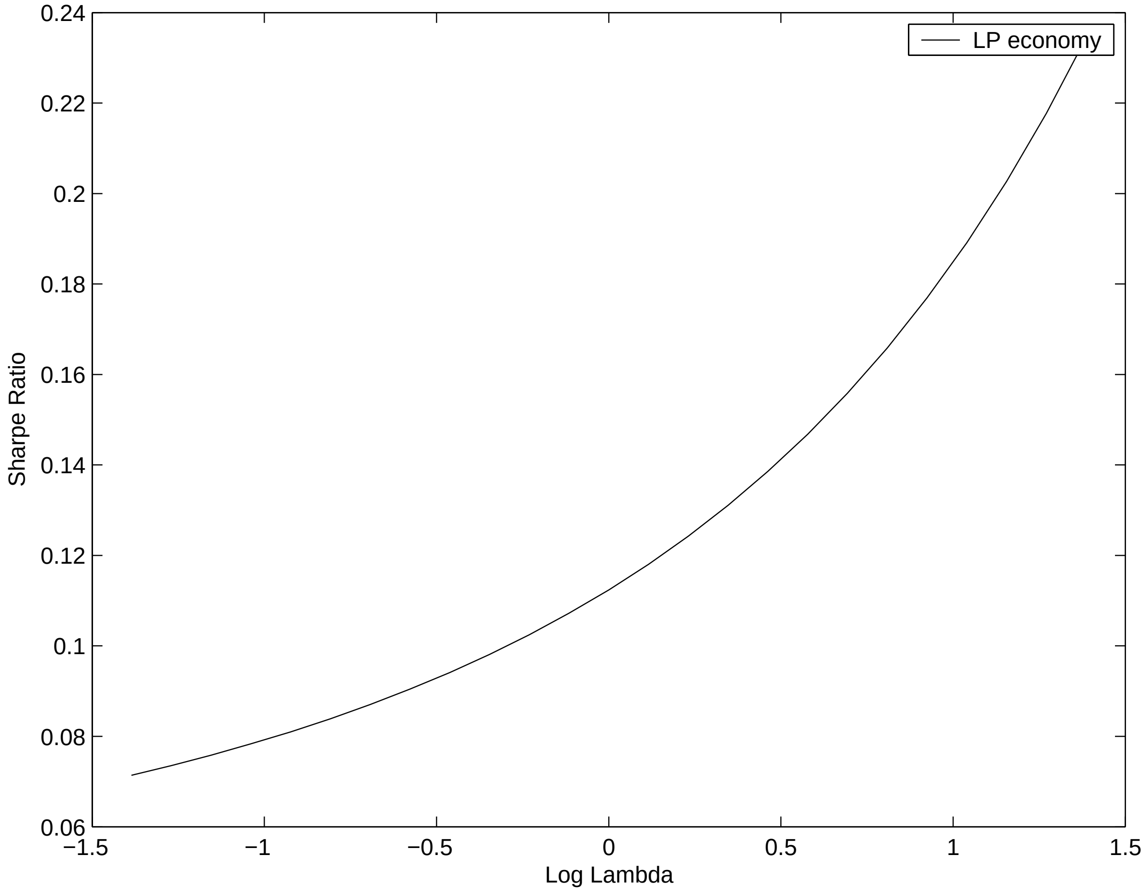

Figure 14: **Sharpe ratio in the limited participation economy.** The x-axis corresponds to log λ. The corresponding range for λ is [1/4,4], which translates into [0.5, 0.85] for the consumption share claimed by agent 2. Parameter values of the economy: agent's relative risk aversion $\gamma_1 = 1$, $\gamma_2 = 3$, impatience parameter $\rho = 0.001$, consumption growth volatility $\bar{\sigma}$=0.0357, expected consumption growth rate $\bar{\mu}$=0.0183, time horizon T=50, the ratio between contribution and aggregate consumption $\bar{x}$=0.12, current aggregate consumption δ=1.

