Bubbles and Panics in a Frictionless Market with Heterogeneous Expectations

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Abstract

When investors have differences of opinion about the payoffs of a stock, Harrison and Kreps (1978) demonstrate the existence of a speculative bubble in the stock price, that is, the stock price can exceed the valuation of the most optimistic investor. A crucial condition that supports this result in their model is that investors are not allowed to short sell the stock. This paper demonstrates that speculative bubbles may arise even without the short sales constraint. The paper also demonstrates that asset panics may arise, that is, the stock price may be lower than the valuations of all individual investors. In particular, even if the short sales constraint binds, asset panics can still arise. This result suggests that Miller's (1977) insight that the short sales constraint causes the stock price to be above the average valuation is not robust in a dynamic framework. In the case of a bubble, our model generalizes the Harrison-Kreps notion of a resale option, namely, investors believe that they can resell the stock later at a higher price. In the case of a panic, our model develops the notion of a waiting option, namely, investors believe that they can purchase the stock later at a lower price.

1 Introduction

It is well known that stock market returns exhibit fat tails. In particular, asset bubbles and panics occur more often than can be justified by changes in the fundamental values of firms. For example, the Dow Jones index dropped by 23% in one day on October 19 of 1987 and 25% on October 24 of 1929. In recent years, the market has experienced the rise and fall of the internet stocks. Ofek and Richardson (2003) report that in the two-year period from early 1998 through February 2000, the Internet sector earned over 1000 percent returns on its public equity, but these returns had completely disappeared by the end of 2000. Spiegel (2004) reviews extensively the stock market performance and offers a comprehensive discussion on bubbles and panics. We argue that the speculative bubbles and panics may be difficult to reconcile with the asset pricing models in which all investors share the same beliefs about asset returns.

Speculative bubbles may arise when investors have differences of opinion about the asset payoffs as well as there are short sales constraints. In a seminal paper, Harrison and Kreps (1978) obtain a remarkable result that the price of an asset can be higher than the valuation of even the most optimistic investor in the market.[1] The intuition is that investors price an asset based not only on the payoffs associated with the ownership of the asset but also on the right to resell the asset. With the short sales constraint and risk-neutral investors, the asset price in every future state is determined by the investor who is most optimistic about the asset payoffs in that state. With differences of opinion, an investor may have the highest valuation today on expected basis, but it is not necessary that he has the highest valuation in all future states. As a result, the stock price may exceed the valuation of the most optimistic investor (based on the expected future prices under the investor's beliefs) or speculative bubbles may occur.[2] In other words, even the most optimistic investor is willing

[1] In this paper we define short sales constraints as any transactions costs or restrictions associated with short selling an asset. In the original Harrison-Kreps model and most of its extensions, investors are simply not permitted to short sell the asset.

[2] Harris and Raviv (1993), Kandel and Pearson (1995), Morris (1996), Biais and Bossaerts (1998), Duffie, Garleanu, and Pederson (2002), Kyle and Lin (2002), Viswanathan (2002), Hong and Stein (2003), Scheinkman and Xiong (2003), and Cao and Ou-Yang (2004) have extended the Harrison-Kreps model in various aspects. For comprehensive

to pay more than his own valuation (expected payoff) for the stock today because in some of the future states, this investor can resell the stock to other investors who have higher valuations.

The Harrison-Kreps model and its extensions have been used to discuss the sharp rise and fall of both the Internet stocks and the NASDAQ index. In those models, however, the stock price can be higher than the valuation of the most optimistic investor but it cannot be lower than the valuation of the most pessimistic investor. In other words, those models cannot be used to explain asset panics.

A key condition that leads to speculative bubbles in the Harrison-Kreps model as well as in its extensions is the presence of the short sales constraint. This paper demonstrates that when investors are risk averse, the short sales constraint is not necessary for speculative bubbles to occur and that speculative panics may also arise. In an influential paper, Miller (1977) argues in a one-period model that the short sales constraint makes it difficult for pessimistic investors to participate in the stock market and that even if these investors believe that the stock price is overvalued, they have no means to bring the price down. As a result, the stock price is determined by the average of the more optimistic investors' valuations. In Miller's model, the equilibrium stock price should never be lower than all investors' valuations. Somewhat strikingly, we find that even when the short sales constraint binds, the stock price may still be lower than all investors' valuations. This result suggests that Miller's insight is not robust in a dynamic setting.

We consider two models with risk-averse investors. In the first model, we impose no restrictions on stock payoffs and assume that investors are myopic and maximize the expected utility of the next-period wealth. There are three periods, 0, 1, and 2. A publicly observed signal, which is correlated with the stock payoff, arrives in period 1. An asset bubble is said to arise when the stock price at time 0 is higher than the valuation based on the most optimistic belief. Similarly, a panics is said to arise when the stock price at time 0 is lower

reviews, see, for example, Rubinstein (2004) and Scheinkman and Xiong (2004). Asset bubbles may also arise in other settings with certain restrictions, such as those of De Long et al. (1990), Allen and Gorton (1993), Allen, Morris, and Postlewaite (1993), and Abreu and Brunnermeier (2003), and DeMarzo, Kaniel, and Kremer (2004).

than the valuation based on the most pessimistic belief. To determine the stock price in the current period, investors must consider the next-period stock price or the resale value of the stock.

We show that under certain conditions, the stock price can be higher than the valuation of the most optimistic investor, and under other conditions, the price can be lower than the valuation of the most pessimistic investor. When investors are risk averse, the stock price today is equal to the average of all investors' expectations of the next-period price, adjusted for the investors' conditional precisions.[3] With differences of opinion about the stock payoff, it is possible that this average of all investors' expectations can be greater than the highest unconditional expectation of the final stock payoff, or lower than the lowest unconditional expectation of the final stock payoff.

Suppose that one of the investors, i, has the highest expectation among all investors. Intuitively, under investor i's belief about the probabilities of the states of the world, investor i's expectation of the other investors' believed payoffs may be higher than his own expected payoff, which is the highest. For example, suppose that there are two investors, i and j, and that there are two states, A and B. Investor i believes that there is a 3/4 probability that A occurs with a payoff 2 and that there is a 1/4 probability that B occurs with a payoff 3. Investor i's expectation of the payoffs is then given by 2.25. Investor j believes that the two states will occur with an equal probability and that the payoffs in A and B are 3 and 1, respectively. The expected payoffs of investor j is then given by 2. Hence, investor i has the higher valuation. When investor i uses his own beliefs to calculate the expected value of the payoffs believed by investor j, investor i arrives at (3/4)*3+(1/4)*1=2.5, which is higher than his own valuation of 2.25. In other words, investor i believes that investor j will value the payoffs higher or that he has an option to resell the stock to investor j in the next period at a higher price.

Similarly, it is possible that under some or all investors' probability beliefs, the stock

[3]For example, if an investor's precision of a signal is high, then he will demand a lower risk premium for the stock, which affects the stock price.

price in the next period is very low so that investors want to wait until the next period to purchase the stock at a low price. In equilibrium, to clear the market in the current period, the stock price today can be lower than the valuations of all investors. In particular, even when the constraint binds in the current period, asset panics can still arise. This result occurs because investors believe that the next-period prices in some states can be so low that the short sales constraint will not bind in those states, although the constraint is allowed. Due to the risk neutrality of investors in Harrison and Kreps (1978), panics will never occur because the stock price in every state of nature is determined by the investor who has the highest valuation in that state, and the short sales constraint binds for all but the most optimistic investor about the state. Again, the investor who has the highest valuation on the expected basis does not have to have the highest valuation in every state of nature. Due to the static nature of Miller (1977), neither bubbles nor panics occur because the stock price is determined by the weighted average of all investors' valuations without any waiting option. When the short sales constraint binds, the stock price is above the valuation based on the average belief because investors with lower valuations cannot participate in the stock trading.

In sum, in the case of asset bubbles, our model generalizes the Harrison-Kreps notion of a resale option to an economy with risk-averse investors but without the short sales constraint. Investors may purchase the stock at a high price today hoping to resell it at an even higher price in the future. In the case of asset panics, our model discovers a waiting option for risk-averse investors. If investors expect the stock price to be low in the future, then they may prefer to wait to purchase the stock later at a low price. Because of this waiting option, asset panics may occur. We further demonstrate that panics may occur even when the short sales constraint binds in the current trading period and is allowed in future trading periods.

One may argue that asset bubbles and panics arise because investors have myopic views in the model. We then show that the results still go through when investors are forward looking in a dynamic model. As in the myopic model, there are three time periods in this economy, 0, 1, and 2. Investors trade in periods 0 and 1. Consumption occurs in period 2.

To obtain closed-form solutions, we assume that investors have negative exponential utility functions and stock payoffs are normally distributed. We derive conditions under which asset bubbles or panics occur without the short sales constraint. Again, bubbles and panics occur due to resale and waiting options, respectively.

Empirically, there is no compelling, direct evidence showing that the short sale constraint is necessary for asset bubbles to form. Reed (2001) and D'Avolio (2002) report that stocks are inexpensive to short in general. Lamont and Thaler (2003) notice that although internet stocks had higher average short interest and were more expensive to short than non-Internet stocks, the average difference in shorting costs was only 1% per year between 1998 and 2000. Jones and Lamont (2002) show that not only are the stocks in their sample overpriced, the magnitude of overpricing cannot be explained by measured shorting costs alone. In addition, Figlewski and Webb (1993) observe that the average short interest is only 0.2% of the outstanding shares for individual stocks. Brunnermeier and Nagel (2004) point out that short sales constraints are not sufficient to explain the failure of rational activity to contain the technology bubble. Lamont and Stein (2004) examine short interest for stock indices and find that total short interest moves in a counter cyclical fashion, that is, short interest actually declines as indices climb, and that short-selling does not particularly help stabilize the overall market. This result is striking because shorting indices is inexpensive and investors do not bear firm-specific risks for trading in indices. These studies suggest that the short sales constraint is perhaps not the main reason that asset bubbles occur.

