A Fully-Rational Liquidity-Based Theory of IPO Underpricing and Underperformance

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Abstract

I present a fully-rational symmetric-information model of an IPO, as well as a dynamic imperfectly competitive model of the aftermarket trading that follows. The model helps explain why IPO share allocations favor large institutional investors. It also helps to explain IPO underpricing, and underperformance, and the large fees charged by underwriters. The critical assumption in the model is that underwriters need to sell a fixed number of shares at the IPO or soon thereafter in the aftermarket, but they want to avoid selling in the aftermarket because there are some aftermarket investors who have market power and can affect the prices received by the underwriter. To maximize revenue and avoid unnecessary aftermarket sales, the underwriter distorts share allocations toward those those investors who have market power, and he sets the offer price at the IPO below the aftermarket price that will prevail shortly after the IPO. In the aftermarket model, I show that there are share allocations that can generate arbitrarily high levels of return underperformance for very long periods of time. In some simulations, the distorted share allocations at the IPO generate return underperformance that persists for more than one year. The underwriter can dilute investor's market power by participating for longer periods of time in the aftermarket. By doing so, he sometimes substantially increase the revenue that is raised by the IPO issuer.

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1 Introduction

Two of the principle functions of a well performing financial system are to facilitate risk sharing among investors, and capital formation by firms. The initial public offering (IPO) process serves both of these functions by allowing the initial owners of a firm to raise capital while simultaneously transferring and sharing some of the firm's risk with the wider investing public.

IPOs are special events in capital markets because the amounts of risk that are transferred during the share allocation process of an IPO dwarfs the amount of risk that is transferred during the regular trading process for individual stocks. If the IPO risk transfer process was fully efficient, then the investors who place the most value on the shares should receive them, and they should pay a high price. Additionally, in the absence of firm specific news or private information, there should be little trading volume after the shares are initially allocated. Relative to this efficient benchmark, IPOs appear to be highly inefficient: share trading is very heavy on the first day after a share has been allocated.[1] Additionally, shares are apparently allocated at too low a price: the closing share price on the first trading day of U.S. IPO's is on average about 17 percent higher than the price at which the shares were allocated earlier in the day. This phenomen, known as IPO underpricing, represents a loss of revenue to the issuer who could presumably do better by selling directly at the high prices that occur in the aftermarket following the IPO.

In addition to underpricing and frequent trading, the returns on newly issued shares underperform; that is, the returns on new issues underperform the market and underperform the returns of shares of firms that have the same risk characteristics, but are not new issues [Loughran and Ritter (1991), Ritter and Welch (2002)]. An additional source of inefficiency is that underwriters charge and receive very high fees for their services; these fees are equal to about 7% of the revenues raised in the new issue.

The goal of this paper is to present a single, fully-rational, symmetric information, theoretical model that helps to explain both IPO underpricing and underperformance. The model is also used to attempt to rationalize the high fees charged by underwriters. The results on underwriters are still highly preliminary, but encouraging. In particular, in some circumstances the underwriters trading activities in the aftermarket were found to add 25% to the total proceeds raised by the issue. The theory is also consistent with the stylized facts that investors are often rationed at the IPO offer price, and institutional investors receive a disproportionate proportion of the share allocation.

The principle insight in the paper is that inefficient risk sharing through the regular trading process that follows the IPO can generate equilibrium underpricing and underperformance, as well as a tilt of initial asset holdings towards institutional investors. The model has three principal features that generate these results. First, the underwriter in the IPO

[1] In Ellis, Michaely, and O'Hara's (2000) study of NASDAQ IPO's, they report that a stock's daily turnover (measured as a percentage of shares traded) on its first trading day following its IPO is equal to about 1/3rd of the turnover that a typical NASDAQ stock experiences over an entire year.