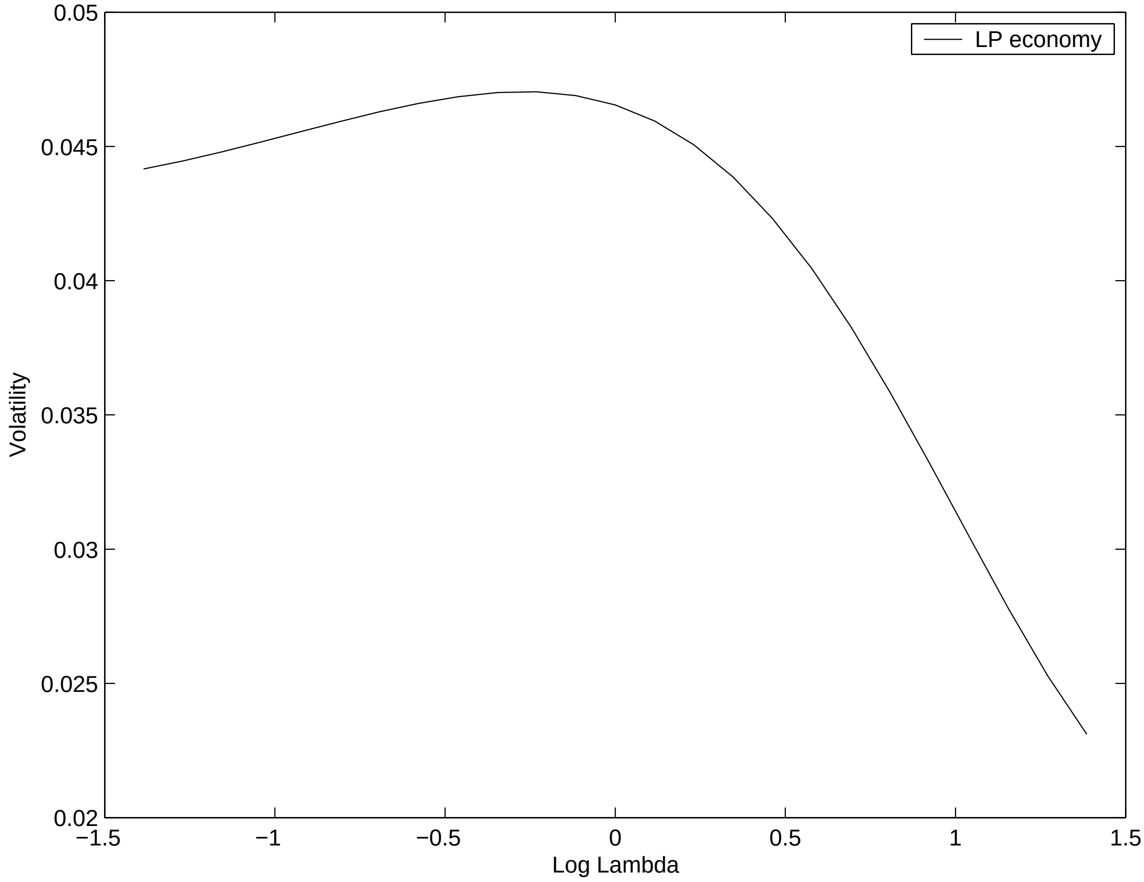

Figure 15: **Stock return volatility in the limited participation economy.** The x-axis corresponds to log λ. The corresponding range for λ is [1/4,4], which translates into [0.5, 0.85] for the consumption share claimed by agent 2. Parameter values of the economy: agent's relative risk aversion $\gamma_1 = 1$, $\gamma_2 = 3$, impatience parameter $\rho = 0.001$, consumption growth volatility $\bar{\sigma}$=0.0357, expected consumption growth rate $\bar{\mu}$=0.0183, time horizon T=50, the ratio between contribution and aggregate consumption $\bar{x}$=0.12, current aggregate consumption δ=1.

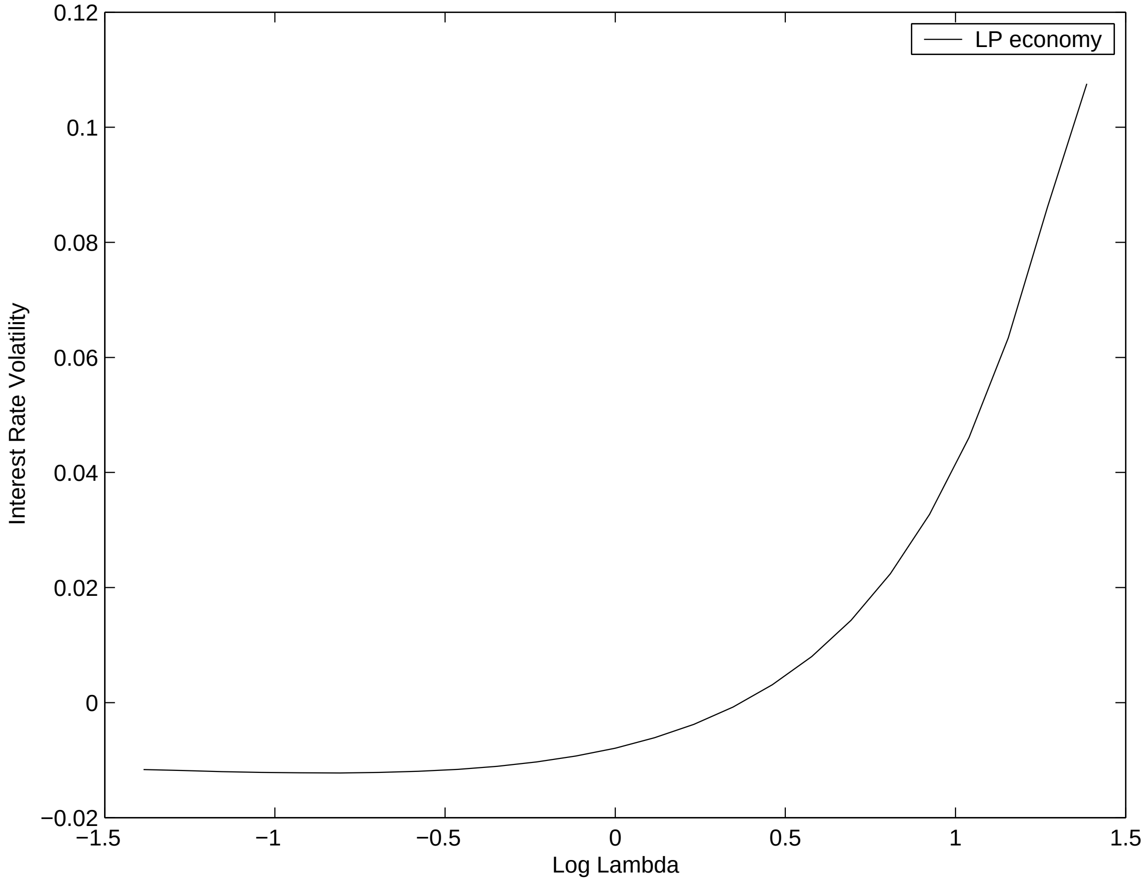

Figure 16: **Interest rate volatility in the limited participation economy.** The x-axis corresponds to log λ. The corresponding range for λ is [1/4,4], which translates into [0.5, 0.85] for the consumption share claimed by agent 2. Parameter values of the economy: agent's relative risk aversion $\gamma_1 = 1$, $\gamma_2 = 3$, impatience parameter $\rho = 0.001$, consumption growth volatility $\bar{\sigma}$=0.0357, expected consumption growth rate $\bar{\mu}$=0.0183, time horizon T=50, the ratio between contribution and aggregate consumption $\bar{x}$=0.12, current aggregate consumption δ=1.