The rest of this paper is organized as follows. When investors are myopic, Section 2 shows that with differences of opinion alone, the asset price in the current period can exceed or be lower than the valuations of all investors. Section 3 compares our results with those of Harrison and Kreps (1978) and Miller (1977). When investors have negative exponential utility functions and maximize their expected utilities through dynamic trading, Section 4 shows that both bubbles and panics can still arise without the short sales constraint. Section 5 concludes the paper. The appendix contains technical proofs.

2 A Myopic Mean-Variance Model

For the simplicity of exposition, we consider a three-period model, with a time line of 0, 1, and 2. There is one risk free bond and one risky stock available for trading. It is assumed that the financial market is populated by investors with the population size normalized to one, each indexed by i where $i \in [0,1]$. Trading takes place because investors have differences of opinion about the terminal payoff (at time 2) of the stock. At time 0, we assume that each investor is endowed with x^i units of the stock and zero units of the bond. Without loss of generality, the interest rate is taken to be zero. The stock payoff at time 2 is v. The per capita supply of the stock is a positive number denoted by x. We assume in this section that investors are myopic, that is, they maximize the expected utility period by period without considering the effects of future periods.

At time 1, a public signal y, which reveals information about the final payoff of the stock v, is made available to investors. Investors have different interpretations of y. For example, they may use different statistical techniques to learn about v through y. We assume that in each period investor i has mean-variance utility function given by

$$U_{ti} = \mathrm{E}_{ti}[W_{(t+1)i}] - \frac{1}{2}\gamma \mathrm{Var}_{ti}[W_{(t+1)i}], \quad t = 0, 1, \tag{1}$$

where γ denotes the investor's risk aversion coefficient and $W_{(t+1)i}$ denotes investor i's wealth at time $(t+1)$.

At time 2, investor i's wealth is given by

$$W_{2i} = W_{1i} + D_{1i}(v - P_1). \tag{2}$$

Here W_{1i} denotes investor i's wealth at time 1 and the second term denotes his profit from investing in the stock, where D_{1i} is the investor's demand for the stock.

From Equation (1), we have that investor i's expected utility at time 1 is given by

$$U_{1i} = W_{1i} + D_{1i}(\mu_{1i} - P_1) - \frac{D_{1i}^2}{2}\gamma\sigma_{1i}^2, \quad t = 0, 1, \tag{3}$$

where $\mu_{1i} \equiv E_{1i}[v]$ denotes investor i's conditional expectation of v at time 1 and $\sigma_{1i}^2 \equiv \mathrm{Var}_{1i}[v]$ denotes investor i's conditional variance of v at time 1. The first-order condition with respect to D_{1i} is given by

$$\mu_{1i} - P_1 - D_{1i}\gamma\sigma_{1i}^2 = 0. \tag{4}$$

The optimal demand at time 1 is then given by

$$D_{1i} = \frac{\mu_{1i} - P_1}{\gamma\sigma_{1i}^2}, \tag{5}$$

Let $\pi_{1i} \equiv 1/\sigma_{1i}^2$. Using the market clearing condition, $\int_i D_{1i}di = x$, we can express the equilibrium price P_1 as

$$P_1 = \mu_1 - \frac{\gamma x}{\pi_1}, \quad \pi_1 = \int_i \pi_{1i}di, \quad \mu_1 = \frac{1}{\pi_1}\int_i \pi_{1i}\mu_{1i}di, \tag{6}$$

where π_1 is the average precision over the entire population of investors and μ_1 is the precision weighted population average expectation of the stock payoff.

At time 1, investor i's wealth is given by

$$W_{1i} = W_{0i} + D_{0i}(P_1 - P_0), \tag{7}$$

where P_0 is the stock price at time 0 (today). Although investors are myopic, their maximization problems are connected through P_1. Similarly, we obtain the investor's demand for the risky stock and the current stock price:

$$D_{0i} = \frac{1}{\gamma}\pi_{0i}[P_1]\left(\mathrm{E}_{0i}[P_1] - P_0\right), \tag{8}$$

$$P_0 = \mu_0[P_1] - \frac{\gamma x}{\pi_0[P_1]}, \tag{9}$$

where $\mu_0[P_1]$ is the precision weighted population average of the expected prices at time 1 (P_1) and $\pi_0[P_1]$ is the population average precision of P_1, $\mathrm{E}_{0i}[P_1]$ represents investor i's conditional expectation of P_1 and $\pi_{0i}[P_1]$ is investor i's conditional precision of P_1, all of which are evaluated at time 0. The expressions for $\mu_0[P_1]$ and $\pi_0[P_1]$ are given by

$$\mu_0[P_1] = \frac{\int \mathrm{E}_{0i}[P_1]\pi_{0i}[P_1]di}{\int \pi_{0i}[P_1]di}, \tag{10}$$

$$\pi_0[P_1] = \int \pi_{0i}[P_1]di. \tag{11}$$

Let P_{ti} denote the stock price at time t if all investors share the same belief of type i. We have the following definitions of bubbles and panics.

Definition 1 *A bubble occurs at time t when the stock price (under heterogeneous beliefs) is higher than the highest stock price that would obtain if all investors were homogeneous. That is*

$$P_t > \max_i \{P_{ti}\}. \tag{12}$$

A panic occurs at time t when the stock price (under heterogeneous beliefs) is lower than the lowest stock price that would obtain if all investors were homogeneous. That is

$$P_t < \min_i \{P_{ti}\}.$$

Let $\pi_{0i}[P_1]$ denote investor i's precision of P_1 and $\pi_0[P_1] \equiv \int_i \pi_{0i}[P_1]di$ denote the population average precision of P_1. We have the expression for the precision weighted population average of expectations

$$\mu_0(P_1) = \frac{\int_i \pi_{0i}[P_1]\mathrm{E}_{0i}[\mu_1]di}{\int_i \pi_{0i}[P_1]di}. \tag{13}$$

We can then rewrite the price at time 0 as

$$P_0 = \mu_0(P_1) - \gamma x \left[\frac{1}{\pi_1} + \frac{1}{\pi_0(P_1)}\right].$$

For a stock bubble to occur at time 0, it is both necessary and sufficient to have

$$\mu_0(P_1) - \gamma x \left[\frac{1}{\pi_1} + \frac{1}{\pi_0(P_1)}\right] > \max_i \{P_{0i}\}.$$

Similarly, for a stock panics to occur at time 0, it is both necessary and sufficient to have

$$\mu_0(P_1) - \gamma x \left[\frac{1}{\pi_1} + \frac{1}{\pi_0(P_1)}\right] < \min_i\{P_{0i}\}.$$

For the simplicity of illustration, we next consider a case in which γx is close to zero or investors are close to risk neutral. In this case, $P_1 = \mu_1$. The condition for a bubble to occur further reduces to

$$\mu_0(\mu_1) > \max_i\{\mu_{0i}\} = \max_i\{\mathrm{E}_{0i}[\mathrm{E}_{1i}[v]]\} = \max_i\{\mathrm{E}_{0i}[v]\}. \tag{14}$$

When investors have homogeneous expectations, this condition cannot be satisfied because no investor's expectation exceeds the maximum expectation. When investors have heterogeneous expectations, we next discuss the necessary conditions for this inequality to hold.

Suppose that

- (1) the conditional and unconditional precisions are deterministic for each investor and
- (2) $\mathrm{E}_{0i}[\mu_{1j}] = \mathrm{E}_{0i}[\mathrm{E}_{1j}[v]] = \mathrm{E}_{0i}[v]$ for all investors $j \neq i$.[4]

We then have

$$\mathrm{E}_{0i}[\mu_1] = \mathrm{E}_{0i}\left[\frac{\int_j \pi_{1j}\mu_{1j}dj}{\int_j \pi_{1j}dj}\right] = \frac{\int_j \pi_{1j}\mathrm{E}_{0i}[v]dj}{\int_j \pi_{1j}dj} = \mathrm{E}_{0i}[v] \leq \max_i\{\mathrm{E}_{0i}[v]\}.$$

As a result, we have that

$$P_0 = \mu_0(\mu_1) \leq \max_i\{\mathrm{E}_{0i}[v]\},$$

which follows from the fact that the weighted average of certain values does not exceed the maximum value. Therefore, for an asset bubble to occur, one of the two conditions must fail.

[4]The precisions do not have to be equal across investors. In a noisy rational expectations equilibrium, Allen, Morris, and Shin (AMS, 2004) are perhaps the first to demonstrate that the law of iterated expectations may not hold for the average expectations. In their model, the demand for the stock is caused by an exogenous supply shock, whereas in our model, the demand for the stock is due to the assumption that investors have heterogeneous beliefs. AMS do not consider the issues as discussed in the current paper, that is, the stock price can exceed the valuation of the most optimistic investor and the stock price can drop below the valuation of the most pessimistic investor.

Suppose that investor i's expectation is the highest among all investors. For a bubble to occur, if condition (1) holds, then it is necessary that $\mathrm{E}_{0i}[\mathrm{E}_{1j}[v]] > \mathrm{E}_{0i}[v]$ for some i such that investor i values investor j's expected payoffs more highly than his own payoffs. In other words, investor i expects investor j to value the stock more highly in the next period. Otherwise, investor i would not pay more than his own valuation for the stock at time 0.