has a fixed number of shares that need to be sold either during the IPO, or shortly thereafter in the IPO aftermarket. Second, I assume that participation in the IPO and aftermarket trading is limited, that is, only a tiny fraction of the economy's investors that could participate in the IPO and trade in the aftermarket actually choose to do so. Third, I assume that trading in the IPO aftermarket is imperfectly competitive—the imperfect competition takes the form that some investors in the IPO aftermarket are large investors whose trades move prices. Because the large investors trades move prices, they cannot buy and sell all that they want at current prices and hence the market is not liquid. This illiquidity has implications for equilibrium expected returns in the aftermarket. In addition, because large investors trades move prices, they have market power in aftermarket trading; this market power provides large investors with bargaining power vis-a-vis the underwriter during the share allocation and price setting process. The bargaining power takes the form that if underwriters set too high an offer price in the IPO, or if they offer large investors too few shares, then the large investors will turn down the offered shares and force the underwriter to instead sell his shares at low prices in the imperfectly competitive IPO aftermarket. To avoid this outcome, the underwriter optimally distorts his initial asset allocations towards large investors with market power in the aftermarket, and he offers shares at a price that is below the price in the IPO aftermarket. Given that the price in the aftermarket will be higher than the IPO offer price, it appears that the underwriter could benefit by offering fewer shares in the initial allocation in order to immediately sell more shares in the aftermarket. However, that would not be incentive compatible for the large investors — they would be better off deviating and forcing the underwriter to dump shares in the aftermarket. In the end, the underwriter would actually lose money by following such a strategy.

If the underwriter did not have to sell shortly after the IPO, but could instead sell shares not allocated at the IPO over a much longer period following the IPO, then doing so, as well as the threat of doing so, dilute the market power of the large investors. In some circumstances these aftermarket "stabilization" activities were found to substantially increase the revenues received by the issuer.

In the IPO aftermarket, investors trade multiple risky assets and a perfectly liquid proxy for the market portfolio over T trading periods. The risky assets are assumed to represent the assets of firms that belong to a particular market segment or group (for example firms that produce semi-conductor parts); the investors in the model are assumed to be the only investors that trade the assets of that market segment. Because the investors in the model represent a small proportion of the investors in the economy, I assume their trades have no impact on the price for the market proxy. As a result, the investors can hedge the market component of their risky assets' returns; and they trade among themselves to share the nonmarket component of the assets returns. In the model I show that the nonmarket component of the returns are priced; that is, they receive a reward for their risk. This reward for risk is a result of the assumption that only a small set of investors trade the assets of the segment; the reward is present even when asset markets are perfectly competitive. An independent contribution of this paper is that it provides a model for why nonmarket risk might be priced. I also that under some circumstances, this feature of the model is a channel that can generate return underperformance following the IPO.

A second channel through which the model generates underperformance is present because of imperfect competition. The most important feature of the IPO aftermarket is that large investors trades have endogenous price impact; i.e. the more shares that a large investor buys or sells of assets in the segment, the more his trade moves their prices. I refer to the large investors as having endogenous price impact because the price impact does not depend on an exogenous transaction cost, instead it depends on his own and other investors risk preferences. Because trades move prices, large investors will not immediately sell an asset because its return underperforms; instead they sell the asset slowly through time to minimize the price impact of their trades.[2] Put differently, from an initial inefficient asset allocation, investors do not immediately trade back to efficient holdings; instead in the adjustment process the imperfect competition model generates equilibrium trajectories of trades and returns. Because of the slow adjustment process, by altering investors initial asset allocations it is possible to generate arbitrarily large amounts of underpricing for all T time periods of aftermarket trade. Whether underperformance actually results and persists depends on how assets are allocated at the IPO. The preliminary results on underpricing are encouraging. In some circumstances, the equilibrium allocations at the IPO generates post-IPO underpricing relative to the return on the market portfolio that persists for longer than one year.

It is important to stress that all of the results in the paper are generated by a model in which all market participants are fully rational. In addition, there is no asymmetric information of any kind.

There is a voluminous literature on IPO underpricing and a smaller literature on underperformance.[3] One strand of the underpricing literature is based on information-asymmetries in asset markets. In Rock (1986), investors who are less well informed about IPO firms quality face adverse selection in the share allocation portion of the IPO process — they are allocated too many shares in bad IPO's and too few shares in good IPOs. In Rock, underpricing is a mechanism to entice these investors to participate in the IPO process. In the bookbuilding literature that begins with Benveniste and Spindt (1989) and has since been refined by many others, some investors have private information about the value of the IPO firm. The IPO is a mechanism that is designed to raise money for the issuer while simultaneously eliciting information from the informed investors in an incentive compatible way.[4] Other rational theories of underpricing are based on the underwriter deliberately underpricing in order to generate trading revenue for himself in the IPO aftermarket (Boehmer and Fishe, 2000), or the underwriter colluding with other investors against the issuer (Bias et. al. 2002).