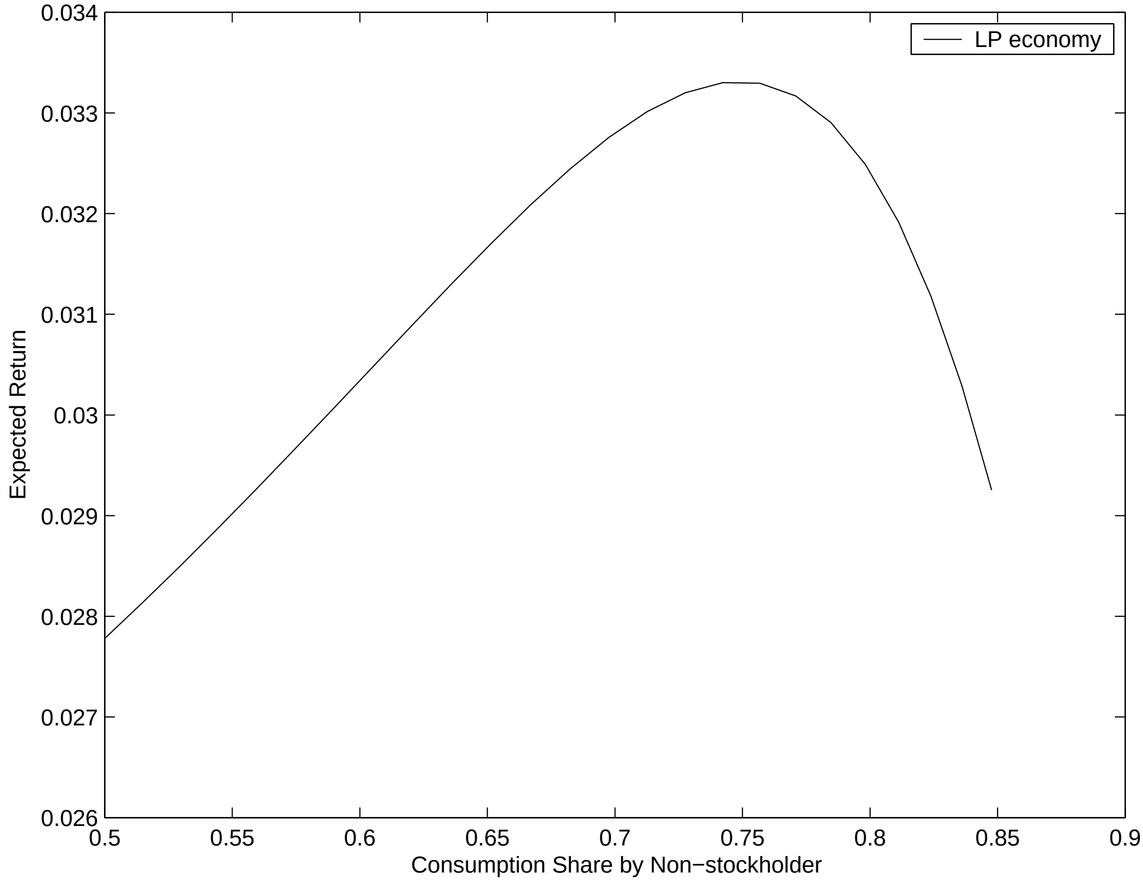

Figure 17: **Expected return in the limited participation economy.** The x-axis corresponds to log λ. The corresponding range for λ is [1/4,4], which translates into [0.5, 0.85] for the consumption share claimed by agent 2. Parameter values of the economy: agent's relative risk aversion $\gamma_1 = 1$, $\gamma_2 = 3$, impatience parameter $\rho = 0.001$, consumption growth volatility $\bar{\sigma}$=0.0357, expected consumption growth rate $\bar{\mu}$=0.0183, time horizon T=50, the ratio between contribution and aggregate consumption $\bar{x}$k v=0.12, current aggregate consumption δ=1.

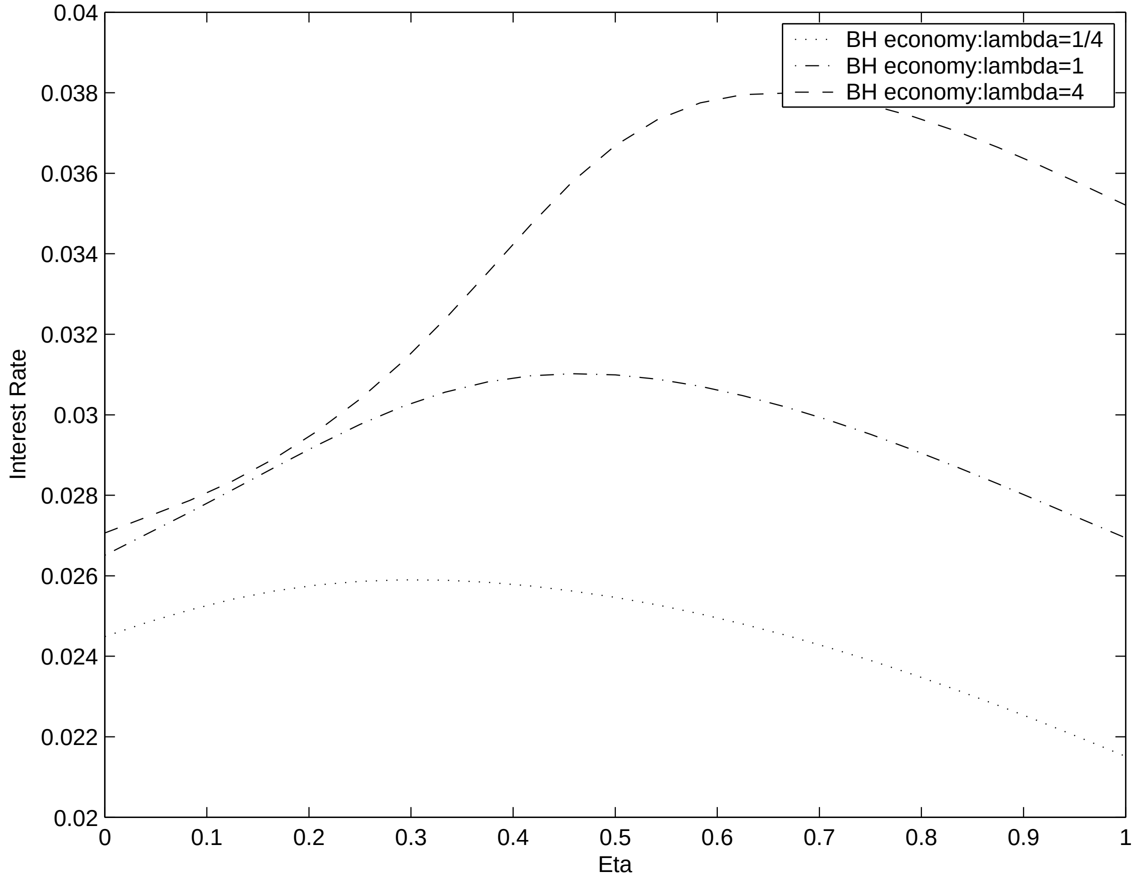

Figure 18: **Interest rate in the buy and hold economy.**
The x-axis corresponds to η, the fraction of the stock held by the buy and hold investor. The dashed line is for buy and hold economy with $\lambda = 4$ (i.e. consumption share claimed by agent 2 equals 85%). The dash-dotted line is for buy and hold economy with $\lambda = 1$ (i.e. consumption share claimed by agent 2 equals 68%). The dotted line is for buy and hold economy with $\lambda = 1/4$ (i.e. consumption share claimed by agent 2 equals 50%). Parameter values of the economy: agent's relative risk aversion $\gamma_1 = 1$, $\gamma_2 = 3$, impatience parameter ρ=0.001, consumption growth volatility $\bar{\sigma}$=0.0357, expected consumption growth rate $\bar{\mu}$=0.0183, time horizon T=50, the ratio between contribution and aggregate consumption $\bar{x}$=0.12, current aggregate consumption δ=1.