Even if condition (2) is satisfied or $\mathrm{E}_{0i}[\mathrm{E}_{1j}[v]] = \mathrm{E}_{0i}[v]$, it is still possible that

$$\mathrm{E}_{0i}[\mu_1] = \mathrm{E}_{0i}\left[\frac{\int_j \pi_{1j}\mu_{1j}dj}{\int_j \pi_{1j}dj}\right] > \max_i\{\mathrm{E}_{0i}[v]\},$$

where π_{1j} must be stochastic. Intuitively, under investor i's probability belief, investor j has high precisions (π_{1j}) in the high payoff states (μ_{1j}) or investor j is willing to pay a high price in those states. Essentially, investor i expects other investors to value the stock more highly than he does. In other words, an investor is willing to buy the stock at a high price today because he expects to resell the stock at an even higher price later.

In sum, for an asset bubble to occur, even if an investor has the highest valuation of a stock under homogeneous beliefs, he expects other investors to value the stock even higher under heterogeneous beliefs. On average, investors are willing to pay a high price today because they believe that they can resell the stock at an even higher price. Similarly, for an asset panic to occur, investors must expect that the stock price in the next period will be even lower, so they would rather wait to buy the stock later. In equilibrium, the stock price today must be low to clear the market. Our analysis indicates that a bubble or a panic can only occur in a dynamic setting in which a resale option or a waiting option exists. We next provide numerical examples for bubbles and panics as well as develop sufficient conditions in a two-state, two-investor economy.

2.1 Numerical Examples

Example 1: A Bubble

Suppose that there are two types of investors, type i and type j, with equal proportion.

The risk free rate is taken to be zero without loss of generality. For investor type i, the final stock payoff is 10 in state H or 0 in state L with equal probability. For investor type j, the final stock payoff is 10 in state H with a probability of 0.45 or 0 in state L with a probability of 0.55. The unconditional expectations are 5 and 4.5 for types i and j, respectively, so under homogeneous beliefs, type i is more optimistic than type j. We next demonstrate that the equilibrium stock price at time 0 can be higher than 5.

At time 1, a public signal arrives with two possible realizations, A and B. For investors of type i, the probability of realization A is $\mathrm{Prob}^i(A) = 1/4$ and the probability of realization B is $\mathrm{Prob}^i(B) = 3/4$. When the signal realization is A, the conditional probability of state H is 4/5. When the signal realization is B, the conditional probability of state H is 2/5. For investors of type j, the probability of state A is 1/2 and the probability of state B is 1/2, that is, $\mathrm{Prob}^j(A) = \mathrm{Prob}^j(B) = 1/2$. When the signal realization is A, the conditional probability of state H is 0.36. When the signal realization is B, the conditional probability of state H is 0.54. Let $\mu_{1a}(s)$ and $\pi_{1a}(s)$ denote, respectively, the conditional mean and conditional precision of investor type a ($a = i, j$) and with signal realization s ($s = A, B$) at time 1.

Given the above data, we can calculate the investors' conditional expectations and precisions. Specifically, we obtain that

$$\mu_{1i}(A) = 8, \quad \mu_{1i}(B) = 4, \quad \mu_{1j}(A) = 3.6, \quad \mu_{1j}(B) = 5.4,$$

$$\pi_{1i}(A) = \frac{1}{16}, \quad \pi_{1i}(B) = \frac{1}{24}, \quad \pi_{1j}(A) = \frac{1}{23.04}, \quad \pi_{1j}(B) = \frac{1}{24.84}.$$

It can be checked that the unconditional expectations of the final payoffs are 5 and 4.5 for investors i and j, respectively.

Assume that all investors are close to risk neutral so that we can ignore the risk premium term in the stock price. When the signal is A at time 1, the stock price is then approximately given by

$$\begin{aligned}
P_1(A) &= [\pi_{1i}(A) + \pi_{1j}(A)]^{-1} \times [\pi_{1i}(A)\mu_{1i}(A) + \pi_{1j}(A)\mu_{1j}(A)] \\
&= \left[\frac{1}{16} + \frac{1}{23.04}\right]^{-1} \times \left[\frac{8}{16} + \frac{3.6}{23.04}\right] = 6.1967.
\end{aligned}$$

When the signal is B, the stock price is given by

$$\begin{aligned}
P_1(B) &= [\pi_{1i}(B) + \pi_{1j}(B)]^{-1} \times [\pi_{1i}(B)\mu_{1i}(B) + \pi_{1j}(B)\mu_{1j}(B)] \\
&= \left[\frac{1}{24} + \frac{1}{24.84}\right]^{-1} \times \left[\frac{4}{24} + \frac{5.4}{24.84}\right] = 4.6880.
\end{aligned}$$

The stock price at time 0 is determined by the average of the conditional expectations of the two investors, adjusted by their conditional precisions. The investors' conditional expectations are

$$\mathrm{E}_{0i}[P_1] = \frac{1}{4} \times 6.1967 + \frac{3}{4} \times 4.6880 = 5.0032,$$

$$\mathrm{E}_{0j}[P_1] = \frac{1}{2}(6.1967 + 4.6880) = 5.4424.$$

Their conditional precisions are

$$\pi_{0i}[P_1] = \left[\frac{1}{4}(6.1967 - 5.0032)^2 + \frac{3}{4}(4.6880 - 5.0032)^2\right]^{-1} = 2.3222,$$

$$\pi_{0j}[P_1] = \left[\frac{1}{2}(6.1967 - 5.4424)^2 + \frac{1}{2}(4.4880 - 5.4424)^2\right]^{-1} = 1.7573.$$

The stock price at time 0 is then given by

$$\begin{aligned}
P_0 &= [\pi_{0i}[P_1] + \pi_{0j}[P_1]]^{-1} [\pi_{0i}[P_1]\mathrm{E}_{0i}(P_1) + \pi_{0j}[P_1]\mathrm{E}_{0j}(P_1)] \\
&= (2.3222 + 1.7573)^{-1} \times (2.3222 \times 5.0032 + 1.7573 \times 5.4424) = 5.1924 > 5.
\end{aligned}$$

Type j investors are willing to acquire the stock at 5.1924 at time 0 because their expected payoff is 5.4424. Type i investors sell the stock because their expected payoff is only 5.0032.

Although the unconditional expected payoff of type i investors is 5, the expected payoff of type j, based on type i's probability beliefs, is given by $1/4 \times 3.6 + 3/4 \times 5.4 = 4.95$. This means that investor i believes that he can buy the stock at a low price in the next period, so he sells the stock today. On the other hand, the expected payoff of type i based on type j's probability beliefs, is given by $1/2 \times 8 + 1/2 \times 4 = 6$. It means that investor j believes that he can resell the stock to investor j at a higher price in the next period, so he buys the stock today at a high price that exceeds the valuation of investor i. In other words, a bubble arises due to the resell option of investor j.

Example 2: A Panic

There are two types of investors, type i and type j, with equal proportion. For type i, the stock payoff is 10 or 0 with equal probability. For type j, the stock payoff is 10 with a probability of 0.55 or 0 with a probability of 0.45. The unconditional expectations are given by 5 and 5.5 for types i and j, respectively, so type i is more pessimistic than type j. We next demonstrate that the equilibrium stock price at time 0 can be lower than 5.

At time 1 there is a signal with two possible realizations A and B. For i, A occurs with 1/4 probability. Conditional on A, 0 occurs with 4/5, and conditional on B, 0 occurs with 2/5. For j, A occurs with 1/2 probability. Conditional on A, 0 occurs with 0.36, and conditional on B, 0 occurs with 0.54.

Given the above data, we can calculate the investors' conditional expectations and precisions. We obtain that

$$\mu_{1i}(A) = 2, \quad \mu_{1i}(B) = 6, \quad \mu_{1j}(A) = 6.4, \quad \mu_{1j}(B) = 4.6,$$

$$\pi_{1i}(A) = \frac{1}{16}, \quad \pi_{1i}(B) = \frac{1}{24}, \quad \pi_{1j}(A) = \frac{1}{23.04}, \quad \pi_{1j}(B) = \frac{1}{24.84}.$$

It can be checked that the unconditional expectations of the final payoffs are 5 and 5.5 for investors i and j, respectively.

Assume that all investors are close to risk neutral so that we can ignore the risk premium term in the stock price. When the signal is A at time 1, the stock price is then approximately given by

$$\begin{aligned} P_1(A) &= [\pi_{1i}(A) + \pi_{1j}(A)]^{-1} \times [\pi_{1i}(A)\mu_{1i}(A) + \pi_{1j}(A)\mu_{1j}(A)] \\ &= \left[\frac{1}{16} + \frac{1}{23.04}\right]^{-1} \times \left[\frac{2}{16} + \frac{6.4}{23.04}\right] = 3.8033. \end{aligned}$$

When the signal is B, the stock price is given by

$$\begin{aligned} P_1(B) &= [\pi_{1i}(B) + \pi_{1j}(B)]^{-1} \times [\pi_{1i}(B)\mu_{1i}(B) + \pi_{1j}(B)\mu_{1j}(B)] \\ &= \left[\frac{1}{24} + \frac{1}{24.84}\right]^{-1} \times \left[\frac{6}{24} + \frac{4.6}{24.84}\right] = 5.3120. \end{aligned}$$

The price at time 0 is determined by the average of the conditional expectations of the two investors, adjusted by their conditional precisions. The investors' conditional expectations are given by

$$\mathrm{E}_{0i}[P_1] = \frac{1}{4} \times 3.8033 + \frac{3}{4} \times 5.3120 = 4.9348,$$

$$\mathrm{E}_{0j}[P_1] = \frac{1}{2}(3.8033 + 5.3120) = 4.5577.$$

Their conditional precisions are

$$\pi_{0i}[P_1] = \left[\frac{1}{4}(3.8033 - 4.9348)^2 + \frac{3}{4}(5.3120 - 4.9348)^2\right]^{-1} = 2.3431,$$

$$\pi_{0j}[P_1] = \left[\frac{1}{2}(3.8033 - 4.5577)^2 + \frac{1}{2}(5.3120 - 4.5577)^2\right]^{-1} = 1.7573.$$

The stock price at time 0 is then given by

$$\begin{aligned} P_0 &= [\pi_{0i}[P_1] + \pi_{0j}[P_1]]^{-1} [\pi_{0i}[P_1]\mathrm{E}_{0i}(P_1) + \pi_{0j}[P_1]\mathrm{E}_{0j}(P_1)] \\ &= (2.3431 + 1.7573)^{-1} \times (2.3431 \times 4.9348 + 1.7573 \times 4.5577) = 4.7733 < 5. \end{aligned}$$

Type i investors buy the stock and type j investors sell the stock. Under type i's probability beliefs, the expected value of type j's payoffs is given by $1/4 \times 6.4 + 3/4 \times 4.6 =$

5.05. Under type j's probability beliefs, the expected value of type i's payoffs is given by $1/2 \times 2 + 1/2 \times 6 = 4$. Although investors i and j value the stock at 5 and 4.5, respectively, they both expect the stock price to be lower than 5 in the next period due to the presence of the other type of investors. Investors do not want to pay a price higher than 4.5 because they believe that they can purchase the stock at a lower price in the next period. This waiting option causes the stock price to be lower than 4.5 or a panic arises.