At the present time I am aware of two theoretical papers on the relationship between illiquidity and underpricing. In Booth and Chua (1996), IPO underpricing is used to en-

[2]Jenkinson and Jones (2004) report that share allocations in IPOs are tilted towards investors who hold onto them instead of rapidly selling them after the IPO.

[3]Recent reviews of this literature are provided by Ritter and Welch (2002) and Ljungvist (2004).

[4]The mechanism involves tilting share allocations towards investors who indicate that they have favorable information by increasing those investors share allocations in the IPO. The IPO offer price is also increasing in favorable information, but it is kept below the expected aftermarket trading price to ensure that investors receive rents from the underwriter in exchange for revealing their information.

courage investors to obtain costly information about the IPO. It is assumed that investors who gather such information become part of the base of investors that trade the IPO shares. The added liquidity is assumed to enhance the value of the firm.[5] A prediction of the Booth and Chua model is that underpricing is positively correlated with liquidity.

In Ellul and Pagano (2003), some investors that participate in the IPO may need to sell their share holdings soon thereafter into an illiquid IPO aftermarket. These investors require a liquidity premium to participate in the IPO. The liquidity premium takes the form of IPO underpricing. A prediction of the Ellul and Pagano model is that liquidity is negatively correlated with underpricing: in a more liquid market less underpricing is required.

The Ellul and Pagano model is essentially a three period model, but it is very rich in some dimensions. For example, it incorporates asymmetric information, illiquidity, and risk averse investors within the same framework. Additionally, their paper contains a substantial empirical section where they show that more illiquidity after the IPO is associated with more IPO underpricing.

My model does not contain any of the information asymmetries or informational costs that are captured by Ellul and Pagano and Booth and Chua. But, my model makes two contributions to the theoretical liquidity literature in other dimensions. First, the Booth and Chua and Ellul and Pagano models contain at most 1 period of aftermarket trade. This is too short a number of trading periods to model underperformance of returns in the IPO aftermarket. By contrast, my model of aftermarket trading is fully dynamic, which allows me to study how illiquidity is related to underperformance. Second, the investors in Booth and Chua and in Ellul and Pagano are competitive. By contrast the large investors in my model are strategic at the IPO and afterwards. Therefore, I use my model to study how the strategic situation between the underwriter and the large investors influences underpricing, and underperformance. I also study how the underwriters actions in the aftermarket can increase the IPO proceeds by altering the strategic environment.

There is a small theoretical literature on underperformance. Ritter and Welch (2002) claim that there are no rational theoretical models of IPO underperformance. If so, then a rational theory of underperformance is a unique contribution of this paper. In a very interesting paper, Ljungqvist, Nanda, and Singh (2003) present a behavioral model of IPO underpricing and underperformance. The key behavioral assumption in their model is that there are irrationally exuberant sentiment investors in the IPO aftermarket; and the demand of these investors may grow through time, or the sentiment might abruptly end and prices would collapse. In the paper they show that if the underwriter is constrained from selling in the IPO aftermarket above the IPO offer price because of legal constraints, then the first best strategy for the underwriter is for the underwriter to sell to one group of regular rational investors at the IPO, and then this group sells to the sentiment investors over time. The

[5] Westerfield (2003) is similar to Booth and Chua in that underpricing is used to change the base of investors in the IPO aftermarket. In Westerfield, there are irrational noise traders, and their presence in the investor base reduces the value of the IPO'd asset because a risk premium is required for noise-trader risk. Underpricing is assumed to reduce the relative share of noise traders in the investor-base, and hence enhances the value of the firm.

presence of the sentiment investors causes underperformance in the IPO aftermarket and it also raises the value of the IPO'd assets for regular investors and leads to a higher IPO offer price. IPO underpricing results in this framework because the regular investors need to break up their sales to the sentiment investors in the aftermarket through time, and in breaking up their sales they run the risk that the sentiment will collapse before they can sell. Underpricing at the IPO is required to compensate regular investors for this risk.