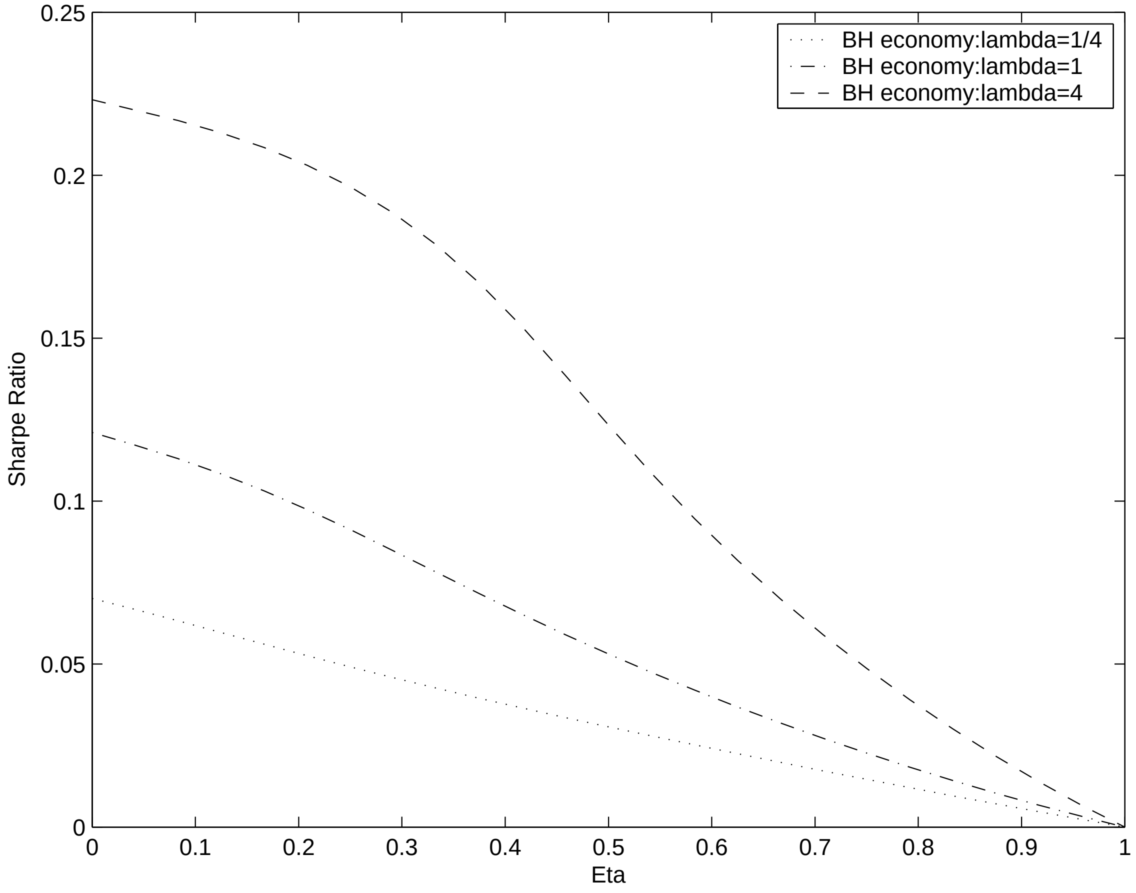

Figure 19: **Sharpe ratio in the buy and hold economy.**
Note that the stock here refers to unlevered equity. The x-axis corresponds to η, the fraction of the stock held by the buy and hold investor. The dashed line is for buy and hold economy with $\lambda = 4$ (i.e. consumption share claimed by agent 2 equals 85%). The dash-dotted line is for buy and hold economy with $\lambda = 1$ (i.e. consumption share claimed by agent 2 equals 68%). The dotted line is for buy and hold economy with $\lambda = 1/4$ (i.e. consumption share claimed by agent 2 equals 50%). Parameter values of the economy: agent's relative risk aversion $\gamma_1 = 1$, $\gamma_2 = 3$, impatience parameter ρ=0.001, consumption growth volatility $\bar{\sigma}$=0.0357, expected consumption growth rate $\bar{\mu}$=0.0183, time horizon T=50, the ratio between contribution and aggregate consumption $\bar{x}$=0.12, current aggregate consumption δ=1.