Example 3: A Bubble with Stochastic Volatility

In the previous two examples, the law of iterated expectations are violated, that is, $\mathrm{E}_{i0}[\mathrm{E}_{j1}[v]] \neq \mathrm{E}_{i0}[v]$. In this example, we show that when the conditional volatility is stochastic, a bubble can still occur even if the law of iterated expectations holds across investors.

There are two types of investors, type i and type j, with equal proportion. For simplicity of exposition, assume that for both types, the stock payoff is 10 with probability 5/8 or 0 with probability 3/8. The expected payoff is then given by 6.25 for both types of investors. We next demonstrate that the equilibrium stock price at time 0 can be higher than 6.25.

At time 1 there is a signal with two possible realizations A and B and both investors believe that A and B will occur with equal probability. For i, conditional on A, 0 occurs with probability 1/4, and conditional on B, 0 occurs with probability 1/2. For j, conditional on A, 0 occurs with probability 1/2 and conditional on B, 0 occurs with probability 1/4. Notice that investors agree on the probabilities of realizations A and B and as a result, the law of iterated expectations holds across investors or $E_{i0}[E_{j1}[v]] = E_{i0}[v]$ and $E_{j0}[E_{i1}[v]] = E_{j0}[v]$.

Given the above data, we can calculate the investors' conditional expectations and precisions. We obtain that

$$\mu_{1i}(A) = 7.5, \quad \mu_{1i}(B) = 5, \quad \mu_{1j}(A) = 5, \quad \mu_{1j}(B) = 7.5,$$

$$\pi_{1i}(A) = \frac{4}{75}, \quad \pi_{1i}(B) = \frac{1}{25}, \quad \pi_{1j}(A) = \frac{1}{25}, \quad \pi_{1j}(B) = \frac{4}{75}.$$

It can be checked that the unconditional expectations of the final payoffs are 6.25 for both types of investors.

Assume that all investors are close to risk neutral so that we can ignore the risk premium term in the stock price. When the signal is A at time 1, the stock price is then approximately given by

$$\begin{aligned} P_1(A) &= [\pi_{1i}(A) + \pi_{1j}(A)]^{-1} \times [\pi_{1i}(A)\mu_{1i}(A) + \pi_{1j}(A)\mu_{1j}(A)] \\ &= \left[\frac{4}{75} + \frac{1}{25}\right]^{-1} \times \left[\frac{4}{75} \times 7.5 + \frac{1}{25} \times 5\right] = 6.4129. \end{aligned}$$

When the signal is B, the stock price is given by

$$\begin{aligned} P_1(B) &= [\pi_{1i}(B) + \pi_{1j}(B)]^{-1} \times [\pi_{1i}(B)\mu_{1i}(B) + \pi_{1j}(B)\mu_{1j}(B)] \\ &= \left[\frac{1}{25} + \frac{4}{75}\right]^{-1} \times \left[\frac{1}{25} \times 5 + \frac{4}{75} \times 7.5\right] = 6.4129. \end{aligned}$$

The price at time 0 is determined by the average of the conditional expectations of the two investors, adjusted by their precisions. Since the prices at time 1 are the same in both states A and B, the stock price at time 0 is also 6.4129. Hence, the stock price with heterogeneous beliefs is larger than 6.25, the price that would obtain if all investors have the same beliefs as either type i or type j. In other words, a bubble can occur even if the law of iterated expectations holds across investors. Again, the investors' perceived resale option lead to the bubble.

2.2 Sufficient Conditions for Bubbles and Panics in a Two-State, Two-Investor Economy

We have examined examples for bubbles and panics to exist. The general sufficient conditions for the existence of bubbles and panics are very complex. To simplify the analysis, we consider only sufficient conditions for the existence of bubbles and panics in a two-state, two-investor economy. Let q_{aA} be the probability of state A for investor $a = i, j$. Let q_{aSH} be the conditional probability of investor $a = i, j$ given signal $S = A, B$. Let the payoff in the high state be V_H and the payoff in the low state be V_L. For simplicity of exposition, assume that both types of investors have the same unconditional expectation, that is,

$$\begin{aligned}\mu &\equiv q_{iA}[q_{iAH}V_H + (1-q_{iAH})V_L] + (1-q_{iA})[q_{iBH}V_H + (1-q_{iBH})V_L] \\ &= q_{jA}[q_{jAH}V_H + (1-q_{jAH})V_L] + (1-q_{jA})[q_{jBH}V_H + (1-q_{jBH})V_L]. \qquad (15)\end{aligned}$$

Given the signals at time 1, the conditional expectations of investors i and j are given by

$$\mu_{i1A} = q_{iAH}V_H + (1-q_{iAH})V_L, \quad \mu_{i1B} = q_{iBH}V_H + (1-q_{iBH})V_L,$$

$$\mu_{j1A} = q_{jAH}V_H + (1-q_{jAH})V_L, \quad \mu_{j1B} = q_{jBH}V_H + (1-q_{jBH})V_L.$$

Suppose that investors have very small risk aversion so that we can ignore the risk premium term in the price functions. Let π_{a1S} denote the conditional precision for investor a given signal S. We have that the stock price at time 1 given signal S is given by

$$P_{1S} = \frac{\pi_{i1S}\mu_{i1S} + \pi_{j1S}\mu_{j1S}}{\pi_{i1S} + \pi_{j1S}}.$$

Similarly, the stock price at time 0, P_0, is approximately given by the precision weighted average of the investors' expectations of the time 1 prices (P_{1S}):

$$P_0 = \left[\frac{\pi_{i0S}\mu_{0iS}(P_{1S}) + \pi_{j0S}\mu_{0jS}(P_{1S})}{\pi_i + \pi_j}\right],$$

where $\mu_{0a}(P_{1S})$ is investor a's precision weighted expectation of P_{1S} and π_a is investor a's conditional precision of P_{1S}.

For the stock price at time 0 to be larger than μ, it is sufficient to have

$$\begin{aligned}&(\pi_i + \pi_j)^{-1}\left\{\pi_i\left[\frac{q_{iA}\pi_{j1A}(\mu_{j1A} - \mu_{i1A})}{\pi_{i1A} + \pi_{j1A}} + \frac{(1-q_{iA})\pi_{j1B}(\mu_{j1B} - \mu_{i1B})}{\pi_{i1B} + \pi_{j1B}}\right]\right. \\ &\left. + \pi_j\left[\frac{q_{jA}\pi_{i1A}(\mu_{i1A} - \mu_{j1A})}{\pi_{i1A} + \pi_{j1A}} + \frac{(1-q_{jA})\pi_{i1B}(\mu_{i1B} - \mu_{j1B})}{\pi_{i1B} + \pi_{j1B}}\right]\right\} > 0. \qquad (16)\end{aligned}$$

Notice that

$$\mu_{i1S} - \mu_{j1S} = (q_{iSH} - q_{jSH})(V_H - V_L).$$

We then have

$$
\begin{aligned}
&(\pi_i q_{iA}\pi_{j1A} - \pi_j q_{jA}\pi_{i1A})(q_{jAH} - q_{iAH})(\pi_{i1B} + \pi_{j1B}) \\
&+(\pi_i q_{iB}\pi_{j1B} - \pi_j q_{jB}\pi_{i1B})(q_{jBH} - q_{iBH})(\pi_{i1A} + \pi_{j1A}) > 0. \qquad (17)
\end{aligned}
$$

If $q_{jAH} - q_{iAH} = 0$ and $q_{jBH} - q_{iBH} = 0$, then there will be no bubbles. In this case, at time 1, the investors' conditional expectations of the final payoffs are the same. As a result, at time 0, investor i's expected value of investor j's time 1 payoffs is the same as the expected value of his own payoffs. Therefore, at time 0, the stock price or the precision weighted average of the investors' expectations of the time 1 prices will not exceed the investors' own valuations.[5]

For simplicity, suppose that $q_{jBH} = q_{iBH}$ or given realization B, the two investors' conditional expectations of the final payoffs are equal, from inequality (17), we must have the following inequality for a bubble to occur:

$$
(\pi_i q_{iA}\pi_{j1A} - \pi_j q_{jA}\pi_{i1A})(q_{jAH} - q_{iAH}) > 0. \qquad (18)
$$

The sufficient conditions are then given by

$$
q_{jHA} > q_{iAH}, \quad \pi_{j1A} > \pi_{i1A}, \quad \pi_i q_{iA} > \pi_j q_{jA}. \qquad (19)
$$

$q_{jHA} > q_{iAH}$ means that at time 1, investor j's valuation is higher than investor i's valuation. $\pi_{j1A} > \pi_{i1A}$ means that at time 1, investor j will trade more aggressively due to a higher conditional precision. $\pi_i q_{iA} > \pi_j q_{jA}$ means that investor i places more weight on realization A, driven by his precision at time 0 and his probability belief of realization A. Therefore, investor i believes that at time 1, investor j will be willing to pay a high price for the stock in state A. As a result, investor i is willing to pay a high price (higher than his own valuation) for the stock today.