The underpricing and underperformance in my model resembles that in Ljungqvist, Nanda, and Singh. In both models, the IPO underwriter sells to one group of investors, and that group of investors turns around and sells their assets over time to a third group. Another similarily is that Ljungqvist, Nanda, and Singh assume that that their regular investors behave strategically by coordinating their actions, while in my paper the amount of underpricing and underperformance is related to the amount of competition in the aftermarket. These strategic aspects of our aftermarket models are very similar. The important difference between our models is that all of the investors in my framework are rational; their demands are derived from first principals; and I make no behavioral assumptions. Nevertheless in my model, I too can generate both underpricing and underperformance, as well as a potential explanation for large underwriter fees.

The rest of the paper proceeds in six parts. Section 2 provides a brief overview of the entire model. Section 3 provides detail on the IPO aftermarket; the section also presents the main results on asset pricing, and return underperformance. Section 4 provides detail on the model of the IPO process and provides intuition for the results on underpricing; section 5 provides all of the simulation evidence including the results on underpricing and underperformance; section 6 reviews the empirical evidence on liquidity and IPO underpricing and underperformance; a final section concludes.

2 Model Overview

Our basic model involves a stylized IPO in which a firm that wishes to raise money by selling X^{IPO} shares of stock through an IPO enlists a single underwriting firm to market the issue. We will assume that any contractual arrangements between the underwriter and the issuer incentivize the underwriter to maximize the revenues from selling the shares through the IPO process or in the aftermarket that follows the IPO.[6] In addition to the underwriter, there is a finite set of M risk-averse investors who have an interest in absorbing the issue. Investor 1 actually represents a continuum of small price-taking investors who can each choose whether or not to participate in the IPO. I will sometimes refer to the small investors as retail investors. Investors 2 through M are large investors whose desired trades in the aftermarket are large enough to move asset prices. The large investors will sometimes be referred to as institutional investors. I assume that the investors who participate in the IPO are the same investors that trade in the aftermarket for shares following the IPO. The

[6] Some of the research in the IPO literature attributes underpricing to agency problems between the underwriter and the issuer [Biais, Bossaerts, and Rochet (2002); Boehmer and Fishe (2000)].

process for setting the IPO offer price and share allocations is modeled as a two-stage game. The first stage closely resembles bookbuilding, which is a process for allocating shares and setting an IPO offer price that is often used in the United States. In the bookbuilding process here, the underwriter learns about demand conditions by learning about the risk preferences, and asset holdings, of the investors in the segment. He also uses his knowledge about the aftermarket to make inferences about investors market power. To rule out the possibility that differences in investors information could drive the results in the model, I assume that information on investors risk preferences, asset holdings, and the entire model of aftermarket trading is publicly available at all points in time; and is common knowledge. Given the demand information, the underwriter sets a uniform IPO price, and proposes take-it or leave it share allocations to each of the investors that is potentially interested in the share offering. In the second stage, investors decide whether to accept their allocations. If some investors turn down their share allocations, then the underwriter sells the remaining shares in the aftermarket. Investors incentives to turn down allocations depend on their market power. The underwriter accounts for this when choosing the IPO offer price and share allocations. The next section formally models the IPO aftermarket; and the following section models the share allocation and price-setting process at the IPO.

3 The IPO aftermarket

The model of trading in the IPO aftermarket is a partial equilibrium extension of Pritsker's (2004) multiple-asset heterogeneous agent model of imperfect competition in asset markets.[7] Investors trade two sets of risky assets and a risk-free asset. The first set of of N^1 risky assets are the shares of firms that belong to a particular market segment. The new issue is one of the assets within the segment. The next N^2 risky assets are perfectly liquid proxies systematic risk factors that are priced in the economy. For symplicity, I assume that the only systematic risk factor is the market portfolio; perfectly liquid proxies for this portfolio could be the returns on the $S\&P$ 500 index or index futures.

I assume that only a tiny fraction of investors in the economy participate in the IPO and trade in the IPO aftermarket. The partial equilibrium aspect of the model is that I assume these investors are so small relative to the entire market that their collective trades have no effect on interest rates or on the market return. In other words, interest rates, and the market return are exogenous. However, the actions of the investors, collectively, and sometimes individually do affect the returns of assets in the first segment. More specifically, the model contains M infinitely lived investors that have potentially heterogeneous risk preferences. As noted above investor 1 represents the trades of small investors that are formally modelled as a continuum of infinitesimal investors who each take prices as given. In addition, there are $M-1$ large investors whose desired trades in the new issue are large enough to move prices and who take their price impact into account when making their trading decisions.