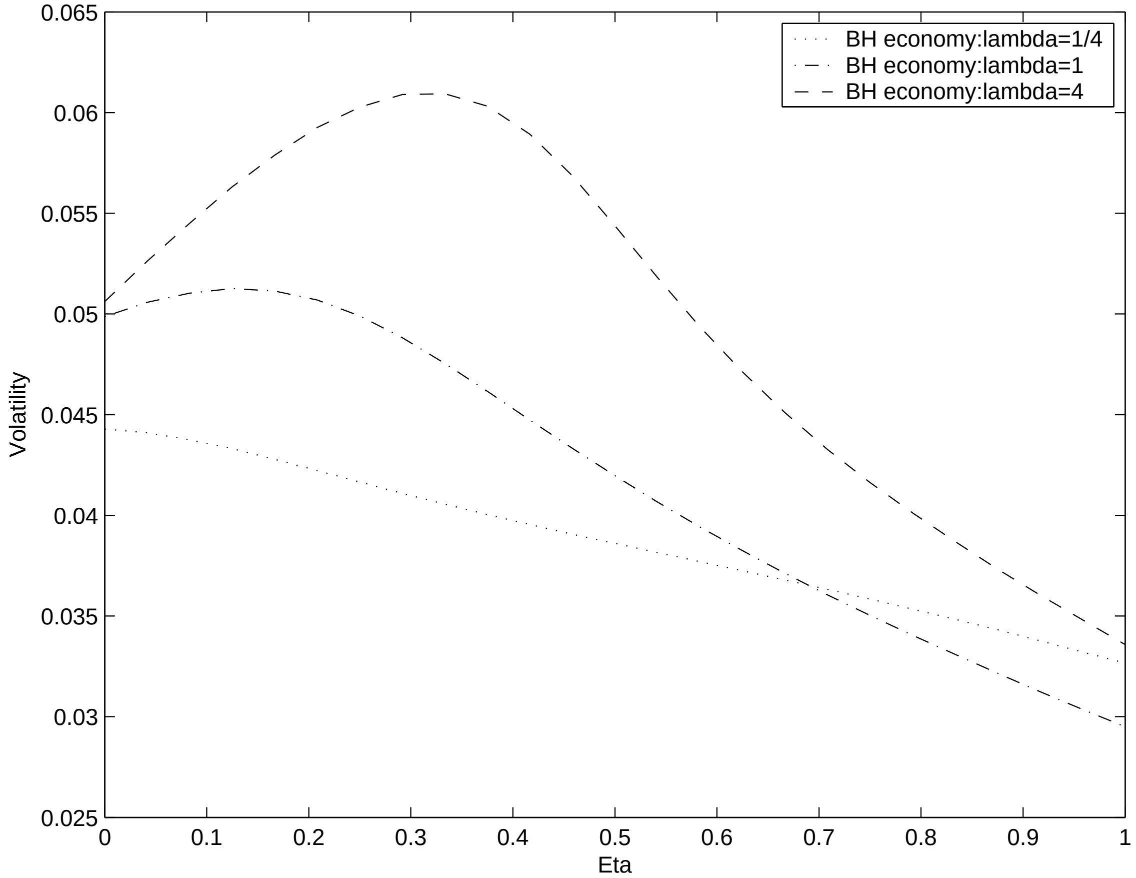

Figure 20: **Stock return volatility in the buy and hold economy.** Note that the stock here refers to unlevered equity. The x-axis corresponds to η, the fraction of the stock held by the buy and hold investor. The dashed line is for buy and hold economy with $\lambda = 4$ (i.e. consumption share claimed by agent 2 equals 85%). The dash-dotted line is for buy and hold economy with $\lambda = 1$ (i.e. consumption share claimed by agent 2 equals 68%). The dotted line is for buy and hold economy with $\lambda = 1/4$ (i.e. consumption share claimed by agent 2 equals 50%). Parameter values of the economy: agent's relative risk aversion $\gamma_1 = 1$, $\gamma_2 = 3$, impatience parameter ρ=0.001, consumption growth volatility $\bar{\sigma}$=0.0357, expected consumption growth rate $\bar{\mu}$=0.0183, time horizon T=50, the ratio between contribution and aggregate consumption $\bar{x}$=0.12, current aggregate consumption δ=1.

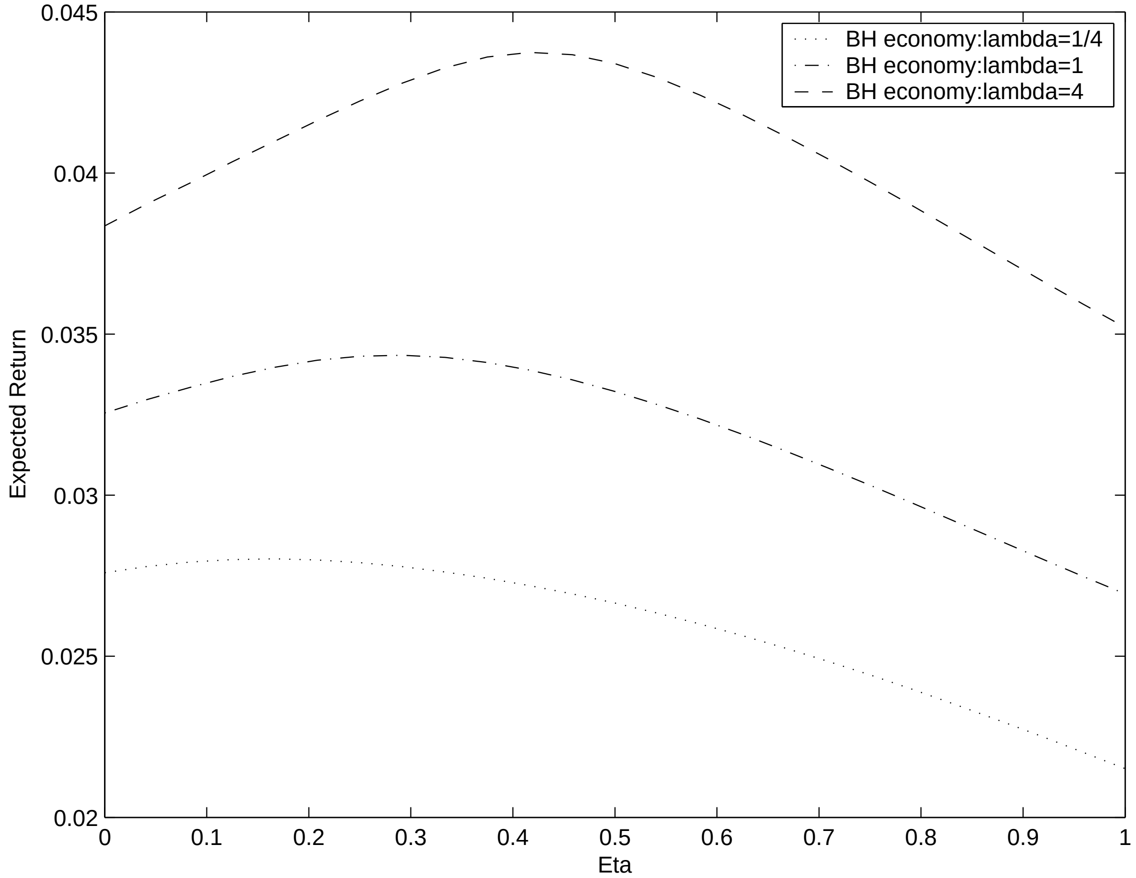

Figure 21: **Expected return in the buy and hold economy.** Note that the stock here refers to unlevered equity. The x-axis corresponds to η, the fraction of the stock held by the buy and hold investor. The dashed line is for buy and hold economy with $\lambda = 4$ (i.e. consumption share claimed by agent 2 equals 85%). The dash-dotted line is for buy and hold economy with $\lambda = 1$ (i.e. consumption share claimed by agent 2 equals 68%). The dotted line is for buy and hold economy with $\lambda = 1/4$ (i.e. consumption share claimed by agent 2 equals 50%). Parameter values of the economy: agent's relative risk aversion $\gamma_1 = 1$, $\gamma_2 = 3$, impatience parameter ρ=0.001, consumption growth volatility $\bar{\sigma}$=0.0357, expected consumption growth rate $\bar{\mu}$=0.0183, time horizon T=50, the ratio between contribution and aggregate consumption $\bar{x}$=0.12, current aggregate consumption δ=1.