Similarly, bubbles can arise under the following conditions:

$$
q_{jAH} < q_{iAH}, \quad \pi_{j1A} < \pi_{i1A}, \quad \pi_i q_{iA} < \pi_j q_{jA}. \qquad (20)
$$

[5] In general, if the unconditional expectation of investor i is higher than that of investor j, then the stock price will not exceed the valuation of investor i.

In other words, when A is realized at time 1, investor i's valuation is higher than investor j's valuation, but at time 0, investor j places more weight on realization A.

In sum, bubbles arise because one type of investors believes that the other type of investors will be willing to pay a higher price for the stock in the future. In general, if both types of investors' unconditional valuations differ, then, for bubbles to occur, the more optimistic investor must believe that under certain conditions, the other type will be willing to pay an even higher price for the stock in the next period. In other words, bubbles occur because some investors believe that they can resell the stock later at an even higher price.

Similarly, for panics to occur, the more pessimistic investor must believe that under certain conditions, the stock price will drop below their valuations in the next period. For example, type i places more weight on realization A at time 0, but this type believes that with realization A at time 1, type j will value the stock very low. As a result, type i believes that he can purchase the stock from type j at a lower price at time 1. In the meantime, type j places more weight in realization B at time 0, but this type believes that with realization B at time 1, type i will value the stock very low. Similarly, type j believes that he can purchase the stock from type i at a low price at time 1. Both types of investors would rather wait until the next period to purchase the stock. For the market to clear at time 0, the current stock price must be sufficiently low and as a result, it is possible that the stock price today is below the valuations of all investors.

3 Comparison with Harrison and Kreps (1978) and Miller (1977)

It is of interest to compare our result with that of Harrison and Kreps (1978) in which short sales constraints are essential and investors are risk neutral. In that model, the stock price at time 1 is given by

$$P_1 = \max_i \{\mathrm{E}_{1i}[v]\},$$

and the stock price at time 0 is given by

$$P_0 = \max_i \{\mathrm{E}_{0i}[P_1]\}.$$

Even with the short sales constraint, if one of the investors in the economy is always the most optimistic one in every state, then the stock price today will not exceed the valuation of this most optimistic investor or bubbles will not arise. In addition to the short sales constraint, the Harrison-Kreps model requires that for a bubble to occur, the most optimistic investor (based on the expected payoffs) must believe that in some of the future realizations, other investors will value the stock more highly than he does. Note that with the short sales constraint, the Harrison-Krpes model cannot generate panics.

To highlight the differences between our model and the Harrison-Kreps model, we revisit the example of a panic presented in subsection 2.1, by imposing the short sales constraint. In that example, we have demonstrated that a panic can occur without the short sales constraint or that the stock price today can be below all of the investors' valuations. We next show that in the Harrison-Kreps framework with risk neutrality and the short sales constraint, a bubble rather than a panic will occur.

We first determine the stock prices at time 1. When the realization is A, the price is equal to the maximum expectation among the investors, which is given by 6.4, the type j investors' expectation. When the realization is B, the price is equal to 6, which is given by type i's expectation. The stock price at time 0 is then given by 6.2, which is a bubble price.

3.1 Panics and the Short Sales Constraint

With the short sales constraint, stock panics do not occur in both Miller (1977) and Harrison and Kreps (1978). In Miller, the short sales constraint prevents pessimistic investors from participating in the stock trading, and as a result, the equilibrium price reflects the views of the more optimistic investors. In Harrison and Kreps, the price is determined by the most optimistic investor in every state. In both models investors do not have an option to wait.

We here demonstrate that even with the short sales constraint, a panic can still occur with risk averse investors. This result suggests that the short sales constraint is neither necessary nor sufficient for bubbles or panics to occur. For tractability, we use a two-investor, two-state economy.

Assume that for investor i the probability of realization A is denoted by $q_{iA} \equiv q > 1/2$, and the probability of realization B is denoted by $q_{iB} = (1-q) < 1/2$. Conditional on state A, the probability of the high value state (H) is $q_{iAH} = q$ and conditional on state B, the probability of the high value state is $q_{iBH} = 1-q$. Similarly, we have $q_{jA} = 1-q$, $q_{jB} = q$, $q_{jAH} = 1-q$, and $q_{jBH} = q$. Further assume that all investors possess the mean variance utility function and that the payoffs in states H and L (the low value state) are given by $V_H = 1$ and $V_L = 0$, respectively.

We solve the equilibrium using backward induction. We first solve for the stock price for the second period. At time 1, investor i's maximization problem is given by

$$\max_{D_{1i}} \left\{ D_{1i}[\mathrm{E}_{1i}[v] - P_1] - \frac{\gamma}{2}\mathrm{Var}_{1i}[v]D_{1i}^2 \right\}, \tag{21}$$

subject to the short sales constraint that $D_{1i} > 0$, where D_{1i} denotes investor i's demand for the stock and $E_{1i}[v] - P_1$ represents the expected profit of owning the stock. The optimal demand is then given by

$$D_{1i} = \frac{(\mathrm{E}_{1i}[v] - P_1)^+}{\gamma \mathrm{Var}_{1i}[v]},$$

where " + " denotes the positive part of a real number. Notice that under our simplified assumptions, the conditional expectations and variances are given by

$$\mathrm{E}_{1i}[v|A] = q, \quad \mathrm{E}_{1j}[v|B] = 1-q,$$

$$\mathrm{Var}_{1i}[v|A] = \mathrm{Var}_{1j}[v|A] = \mathrm{Var}_{1i}[v|B] = \mathrm{Var}_{1j}[v|B] = q(1-q).$$

When state A is realized, investor i is more optimistic and the short sales constraint will not be binding for investor i but the constraint could be binding for investor j. When the short sales constraint does not bind for investor j, we have

$$D_{1iA} = \frac{q - P_{1A}}{\gamma q(1-q)}, \quad D_{1jA} = \frac{1 - q - P_{1A}}{\gamma q(1-q)}.$$

Applying the market clearing condition, $x = 1/2D_{1iA} + 1/2D_{1jA}$,[6] we obtain the equilibrium stock price:

$$P_{1A} = \frac{1}{2} - \gamma q(1-q)x. \tag{22}$$

From the expression for D_{1jA}, when the short sales constraint does not bind or $D_{1jA} > 0$, we must have

$$1 - q - P_{1A} \geq 0 \quad \text{or} \quad \gamma q(1-q)x > q - \frac{1}{2}.$$

On the other hand, when

$$\gamma q(1-q)x \leq q - \frac{1}{2},$$

the short sales constraint binds. We then have

$$D_{1iA} = \frac{q - P_{1A}}{\gamma q(1-q)}, \quad D_{1j} = \frac{(1 - q - P_{1A})^+}{\gamma q(1-q)} = 0.$$

The market clearing condition yields

$$P_{1A} = q - 2\gamma q(1-q)x. \tag{23}$$

We can express the equilibrium stock price in a compact form as follows:

$$P_{1A} = \frac{1}{2} - \gamma q(1-q)x + \left[q - \frac{1}{2} - \gamma q(1-q)x\right]^+. \tag{24}$$

Due to the assumption of symmetry between states A and B, the equilibrium stock price, when state B is realized, is equal to P_{1A}, that is,

$$P_{1B} = \frac{1}{2} - \gamma q(1-q)x + \left[q - \frac{1}{2} - \gamma q(1-q)x\right]^+.$$

[6] Note that x is the per capital supply of the stock. 1/2 is from the assumption that both types of investors are equally populated.

Since the risk free rate is assumed to be zero, we have that the equilibrium stock price at time 0 is given by

$$P_0 = \frac{1}{2} - \gamma q(1-q)x + \left[q - \frac{1}{2} - \gamma q(1-q)x\right]^+ . \tag{25}$$

Notice that in the homogeneous belief case in which all investors share the same belief as investor i, investor i holds the total supply of the stock, x, under all conditions. Following the same procedure as in the case of heterogeneous beliefs, we have

$$P_{1iA} = q - \gamma q(1-q)x, \quad P_{1iB} = 1 - q - \gamma q(1-q)x,$$

$$\begin{aligned} P_{0i} &= P_{1iA} + (1-q)P_{1iB} - \gamma q(1-q)(P_{1iA} - P_{1iB})^2 x \\ &= 1 - 2q + 2q^2 - \gamma q(1-q)x - \gamma a(1-q)(2q-1)^2 x = P_{0j}. \end{aligned}$$

For a panic to occur or for P_0 to be lower than P_{0i}, we must have

$$\frac{1}{2} - \gamma q(1-q)x + (q - \frac{1}{2} - \gamma q(1-q)x)^+ < 1 - 2q + 2q^2 - \gamma q(1-q)x - \gamma a(1-q)(2q-1)^2 x,$$

which reduces to

$$(2q-1)^2 \left[\frac{1}{2} - \gamma x q(1-q)\right] > \left[q - \frac{1}{2} - \gamma q(1-q)x\right]^+ . \tag{26}$$

Based on inequality (26), we consider three cases that contrast our model with those of Miller and Harrison and Kreps.

(1) We first examine the case in which investors are close to risk neutral or the term $\gamma x q(1-q)$ is negligible. Inequality (26) reduces to

$$\frac{1}{2}(2q-1)^2 > q - \frac{1}{2} > 0 \quad \text{or} \quad \left(q - \frac{1}{2}\right) > \frac{1}{2},$$

which cannot be satisfied because of the assumption that $1/2 < q < 1$. In other words, panics will never occur in the Harrison-Kreps model.