[7] Closely related models of imperfect competition in asset markets include Urosevic (2002a & b), DeMarzo and Urosevic (2000), and Vayanos (2001).

Although the investors and assets are infinitely lived, I assume that shares of assets in the first segment can only be traded for a large but finite number of periods T. After period T investors continue to hold their first segment shares; they continue to trade all other assets; they continue to receive dividends; and they continue to consume. The assumption that there is a final period of trade T facilitates solution of the model through the use of backwards induction techniques from the last period of trade. That said, assuming there is no trade after period T is tantamount to an assumption that market liquidity for assets in the first segment eventually dries up. The time until the liquidity dries up influences the dynamic behavior of the model.

The Assets

Investors trade in a risk-free asset and two sets of risky assets. Because the investors represent a tiny fraction of the economy's investors, I assume their actions cannot influence the return on the risk-free asset. For simplicity, the gross per-period risk free rate of return is fixed at $r > 1$. The time t prices of the first and second set of risky assets are denoted $P^1(t)$ and $P^2(t)$ respectively. $P(t)$ denotes the stacked vector of risky asset prices at time t. Similar naming conventions will be followed throughout the rest of the paper. The risky assets pay dividends $D(t)$ in each period and dividends are distributed i.i.d. normally through time:

$$D(t) \sim \text{i.i.d.} \quad \mathcal{N}(\bar{D}, \Omega) \tag{1}$$

Because dividends are normally distributed, the risky assets are not limited liability instruments; and hence their share price can drop below zero. Because of this possibility, the returns in excess of the risk free rate are best expressed in units of return per share instead of units of return per dollar invested. This means that assets excess return over the riskless rate per share are given by the vector:

$$Z(t) = P(t) + D(t) - rP(t-1) \tag{2}$$

Because the model is partial equilibrium, I assume that $P^2(t)$ is exogenous, and for simplicity, fixed for all $t = 1, \ldots \infty$. This implies that excess returns on the market portfolio are i.i.d. through time with mean $\bar{Z}^2$ and variance $\text{Var}(Z)^2$.

It is useful to decompose the return on the in the first segment into a systematic component that is perfectly correlated with the market and into a nonmarket component $e(t)$:

$$Z^1(t) = \beta_{12} Z^2(t) + e(t) \tag{3}$$

where $\beta_{12} = \frac{\text{Cov}[Z^1(t), Z^2(t)]}{\text{Var}[Z^2(t)]}$ is asset 1's beta coefficient in the CAPM.

The systematic component of the first set of assets' return is hedgeable risk because its risk can be offset by trading asset 2. The nonmarket component of returns is not hedgable, but it can usually be diversified under the assumption that a very large number of investors can each take a very small piece of the nonmarket risk. In the current setting, my assumption that the risk is shared across a set of M investors means that although $e(t)$ is nonmarket risk, it is not diversifiable. Therefore, the expected return for holding it will not necessarily be equal to 0. The variance of $e(t)$ is denoted Ω_e; in equilibrium it turns out to be constant through time; in equilibrium β_{12}, the vector of CAPM betas, is constant through time as well.

Investors

There are M investors in the model. With great loss of generality, each investor m is assumed to have a per period utility of consumption that takes the CARA form with absolute risk aversion parameter A_m. Investors choose their consumption and asset holdings to maximize their discounted expected CARA utility of consumption:

$$U_m(C_m(1), ...C_m(\infty)) = \sum_{t=1}^{\infty} -\delta^t e^{-A_m C_m(t)}. \tag{4}$$

Investor $m's$ holdings of risky assets at the beginning of time t is denoted by $Q_m(t)$ which is the stacked vector of his holdings of both sets of risky assets. The change in his risky asset holdings is denoted by $\Delta Q_m(t)$ $(= Q_m(t+1) - Q_m(t))$. In each period investors end of period wealth that is not consumed or held as stock, is put in riskfree bonds. Under this assumption, investors choose their consumption and risky asset holdings subject to the standard set of intertemporal budget constraints:

$$W_m(t) = Q_m(t-1)'Z(t) + r[W_m(t-1) - C_m(t-1)] \quad t = 1, \ldots T, \tag{5}$$

where $W_m(t)$ denotes total wealth at the beginning of time t.