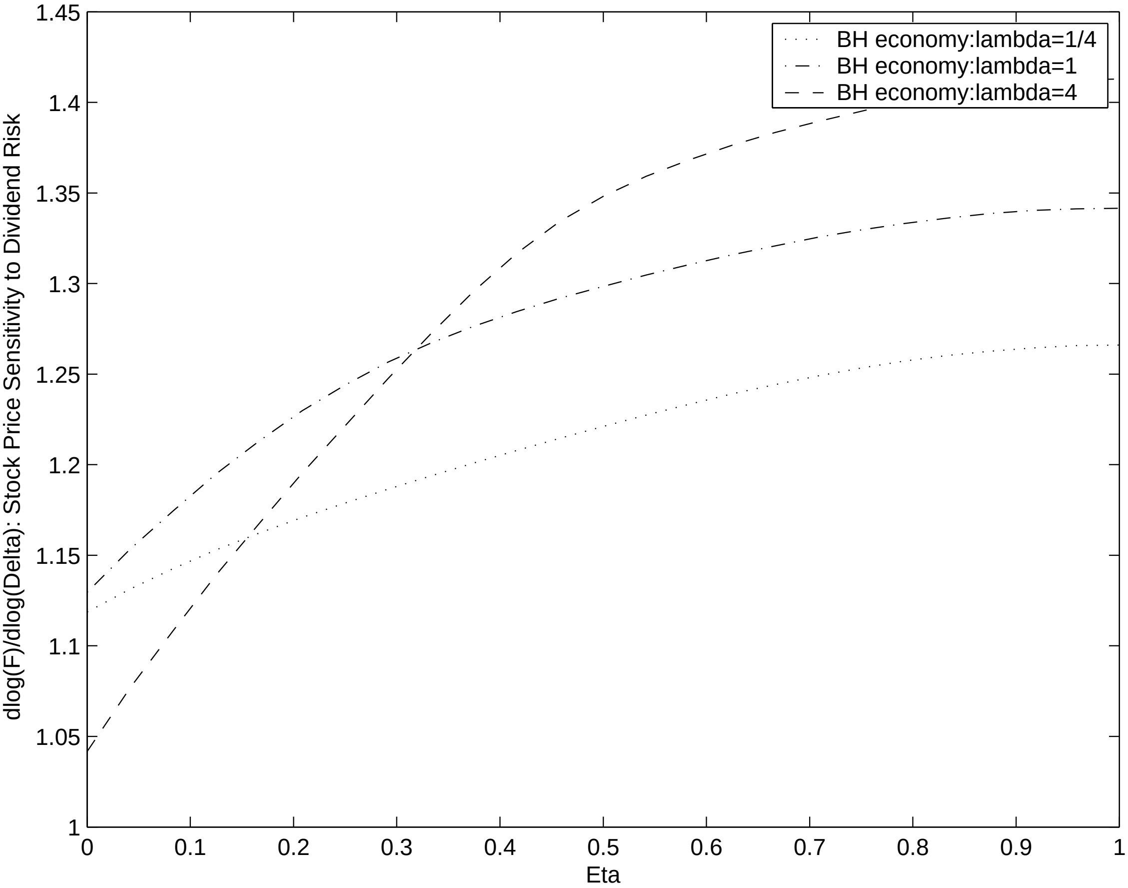

Figure 22: **Stock price sensitivity to dividend risk in the buy and hold economy.** The x-axis corresponds to η, the fraction of the stock held by the buy and hold investor. The dashed line is for buy and hold economy with $\lambda = 4$ (i.e. consumption share claimed by agent 2 equals 85%). The dash-dotted line is for buy and hold economy with $\lambda = 1$ (i.e. consumption share claimed by agent 2 equals 68%). The dotted line is for buy and hold economy with $\lambda = 1/4$ (i.e. consumption share claimed by agent 2 equals 50%). Parameter values of the economy: agent's relative risk aversion $\gamma_1 = 1$, $\gamma_2 = 3$, impatience parameter ρ=0.001, consumption growth volatility $\bar{\sigma}$=0.0357, expected consumption growth rate $\bar{\mu}$=0.0183, time horizon T=50, the ratio between contribution and aggregate consumption $\bar{x}$=0.12, current aggregate consumption δ=1.

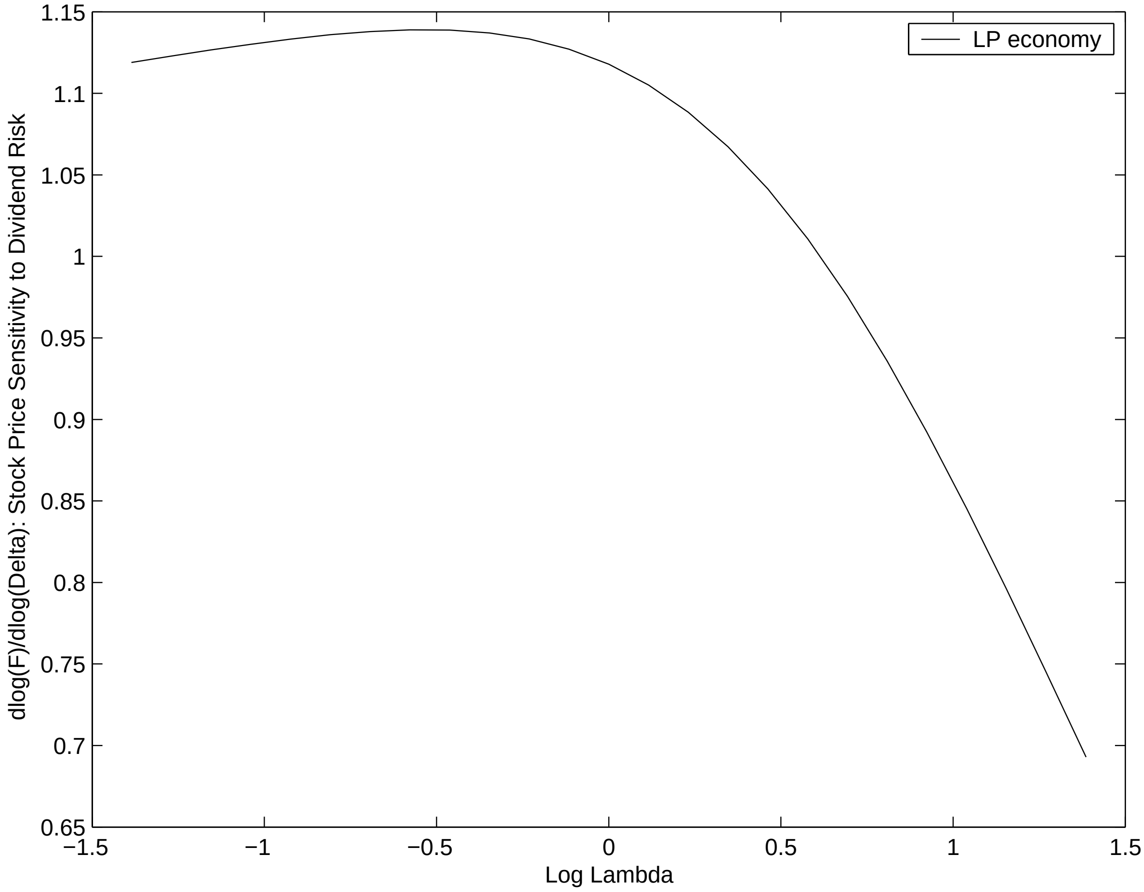

Figure 23: **Stock price sensitivity to dividend risk in the limited participation economy.**

The x-axis corresponds to log λ. The corresponding range for λ is [1/4,4], which translates into [0.5, 0.85] for the consumption share claimed by agent 2. Parameter values of the economy: agent's relative risk aversion $\gamma_1 = 1$, $\gamma_2 = 3$, impatience parameter $\rho = 0.001$, consumption growth volatility $\bar{\sigma}$=0.0357, expected consumption growth rate $\bar{\mu}$=0.0183, time horizon T=50, the ratio between contribution and aggregate consumption $\bar{x}$=0.12, current aggregate consumption δ=1.