(2) We consider the case in which $\left[q - \frac{1}{2} - \gamma q(1-q)x\right]^{+} > 0$ or the short sales constraint binds for investor j. In this case, $rq(1-q)x < 1/2$. Rearranging expression (26), we have

$$[1 - (2q-1)^2]\gamma x q(1-q) > \left(q - \frac{1}{2}\right)(2 - 2q),$$

which reduces to

$$\gamma x q(1-q) > \frac{q - \frac{1}{2}}{2q}.$$

Consequently, when

$$\frac{q - \frac{1}{2}}{2q^2(1-q)} < \gamma x < \frac{q - \frac{1}{2}}{q(1-q)},$$

panics occur even when the short sales constraint binds. Note that the bounds depend on our normalization of $V_H = 1$.

(3) When the short sales constraint does not bind, the equilibrium stock price is lower than the price when the constraint binds, so it is easier to obtain panics in this case than in case 2.

We summarize the results obtained above in the following theorem.

Theorem 1 *When investors are risk averse, panics can still occur in the presence of the short sales constraint.*

Recall that we have demonstrated that without the short sales constraint, both bubbles and panics can arise even when the total supply of the stock is zero or investors are close to risk neutral so that the risk premium term in the stock price is negligible. This theorem presents a striking result, that is, even if the short sales constraint binds in both states at time 1, panics can still occur. This result holds only when investors are risk averse or when the total supply of the stock is nonzero.

Suppose that the short sales constraint binds for investor i in state B and for investor j in state A. At time 1, if state A occurs, with the short sales constraint binding for investor

j, then investor i must hold all of the stock supply for the market to clear. The stock price can be very low because investor i may demand a high risk premium for holding the entire supply of the stock.[7] If state B occurs at time 1, then the stock price is higher than investor i's conditional expectation. At time 0, however, the weighted average of the stock prices in states A and B, adjusted by investor i's conditional precisions, can be lower than the stock price at time 0 when both types of investors share the same beliefs as investor i. Similarly, at time 0, the weighted average of the stock prices in states A and B, adjusted by investor j's conditional precisions, can be lower than the stock price at time 0 when both investors share the same beliefs as investor j. In other words, panics can occur at time 0 even when the short sales constraint binds in both states at time 1.

In a one-period arrangement, if the short sales constraint binds for some investors, then the stock price must be higher than those investors' expected values of the stock. Panics will never occur in this case.

4 A Dynamic Sequential Equilibrium

So far we have obtained the result that asset bubbles and panics can arise without short sales constraints when investors are myopic. In this section, we show that the investors' myopic nature is not crucial for bubbles and panics to occur.

We consider a fully dynamic model in which investors are forward looking and take into account both immediate capital gains and future returns. To obtain closed form solutions, we assume that there is a continuum of risk averse investors with negative exponential utility functions, $-\exp(-\gamma W_{2i})$, where γ is the investors' risk aversion coefficient and W_{2i} is investor i's terminal wealth or consumption. We consider a three-period model. Let v denote the value of the asset at the end of period 2. Assume that v is normally distributed and that for investor i, the unconditional mean of v is μ_i and the unconditional precision of v is h_i.[8]

[7] When investors i and j have homogeneous beliefs, the short sales constraint is never binding in order for the market to clear. In other words, both investors share the stock-related risk, and as a result, due to a lower risk premium, the stock price can be higher than that when investors have heterogeneous beliefs.

[8] Similar results can also be obtained in closed form solution when the asset payoff and the signal have binomial

Investors trade in periods 0 and 1. At time 1, a public signal y arrives. Investors have different interpretations about the relationship between v and y, which generates trades among investors. In particular, investor i believes the following relationship:

$$a_i y - m_i = v + \epsilon_i, \;\; i \in [0,1], \tag{27}$$

where a_i and m_i are constants and ϵ_i is normally distributed with mean 0 and precision n_i. With some normalization, we assume that $\int_i n_i m_i di = 0$, $\int_i n_i di = n$, and $\int_i a_i n_i di = n$.[9] We further define $h \equiv \int_i h_i di$ and $\mu \equiv \int_i h_i \mu_i di/h$.

We solve this dynamic maximization problem backward. We first solve the problem at time 1 for the second period. We then take the solutions for the second period as given and use them to solve the problem for the first period. We next provide a few key steps that are necessary to understand the equilibrium results to be presented in Theorem 2.

At time 1, there is only one period left, we obtain the following equilibrium price:

$$P_1 = \frac{h\mu + ny}{h+n} - \frac{\gamma x}{(h+n)},$$

that is, the price is the precision weighted average expectation minus the risk premium. At time 0, the expected utility for investor i has the following form:

$$\begin{aligned} \mathrm{E}_{1i}[U_i] &= -\exp\left\{-\gamma\left[W_{0i} + D_{0i}(P_1 - P_0) + D_{1i}(\mathrm{E}_{1i}[v] - P_1)\right] + \frac{\gamma^2}{2}\mathrm{Var}_{1i}[v]D_{1i}^2\right\} \\ &= -\exp\left\{-\gamma\left[W_{0i} + D_{0i}(P_1 - P_0)\right] - \frac{h_i + n_i}{2}\left[\frac{h_i\mu_i + n_i(a_i y - m_i)}{h_i + n_i}\right.\right. \\ &\quad \left.\left. - \frac{h\mu + ny}{h+n} + \frac{\gamma x}{(h+n)}\right]^2\right\}. \end{aligned} \tag{28}$$

distributions. However, the normality assumption allows the stock price to be expressed in terms of means and variances which make it easier to understand the intuition behind the results. We focus on the normality case here but the results for the binomial case are available on request.

[9] We can always redefine the signals for these normalization conditions to be satisfied. Suppose that $\int_i m_i n_i di = \bar{m}n$ and $\int_i a_i n_i di = \bar{a}n$. Let $y' = \bar{a}y - \bar{m}$, $a_i' = a_i/\bar{a}$, and $m_i' = m_i - a_i\bar{m}/\bar{a}$. Then $a_i'y' - m_i' = a_i y - m_i = v + \epsilon_i$ and we have $\int_i a_i' n_i di = n$ and $\int_i n_i m_i' di = 0$.

Taking the expectation with respect to y at time 0, we have

$$\mathrm{E}_{0i}[U_i] = \mathrm{E}_{0i}[\mathrm{E}_{1i}[U_i]] \propto -\int_y \exp\left[-\frac{(a_i y - \mu_i - m_i)^2}{2(h_i + n_i)/h_i n_i}\right] \mathrm{E}_{1i}[U_i] dy. \tag{29}$$

Combining terms, we can rewrite the expected utility as

$$\begin{aligned}
\mathrm{E}_{0i}[U_i] &\propto -\int_y \exp[-\alpha_i y^2 - \beta_i y - \delta_i] dy \\
&= -\int_y \exp[-\alpha_i \left(y - \frac{\beta_i}{2\alpha_i}\right)^2 + \frac{\beta_i^2}{4\alpha_i} - \delta_i] dy \\
&\propto -\exp\left[\frac{\beta_i^2}{4\alpha_i} - \delta_i\right],
\end{aligned} \tag{30}$$

where the expressions for α_i, β_i, and δ_i are given by

$$\alpha_i = \left[\frac{a_i^2 n_i}{2} - \frac{a_i n_i n}{h+n} + \frac{n^2(h_i + n_i)}{2(h+n)^2}\right], \tag{31}$$

$$\begin{aligned}
\beta_i = &\left(D_{0i}\frac{\gamma n}{h+n} - \frac{a_i h_i n_i (m_i + \mu_i)}{h_i + n_i}\right. \\
&\left. +(h_i + n_i)\left[\frac{h_i \mu_i - n_i m_i}{h_i + n_i} - \frac{h\mu - \gamma x}{h+n}\right]\left[\frac{a_i n_i}{h_i + n_i} - \frac{n}{h+n}\right]\right),
\end{aligned} \tag{32}$$

$$\delta_i = \gamma D_{0i}\left[\frac{h\mu - \gamma x}{h+n} - P_0\right]. \tag{33}$$

Dropping irrelevant terms, investor i's expected utility is proportional to

$$\mathrm{E}_{0i}[U_i] \propto -\exp\left[\frac{\beta_i^2 - 4\alpha_i \delta_i}{4\alpha_i}\right]. \tag{34}$$

The objective of investor i is to maximize $\mathrm{E}_{0i}[U_i]$ with respect to his demand function D_{0i}. The first-order condition is given by

$$\frac{\beta_i}{2\alpha_i}\frac{\partial \beta_i}{\partial D_{0i}} - \frac{\partial \delta_i}{\partial D_{0i}} = 0, \tag{35}$$

where

$$\frac{\partial \beta_i}{\partial D_{0i}} = \frac{\gamma n}{h+n}, \quad \frac{\partial \delta_i}{\partial D_{0i}} = \gamma \left[\frac{h\mu - \gamma x}{h+n} - P_0 \right]. \tag{36}$$

The optimal demand can then be determined. The following theorem summarizes the equilibrium results.