Although the budget constraint will be formally satisfied for all investors, the interpretation of $W_m(t)$ is different for large and small investors. In particular because small investors are infinitesimal, a small investor can liquidate his wealth at its pre-liquidation market value. By contrast, if a large investor attempted to liquidated his holdings of asset 1, his trades would move the price; therefore he would not be able to recover full value. Because large investors cannot recover all of their wealth by immediately selling, in their portfolio choices they make a distinction between perfectly liquid wealth which can be sold without any loss in value, and illiquid wealth. Consequently, it is useful to express large investors set of intertemporal budget constraints in terms of the evolution of liquid wealth. Investor $m's$ liquid wealth at the beginning of time t, denoted by $W_{ml}(t)$, consists of dividends on their beginning of time t share holdings plus the value of their bond portfolio plus the market value of asset 2. The intertemporal budget constraints expressed in terms of liquid wealth have form:

$$W_{ml}(t) = Q_m^1(t)'D^1(t) + Q_m^2(t)'Z^2(t) \\ + r\left[W_{ml}(t-1) - \Delta Q_m^1(t-1)'P^1(t-1) - C_m(t-1)\right] \quad t = 1,\ldots T. \tag{6}$$

It will turn out that when there is imperfect competition, the stacked vector of investors holdings of assets in the first segment is a crucial state variable. This state variable is denoted by denoted by $Q^1(t) = \text{vech}(Q_1^1(t)', \ldots, Q_M^1(t)')'$.

Trading Dynamics

In each time period $t \leq T$, investors enter the period with with their holdings $Q_m(t), m = 1,\ldots M$. They receive dividends on their risky asset holdings; their risky asset trades $\Delta Q_m(t)$ and risky asset prices $P(t)$ are jointly determined; investors then make their consumption choices, and then the period ends.

The process of trade for the first set of assets is modeled as a dynamic Cournot-Stackelberg game of full information. In each period $t \leq T$, the strategic environment is described by the state variable $(Q^1(t), t)$. Given the strategic environment, the set of small investors individual asset demands form a price schedule that describes the set of market clearing prices for the first set of assets at which the small investors are willing to absorb all possible quantities of the large investors orderflow for those assets. Given this price schedule, large investors play a Cournot game in period t in which they take the price schedule and other investors trades as given. They then choose their own trades while accounting for the effect that their trades have on prices. Large investors equilibrium trades in each period are a Cournot Nash equilibrium within the period. The entire model of trading is solved by backwards induction from period T; therefore investors optimal trading strategies are subgame perfect. It important to emphasize that although small investors take prices as given, both the large and small investors are perfectly rational and take the strategic environment into account when forming their asset demands. It is also important to emphasize that although the discussion focuses on investors demand for the first set of assets, implicit in their demands are their optimal choices for their holdings of the market porfolio.

To illustrate the derivation of the price schedule at period t, without loss of generality assume that an equilbrium price function has been derived for time $t+1$ that maps investors holdings of the first set of assets at the beginning of period $t+1$ into equilibrium prices during time $t+1$.[8] The presence of such a price function is necessary so that small investors can compute their expected future wealth at time $t+1$. Given the price function at time $t+1$, imagine that at the beginning of period t the state ariable carried over from the end of the previous period is $Q^1(t)$. After the period begins, large investors submit risky-asset orderflow $\Delta Q_m^1(t)$, $m = 2, \ldots M$. Based on this orderflow, there exists a market clearing

[8]This is without loss of generality because I derive equilibrium price functions for all trading periods using dynamic programming from time infinity until the last period of trade, and then backwards induction from the last period of trade.

price $P^1(.,t)$, for which the risky asset trade vector $\Delta Q_s(t)$, of each infinitesimal investor s, $s \in [0,1]$, solves the maximization problem:

$$\max_{C_s(t),\Delta Q_s(t)} -e^{-A_s C_s(t)} + \delta E_t\{V_s(W_s(t+1); Q^1(t) + \Delta Q^1(t), t+1)\}, \tag{7}$$

subject to the budget constraint,

$$W_s(t+1) = Q_s(t+1)' Z(t+1) + r[W_s(t) - C_s(t)]$$

where, $Q_s(t+1) = Q_s(t) + \Delta Q_s(t)$.