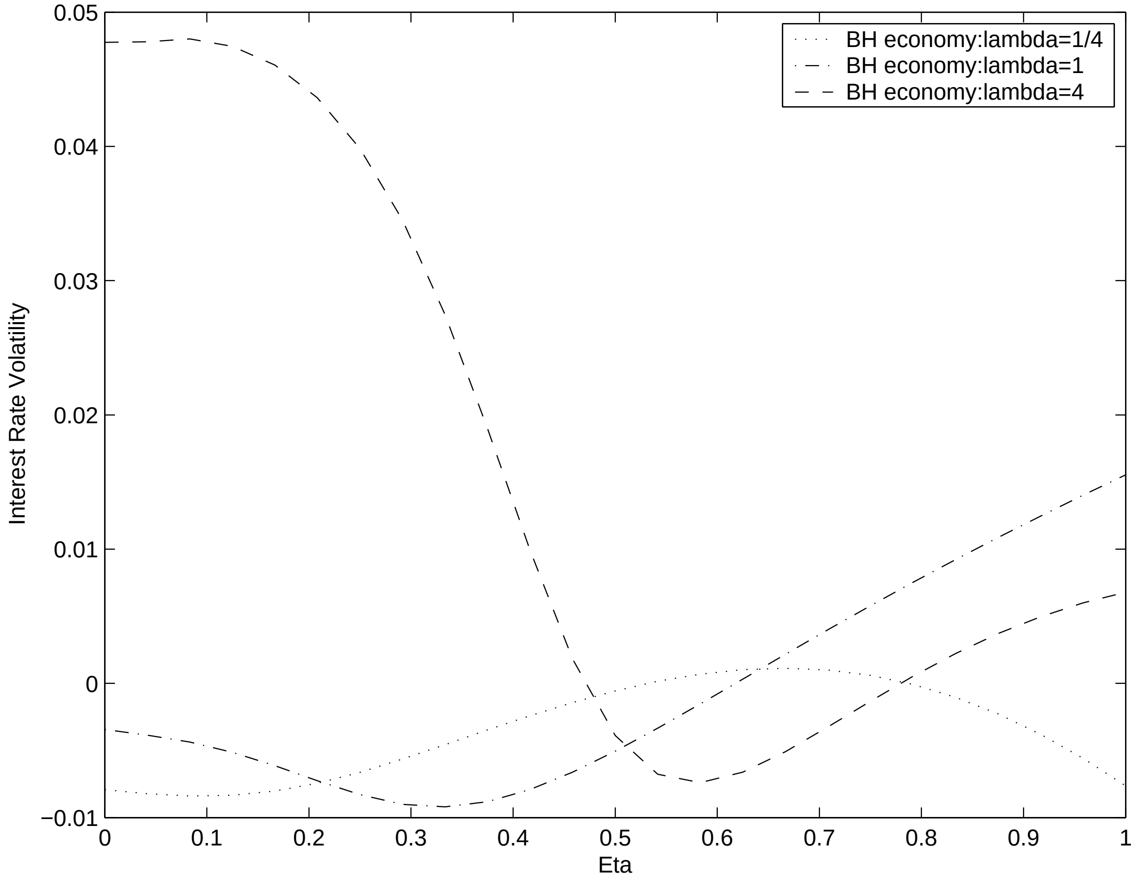

Figure 24: **Interest rate volatility in the buy and hold economy.** (Note that a negative value simply means the interest rate is counter-cyclical in that region.) The x-axis corresponds to η, the fraction of the stock held by the buy and hold investor. The dashed line is for buy and hold economy with $\lambda = 4$ (i.e. consumption share claimed by agent 2 equals 85%). The dash-dotted line is for buy and hold economy with $\lambda = 1$ (i.e. consumption share claimed by agent 2 equals 68%). The dotted line is for buy and hold economy with $\lambda = 1/4$ (i.e. consumption share claimed by agent 2 equals 50%). Parameter values of the economy: agent's relative risk aversion $\gamma_1 = 1$, $\gamma_2 = 3$, impatience parameter ρ=0.001, consumption growth volatility $\bar{\sigma}$=0.0357, expected consumption growth rate $\bar{\mu}$=0.0183, time horizon T=50, the ratio between contribution and aggregate consumption $\bar{x}$=0.12, current aggregate consumption δ=1.