Theorem 2 *There exists a dynamic sequential equilibrium in which the stock price and demand in the first period are given by*

$$\begin{aligned} D_{0i} &= \frac{(h+n)^2}{\gamma n^2} \left\{ \left[\frac{h\mu - \gamma x}{h+n} - P_0 \right] \left[a_i^2 n_i - \frac{2a_i n_i n}{h+n} + \frac{n^2(h_i+n_i)}{(h+n)^2} \right] \right. \\ &+ \frac{n}{h+n} \left[a_i n_i m_i + a_i n_i \frac{h\mu}{h+n} + \frac{n}{h+n} \left[h_i \mu_i - n_i m_i - \frac{h\mu(h_i+n_i)}{h+n} \right] \right. \\ &\left. \left. + \frac{\gamma x (h_i+n_i)}{(h+n)} \left[\frac{a_i n_i}{h_i+n_i} - \frac{n}{h+n} \right] \right] \right\}, \end{aligned} \tag{37}$$

$$P_0 = \frac{(h\mu - \gamma x)\int_i a_i^2 n_i di + n \int_i a_i n_i m_i di}{h \int_i a_i^2 n_i di + n \int_i (a_i - 1)^2 n_i di}. \tag{38}$$

The equilibrium stock price and demand in the second period are given by

$$P_1 = \frac{h\mu + ny}{h+n} - \frac{\gamma x}{(h+n)}, \tag{39}$$

$$D_{1i} = \frac{1}{\gamma}[h_i \mu_i + n_i(a_i y - m_i) - (h_i + n_i)P_1]. \tag{40}$$

The detailed proof of this theorem is given in the appendix. It can be shown that in the homogeneous belief case in which all investors share the same belief as investor i, the equilibrium stock price would be determined according to investor i's belief about the asset value. It can be shown that the equilibrium stock price in this homogeneous case is given by

$$P_{0i} = \mu_{0i} - \gamma x / h_i,$$

where $\mu_{0i} = \mathrm{E}_{0i}[v]$. Let

$$P_{Max} = \max_i \{P_{0i}\}, \quad P_{Min} = \min_i \{P_{0i}\}.$$

For a bubble to occur, we must have

$$P_0 > P_{Max},$$

which reduces to

$$n \int_i a_i n_i m_i di > \left[h \int_i a_i^2 n_i di + n \int_i (a_i - 1)^2 n_i di \right] P_{Max} - (h\mu - \gamma x) \int_i a_i^2 n_i di. \tag{41}$$

For a panic to occur, we must have

$$P_0 < P_{Min},$$

which reduces to

$$n \int_i a_i n_i m_i di < \left[h \int_i a_i^2 n_i di + n \int_i (a_i - 1)^2 n_i di \right] P_{Min} - (h\mu - \gamma x) \int_i a_i^2 n_i di. \tag{42}$$

Notice that the left hand side of (41) or (42) depends on m_i, whereas the right hand side does not depend on m_i. Consequently, one can always adjust m_i so that the inequality is satisfied. For example, suppose that there are two groups of investors with equal proportion. The first group believes that $1.6y = 5 + v + \epsilon_1$ where the variance of ϵ_1 is 1. The second group believes that $0.4y = v - 3 - \epsilon_2$ where the variance of ϵ_2 is also 1. Assume that all other parameter values are given by 1. A calculation shows that inequality (41) holds. The first group of investors believes that signal y has a high mean and a low variance while the second group believes that the signal has a low mean and a high variance.

We next demonstrate that the first group is willing to pay a high price at time 0 because this group believes that the stock price will be high in the next period. To simplify the discussion, we further assume that $n_i = n$, $h_i = h$, and $x = 0$, where $x = 0$ removes the risk premium term from the price function. Under these conditions, the equilibrium stock price at time 0 reduces to

$$P_0 = \frac{h\mu \int_i a_i^2 di + n \int_i a_i m_i di}{h \int_i a_i^2 di + n \int_i (a_i - 1)^2 di}. \tag{43}$$

In the homogeneous case in which all investors share the same belief as investor i, the equilibrium stock price is given by

$$P_{0i} = \mu_{0i} \equiv \mu_i.$$

Let $\mu_{Max} = \max_i\{\mu_i\}$ and $\mu_{Min} = \min_i\{\mu_i\}$, then $P_{Max} = \mu_{Max}$ and $P_{Min} = \mu_{Min}$.

The condition for the existence of a bubble becomes

$$P_0 > P_{Max} = \mu_{Max},$$

which reduces to

$$n\int_i a_i m_i di > \left[h\int_i a_i^2 di + n\int_i (a_i-1)^2 di\right]\mu_{Max} - h\mu\int_i a_i^2 di.$$

Similarly, For a panic to occur, we must have

$$P_0 < P_{Min} = \mu_{Min},$$

which reduces to

$$n\int_i a_i m_i di < \left[h\int_i a_i^2 di + n\int_i (a_i-1)^2 di\right]\mu_{Min} - h\mu\int_i a_i^2 di.$$

As in the myopic model, for a bubble to occur today, investors including the most optimistic one must believe that he can sell the stock at an even higher price in the future. To capture this notion, we examine the conditional expectation at time 0 of the stock prices at time 1, based on investor i's belief. We obtain that

$$\mathrm{E}_{0i}[P_1] = \frac{h\mu + n\mathrm{E}_{0i}[y]}{h+n} = \frac{h\mu + n(\mu_i - m_i)/a_i}{h+n}.$$

Notice that this conditional expectation is different from the unconditional expectation of investor i, which is given by μ_{0i}. Indeed, it is possible that

$$\mathrm{E}_{0i}[P_1] = \frac{h\mu + nE_{0i}[y]}{h+n} = \frac{h\mu + n(\mu_i + m_i)/a_i}{h+n} > \mu_{Max},$$

which holds when

$$a_i m_i > \frac{a_i^2}{n}\left[(h+n)\mu_{Max} - h\mu\right] - a_i\mu_i. \tag{44}$$

Recall that in expression (44), $\mu_i = \mathrm{E}_{0i}[v]$, $\mu_{Max} = \max_i\{\mu_{0i}\}$, and $\mu = \int_i h_i\mu_i di/h = \int_i \mu_i di$. If inequality (44) holds, then it means that all investors' time 0 expected value of the time 1 stock prices is larger than the most optimistic investor's unconditional expected value of the stock payoff. In other words, investors expect the stock price to be higher in the next period so that they are willing to pay a higher price (than their valuations) today for the stock.

For an example, suppose that there are two types of investors, i and j, with equal proportion. Also suppose that $h = n = 1$ and that $\mu_i = 5$ and $\mu_j = 6$. We then have $\mu = 1/2 * (5+6) = 5.5$ and $\mu_{Max} = 6$. Inequality (44) becomes

$$a_i m_i > 5.5a_i^2 - 5a_i, \quad a_j m_j > 5.5a_j^2 - 6a_j.$$

It can be seen that there are numerous sets of a's and m's that satisfy the two inequalities. Under those conditions, both investors expect the next period price to be higher than 6 so that they are willing to pay more than 6 today for the stock.

Similarly, it is possible that

$$\mathrm{E}_{0i}[P_1] = \frac{h\mu + n\mathrm{E}_{0i}[y]}{h+n} = \frac{h\mu + n(\mu_i + m_i)/a_i}{h+n} < \mu_{Min},$$

which holds when

$$a_i m_i < \frac{a_i^2}{n}\left[(h+n)\mu_{Min} - h\mu\right] - a_i\mu_i.$$

There are some papers in the literature that employ risk-averse investors and without the short sales constraint. See, for example, Kandel and Pearson (1995), Allen, Morris, and Shin (2004), and Cao and Ou-Yang (2004). Those papers, however, do not arrive at bubbles and panics. We next offer a necessary condition for the existence of bubbles and panics. It shall be seen that none of the papers on both difference of opinion and asymmetric information satisfy this condition.

Theorem 3 *If investors interpret the signals according to Equation (27), then a necessary condition for bubbles and panics to occur in a normal-exponential framework is that some of the a_i's in Equation (27) are different from 1.*

Proof: Start with the expression for P_0,

$$P_0 = \frac{h\mu \int_i a_i^2 di + n \int_i a_i m_i di}{h \int_i a_i^2 di + n \int_i (a_i - 1)^2 di}.$$

If $a_i = 1$, then we have that $\int_i a_i m_i di = \int_i m_i di = 0$ and that

$$P_0 = \mu, \quad \text{and} \quad P_{Min} \leq P_0 \leq P_{Max}.$$

Because μ is the precision weighted average of all investors' expectations, it cannot be higher (lower) than the highest (lowest) expectation. **Q.E.D.**

Previous models universally use the relation

$$y - m_i = v + \epsilon_i$$

in which investors differ about the value of m_i and the precision of ϵ_i. This is the reason that they are unable to arrive at bubbles and panics with risk-averse investors.

5 Conclusion

It has been widely believed or even taken for granted that the short sales constraint is crucial for asset bubbles to occur [Harrison and Kreps (1978)] and that the short sales constraint can cause the stock price to be biased upward [Miller (1977)]. In this paper, we demonstrate that the insight of Harrison and Kreps is still robust even without the short sales constraint. Investors' differences of opinion about the stock payoffs alone can lead them to believe that the stock price in the future will be even higher, so that they are willing to pay a higher (than the most optimistic investor's valuation) price today for the stock. Consequently, an asset bubble arises. Our model generalizes the Harrison-Kreps notion of a resale option with risk-averse investors but without the short sales constraint. We also demonstrate that asset panics can occur due to a waiting option. In other words, differences of opinion can cause investors to believe that the future stock price will be even lower so that they would want to wait to participate in the market. Consequently, the stock price today must be low to induce investors to participate. We further show that the waiting option may be so valuable that the current stock price can be lower than all investors' valuations even with the short sales constraint binding. This result suggests that Miller's intuition that the short sales constraint causes the stock price to be biased upward may not be robust in a dynamic setting with the possibility of a waiting option.

Our results imply that it may be reasonable to observe that assets can be priced higher than the level that cannot be accounted for by shorting costs [Jones and Lamont (2002)] and that short interest may go down while certain asset bubbles are forming [Lamont and Stein (2004)]. Although shorting costs are generally very low already, they will inevitably be even lower in the future with the continual development of financial markets. Our exercise illustrates that both bubbles and panics may still arise if investors develop divergent views about the stock payoffs.

Appendix: Proof of Theorem 2

We solve this dynamic maximization problem backward. We first solve the problem at time 1 for the second period. We then take the solutions for the second period as given and use them to solve the problem for the first period.