Equation (7) represents the portfolio choice and consumption problem of each small investor in its dynamic programming form. The arguments of small investors value function are time, their future wealth, and the state variable $Q^1(t+1) = Q^t + \Delta Q^1(t)$. Note that the state variable $Q^1(t+1)$ affects the demand of each small investor, but because each small invstor is infinitesimal, his asset demands do not affect the state variable.

For the price schedule $P(.,t)$ to be market clearing, each small investors net purchases of the first set of risky assets, denoted by $\Delta Q^1_s(t)$ must satisfy equation (7) and prices must be set so that the net orderflow of the small and large investors sums to 0.

$$\int_0^1 \Delta Q^1_s(t)\ ds + \sum_{m=2}^{M} \Delta Q^1_m(t) = 0 \tag{8}$$

The price schedule must also be consistent with an additional internal consistency condition for small investors orderflow. Recall that small investors are infinitesismal. This means that they take the orderflow of the other small investors as given and treat it as a state-variable. For small investors beliefs about the state variable to be internally consistent, $\Delta Q^1_1(t)$, their beliefs about the net trades of all small investors in equation (7), must be consistent with the optimal behavior of small investors conditional on their beliefs; i.e. internal consistency requires that[9]:

$$\Delta Q^1_1(t) = \int_0^1 \Delta Q^1_s(t) ds \tag{9}$$

For any given set of trades by the large investors, I solve for equilibrium prices which satisfy the market clearing and internal consistency conditions. Each such price $P^1(.,t) = P^1(\Delta Q^1(t), Q^1(t), t)$ is one point on the price schedule which is faced by the large investors. The full price schedule is found by solving the above problem for all possible $Q^1(t)$ and all possible $\Delta Q^1(t)$. The resulting price schedule turns out to a linear function of the elements of $Q^1_m(t)$ and $\Delta Q^1_m(t)$, $m = 2, \ldots M$:

[9] $\Delta Q^1_1(t)$ corresponds to the first row of the $Q^1(t) + \Delta Q^1(t)$ argument of the small investors value function in equation (7).

$$P^1(.,t) = \frac{1}{r}\left(\beta_0(t) - \beta_{Q^1}(t)Q^1(t) - \sum_{m=2}^{M}\beta_m(t)\Delta Q^1_m(t)\right), \tag{10}$$

where $(1/r)\beta_m(t)$ is the slope of the price schedule with respect to ΔQ^1_m, large investor m's orderflow for asset 1 at time t.

Given the demand curve in equation (10), large investors solve the dynamic programming problem:

$$\max_{\substack{C_m(t),\\ \Delta Q_m(t)}} \quad -e^{-A_m C_m(t)} - \delta\, \mathrm{E}_t\, V_m(W_{ml}(t+1), Q^1(t) + \Delta Q^1(t), t+1) \tag{11}$$

subject to the budget constraint:

$$\begin{aligned} W_{ml}(t+1) = {} & Q^1_m(t+1)'D^1(t) + Q^2_m(t+1)'Z^2(t) \\ & + r\left[W_{ml}(t) - \Delta Q^1_m(t)'P^1(.,t) - C_m(t)\right] \end{aligned} \tag{12}$$

where $Q_m(t+1) = Q_m(t) + \Delta Q_m(t)$.

In equation (11), the arguments of large investors value function include time, liquid wealth $W_{ml}(t+1)$, and the state variable $Q^1(t+1)$. Note that each large investors trades affect the state variable and they account for their own effect on the state variable when trading. Finally, $P^1(.,t)$, the price schedule faced by large investors from equation (10) appears in the budget constraint and large investors account for the impact of their own trades on prices when they trade.

The trade and consumption choices of large and small investors are an equilibrium, if given those choices, small investors investors trades and consumption choices solve equation (7), large investors trade and consumption choices satisfy equation (11), and investors choices satisfy the market clearing and internal consistency conditions given in equations (8) and(9). Finally, large and small investors value functions in every time period have to satisfy the Bellman equation.