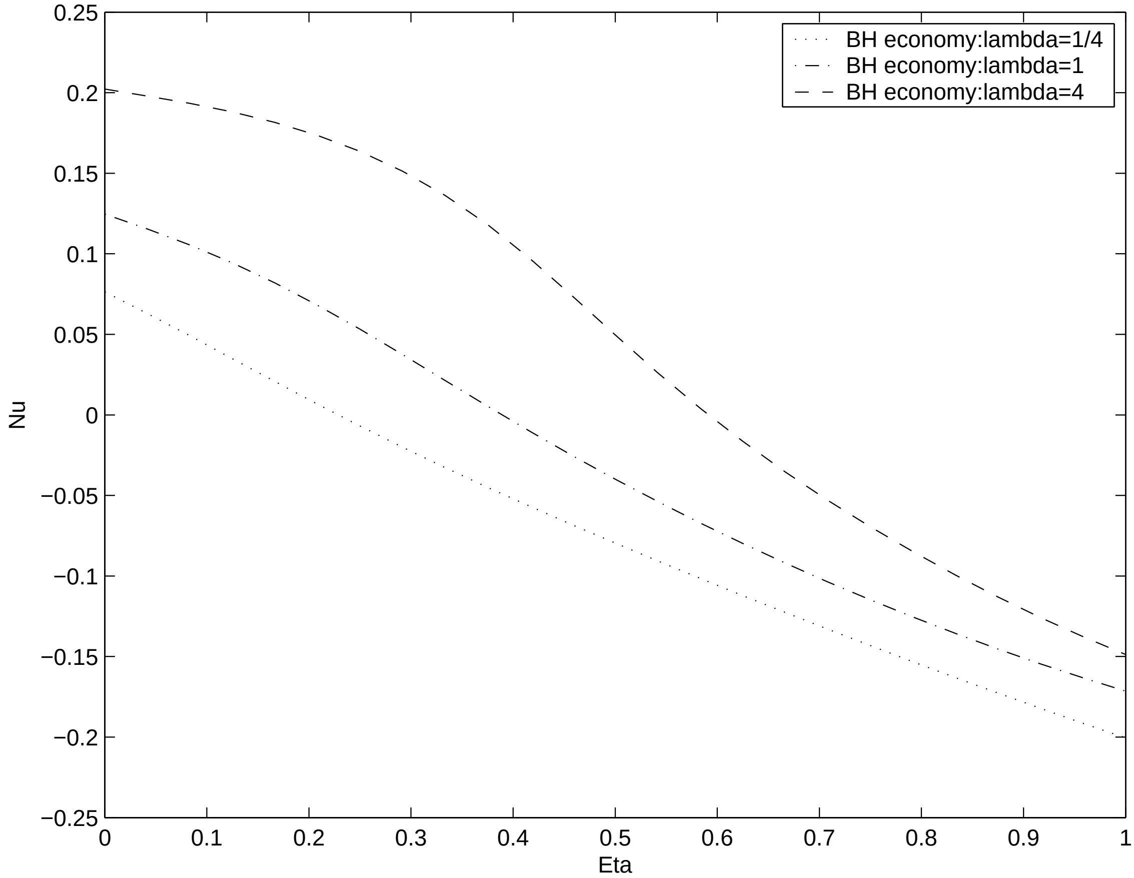

Figure 25: **Shadow process ν (volatility of the stochastic weight) in the buy and hold economy.**
The x-axis corresponds to η, the fraction of the stock held by the buy and hold investor. The dashed line is for buy and hold economy with $\lambda = 4$ (i.e. consumption share claimed by agent 2 equals 85%). The dash-dotted line is for buy and hold economy with $\lambda = 1$ (i.e. consumption share claimed by agent 2 equals 68%). The dotted line is for buy and hold economy with $\lambda = 1/4$ (i.e. consumption share claimed by agent 2 equals 50%). Parameter values of the economy: agent's relative risk aversion $\gamma_1 = 1$, $\gamma_2 = 3$, impatience parameter ρ=0.001, consumption growth volatility $\bar{\sigma}$=0.0357, expected consumption growth rate $\bar{\mu}$=0.0183, time horizon T=50, the ratio between contribution and aggregate consumption $\bar{x}$=0.12, current aggregate consumption δ=1.

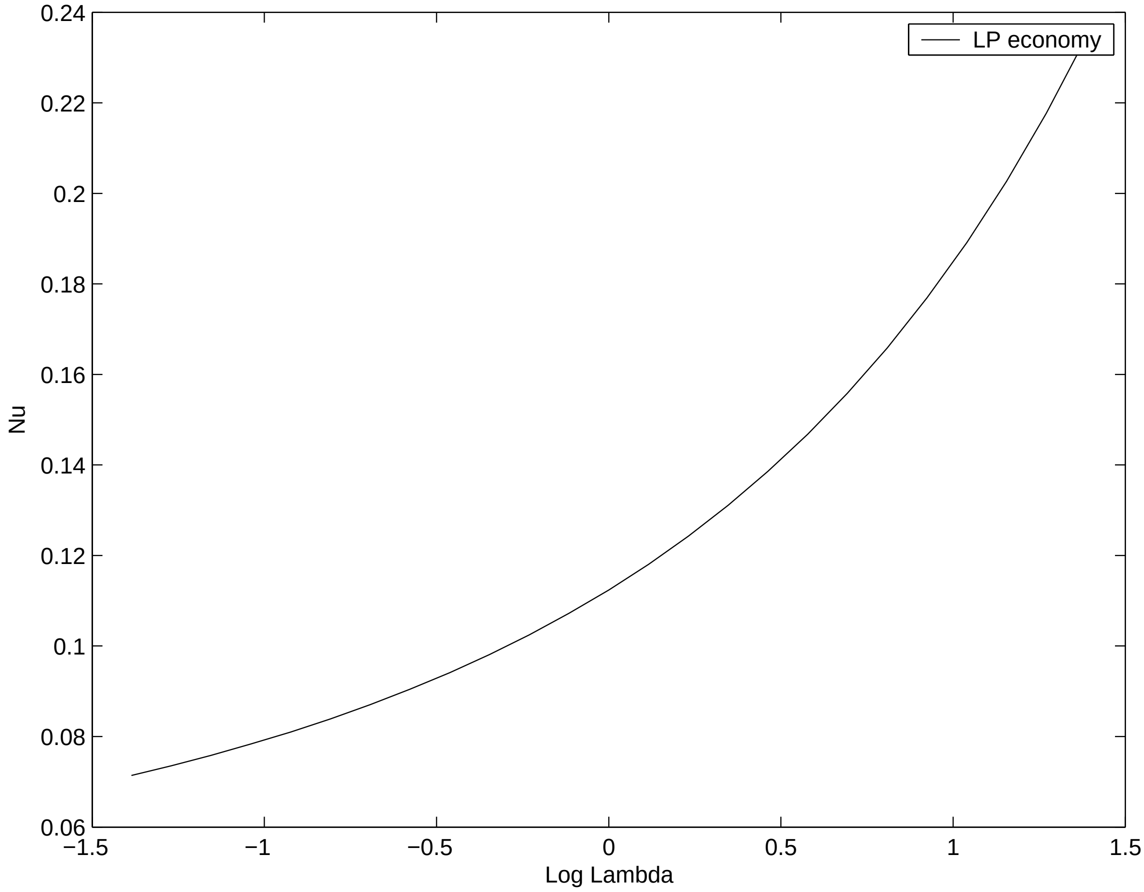

Figure 26: **Shadow process ν (volatility of the stochastic weight) in the limited participation economy.**
The x-axis corresponds to log λ. The corresponding range for λ is [1/4,4], which translates into [0.5, 0.85] for the consumption share claimed by agent 2. Parameter values of the economy: agent's relative risk aversion $\gamma_1 = 1$, $\gamma_2 = 3$, impatience parameter $\rho = 0.001$, consumption growth volatility $\bar{\sigma}$=0.0357, expected consumption growth rate $\bar{\mu}$=0.0183, time horizon T=50, the ratio between contribution and aggregate consumption $\bar{x}$=0.12, current aggregate consumption δ=1.