At time 1, there is only one period left, investor i's conditional expectation of the stock payoff v is the precision weighted average of the unconditional mean and the signal:

$$\mathrm{E}_{1i}[v] = \frac{h_i\mu_i + n_i(v+\epsilon_i)}{h_i+n_i} = \frac{h_i\mu_i + n_i(a_iy - m_i)}{h_i+n_i},$$

and the conditional precision of v is the sum of the unconditional precision and the precision of the signal:

$$\mathrm{Var}_{1i}[v] = (h_i+n_i)^{-1}.$$

Let W_{1i} denote investor i's wealth at time 1. Investor i's optimization problem is given by

$$\max_{D_{1i}} \mathrm{E}_{1i}[U_i] = \max_{D_{1i}} \mathrm{E}_{1i}\left[-\exp(-\gamma W_{2i})\right] = \max_{D_{1i}} \mathrm{E}_{1i}\left[-\exp\left\{-\gamma\left[W_{1i} + D_{1i}(v-P_1)\right]\right\}\right],$$

where D_{1i} denotes investor i's demand for the risky stock, P_1 is the equilibrium stock price at time 1, and $W_{2i} = W_{1i} + D_{1i}(v - P_1)$ represents investor i's wealth or consumption at the terminal date $t = 2$. Because v is normally distributed, this maximization problem reduces to a mean-variance problem:

$$\max_{D_{1i}} \left[-\exp\left\{-\gamma\left[W_{1i} + D_{1i}(\mathrm{E}_{1i}[v] - P_1)\right] + \frac{\gamma^2}{2}\mathrm{Var}_{1i}[v]D_{1i}^2\right\}\right].$$

The first-order condition (FOC) for the optimal demand at time 1 is given by

$$\mathrm{E}_{1i}[v] - P_1 - \gamma\mathrm{Var}_{1i}[v]D_{1i} = 0,$$

yielding

$$D_{1i} = \frac{\mathrm{E}_{1i}[v] - P_1}{\gamma\mathrm{Var}_{1i}[v]} = \frac{1}{\gamma}[h_i\mu_i + n_i(a_iy - m_i) - (h_i+n_i)P_1].$$

In equilibrium, the aggregate demand equals the supply. We have

$$x = \int_i D_{0i} di = \int_i \frac{1}{\gamma}[h_i\mu_i + n_i(a_i y - m_i) - (h_i + n_i)P_1]di = \frac{1}{\gamma}[h\mu - n - (h+n)P_1].$$

Notice that we have used the normalization conditions that $\int_i h_i di = h$, $\int_i n_i m_i di = 0$, $\int_i a_i n_i di = n$, $\int_i n_i di = n$, and $\int_i h_i \mu_i di = h\mu$. Solving for the equilibrium price, we arrive at

$$P_1 = \frac{h\mu + ny}{h+n} - \frac{\gamma x}{(h+n)},$$

that is, the price is the precision weighted average expectation minus a risk premium term determined by the investor's risk aversion and the supply of the risky stock.

We can now solve investor i's demand function D_{0i} at time 0 using backward induction. Investor i's wealth at time 1 is given by

$$W_{1i} = W_{0i} + D_{0i}(P_1 - P_0),$$

where P_0 is the equilibrium stock price at time 0. At time 1 investor i's expected utility over his terminal wealth has the following form:

$$\begin{aligned} \mathrm{E}_{1i}[U_i] &= -\exp\left\{-\gamma\left[W_{0i} + D_{0i}(P_1 - P_0) + D_{1i}(\mathrm{E}_{1i}[v] - P_1)\right] + \frac{\gamma^2}{2}\mathrm{Var}_{1i}[v]D_{1i}^2\right\} \\ &= -\exp\left\{-\gamma\left[W_{0i} + D_{0i}(P_1 - P_0)\right] - \frac{h_i + n_i}{2}\left[\frac{h_i\mu_i + n_i(a_i y - m_i)}{h_i + n_i}\right.\right. \\ &\qquad \left.\left. - \frac{h\mu + ny}{h+n} + \frac{x}{\tau(h+n)}\right]^2\right\}. \end{aligned} \tag{45}$$

Taking the expectation with respect to y at time 0 and using the law of iterated expectations, we obtain investor i's expectation at time 0:

$$\mathrm{E}_{0i}[U_i] = \mathrm{E}_{0i}[\mathrm{E}_{1i}[U_i]] \propto -\int_y \exp\left[-\frac{(a_i y - \mu_i - m_i)^2}{2(h_i + n_i)/h_i n_i}\right]\mathrm{E}_{1i}[U_i]dy, \tag{46}$$

where the exponential function in the integral is the density function of y which is normally distributed. Combining terms, we can rewrite investor i's expected utility at time 0 as

$$\mathrm{E}_{0i}[U_i] \propto -\int_y \exp[-\alpha_i y^2 - \beta_i y - \delta_i] dy$$

$$= -\int_y \exp\left[-\alpha_i \left(y - \frac{\beta_i}{2\alpha_i}\right)^2 + \frac{\beta_i^2}{4\alpha_i} - \delta_i\right] dy \propto -\exp\left[\frac{\beta_i^2}{4\alpha_i} - \delta_i\right], \tag{47}$$

where the expressions for α_i, β_i, and δ_i are given by

$$\alpha_i = \left[\frac{a_i^2 n_i}{2} - \frac{a_i n_i n}{h+n} + \frac{n^2(h_i+n_i)}{2(h+n)^2}\right], \tag{48}$$

$$\begin{aligned}\beta_i \quad = \quad & \Bigg(\gamma D_{0i}\frac{n}{h+n} - \frac{a_i h_i n_i (m_i+\mu_i)}{h_i+n_i} \\ & +(h_i+n_i)\left[\frac{h_i\mu_i - n_i m_i}{h_i+n_i} - \frac{h\mu - \gamma x}{h+n}\right]\left[\frac{a_i n_i}{h_i+n_i} - \frac{n}{h+n}\right]\Bigg),\end{aligned} \tag{49}$$

$$\delta_i = \gamma D_{0i}\left[\frac{h\mu - \gamma x}{h+n} - P_0\right]. \tag{50}$$

Dropping irrelevant terms, investor i's expected utility is proportional to

$$E_{0i}[U_i] \propto -\exp\left[\frac{\beta_i^2 - 4\alpha_i\delta_i}{4\alpha_i}\right]. \tag{51}$$

The objective of investor i is to maximize $E_{0i}[U_i]$ with respect to his demand function D_{0i}. From Equation (51), the FOC is given by

$$\frac{\beta_i}{2\alpha_i}\frac{\partial \beta_i}{\partial D_{0i}} - \frac{\partial \delta_i}{\partial D_{0i}} = 0, \tag{52}$$

where

$$\frac{\partial \beta_i}{\partial D_{0i}} = \left(\gamma\frac{n}{h+n}\right), \quad \frac{\partial \delta_i}{\partial D_{0i}} = \gamma\left[\frac{h\mu - \gamma x}{h+n} - P_0\right]. \tag{53}$$

The FOC reduces to

$$
\begin{aligned}
&\frac{n}{h+n}\left[\gamma D_{0i}\frac{n}{h+n}-a_in_im_i-a_in_i\frac{h\mu}{h+n}-\frac{n}{h+n}\left[h_i\mu_i-n_im_i-\frac{h\mu(h_i+n_i)}{h+n}\right]\right.\\
&\left.+\frac{\gamma x(h_i+n_i)}{(h+n)}\left[\frac{a_in_i}{h_i+n_i}-\frac{n}{h+n}\right]\right]\\
&-\left[\frac{h\mu-\gamma x}{h+n}-P_0\right]\left[a_i^2n_i-\frac{2a_in_in}{h+n}+\frac{n^2(h_i+n_i)}{(h+n)^2}\right]=0. \qquad (54)
\end{aligned}
$$

We then arrive at the optimal demand for investor i at time 0:

$$
\begin{aligned}
D_{0i} &= \frac{(h+n)^2}{\gamma n^2}\left\{\left[\frac{h\mu-\gamma x}{h+n}-P_0\right]\left[a_i^2n_i-\frac{2a_in_in}{h+n}+\frac{n^2(h_i+n_i)}{(h+n)^2}\right]\right.\\
&+\frac{n}{h+n}\left[a_in_im_i+a_in_i\frac{h\mu}{h+n}+\frac{n}{h+n}\left[h_i\mu_i-n_im_i-\frac{h\mu(h_i+n_i)}{h+n}\right]\right.\\
&\left.\left.+\frac{\gamma x(h_i+n_i)}{(h+n)}\left[\frac{a_in_i}{h_i+n_i}-\frac{n}{h+n}\right]\right]\right\}. \qquad (55)
\end{aligned}
$$

Using the market clearing condition, $x=\int_i D_{0i}di$, we have

$$
\begin{aligned}
x &= \int_i D_{0i}di=\frac{(h+n)^2}{\gamma n^2}\left\{\left[\frac{h\mu-\gamma x}{h+n}-P_0\right]\left[\int_i a_i^2n_idi-\frac{n^2}{h+n}\right]\right.\\
&\left.+\frac{n}{h+n}\left[\int_i a_in_im_idi+n\frac{h\mu}{h+n}\right]\right\}, \qquad (56)
\end{aligned}
$$

which reduces to

$$
P_0\left[\int_i a_i^2n_idi-\frac{n^2}{h+n}\right]=\frac{h\mu-\gamma x}{h+n}\left[\int_i a_i^2n_idi\right]+\frac{n}{h+n}\int_i a_in_im_idi.
$$

Consequently, we arrive at the equilibrium stock price at time 0:

$$
P_0=\frac{(h\mu-\gamma x)\int_i a_i^2n_idi+n\int_i a_in_im_idi}{h\int_i a_i^2n_idi+n\int_i(a_i-1)^2n_idi}. \qquad (57)
$$

Q.E.D.

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