The form of investors value functions, and the form of the equilibrium price function in each period is given in the following proposition:

Proposition 1 *Each small investors value function for entering period $t \leq T$ with wealth W_s, when the state vector of investors holdings of illiquid assets is Q^1 is given by:*

$$\begin{aligned} V_s(W_s, Q^1, t) &= -K_1(t)\ F(Q^1, t)\ e^{-A_s(t)W_s}, \\ \textit{where } F(Q^1, t) &= e^{-Q^1(t)'\bar{v}_s(t) - Q^1(t)'\theta_s(t)Q^1(t)}. \end{aligned} \tag{13}$$

Large investor m's value function for entering period $t \leq T$ when the state vector of illiquid asset holdings is Q^1 and his liquid wealth is W_{ml} is given by:

$$V_m(W_{ml}, Q^1, t) = -K_m(t)e^{-A_m(t)W_{ml} - A_m(t)Q^{1\prime}\Lambda_m(t) + .5A_m(t)^2 Q^{1\prime}\Xi_m(t)Q^1} \qquad m = 2, \ldots M, \tag{14}$$

and the price function for illiquid assets has the functional form:

$$P^1(t) = \frac{1}{r}(\alpha(t) - \Gamma(t)Q^1) \tag{15}$$

In the small investors value function, the parameters $A_s(t)$, $\bar{v}_s(t)$ and $\theta_s(t)$ are a scalar, an $N^1M \times 1$ vector, and an $N^1M \times N^1M$ matrix respectively. The parameters $A_m(t)$, $\Lambda_m(t)$, and $\Xi_m(t)$ from large investors value functions are similarly dimensioned. The parameters of the value functions in each time period are the solution of a system of nonlinear Riccati difference equations that are solved backwards from date T. The details are in the appendix.

The main purpose of presenting the imperfect competition model is to study how imperfect competition affects asset pricing. This is done in the next section.

3.1 Asset Pricing

The main results on asset pricing can best be interpreted when compared against a competitive benchmark in which the setup of the model is essentially the same except that all investors in the model are price-takers.

Asset Pricing with Perfect Competition

The results on asset pricing in a competitive framework are provided in the next proposition:

Proposition 2 *If the first set of risky assets is traded in a perfectly competitive environment in which all investors take asset prices as given, then the equilibrium expected excess return for the assets has a 2-factor structure:*

$$\bar{Z}^1(t) = \beta_{12}\bar{Z}^2(t) + \lambda_{[X^1]}\Omega_e X^1, \tag{16}$$

with market price of risk

$$\lambda_{[X^1]} = \frac{1 - (1/r)}{\sum_{m=1}^{M} 1/A_m}. \tag{17}$$

Additionally, investors equilibrium holdings of asset 1 in each period are constant. The equilibrium asset holdings of investor m are denoted Q_m^{1W} and given by

$$Q_m^{1W} = \frac{(1/A_m)X^1}{\sum_{m=1}^{M}(1/A_m)} \tag{18}$$

Proof: See section D.1 of the appendix.

Equation (16) shows that the excess return for asset 1 consists of a reward for the component of its return that is correlated with the market plus an additional reward for its nonmarket return. The main focus of the analysis is on the reward for non-market risk. To interpret this reward, note the investors can hedge the market component of the return of each in the segment. Therefore, one can view the investors as trading in a submarket in which they share the nonmarket risk of the N^1 assets, where the nonmarket risk component is $e(t) = Z^1(t) - \beta_{12} Z^2(t)$. $e(t)$ is not referred to as idiosyncratic risk because $e(t)$ could be correlated across firms within the segment (for example if they are in same industry). Recall X^1 is the $N_1 \times 1$ vector that represents the total supply of shares outstanding for the first set of risky assets. The quantity $X^{1\prime} e(t)$ can be interpreted as the "market portfolio" of nonmarket risk that is shared by the investors. The expression $\Omega_e X^1$ denotes the covariance of each assets nonmarket return with the "market portfolio" of nonmarket risk. Because the nonmarket risks turn out to be normally distributed (because dividends are normally distributed) and because the nonmarkets risks are exclusively traded within a single submarket by investors who have CARA utility, intuition suggests that the expected nonmarket component of returns should satisfy a CAPM-like pricing relationship in which the nonmarket risks are priced based on their covariances with the "market portfolio" of nonmarket risk.[10] This intuition is precisely what the second term on the right hand side of equation (16) confirms. In the equation, the price for nonmarket risk, $\Lambda_{[X^1]}$, depends on the sum-total of investors risk tolerances $(1/A_m)$. I refer to this quantity as the risk bearing capacity in the submarket.

Because the submarket is perfectly competitive, risk sharing among market participants is efficient; and investors efficient risky asset holdings are intuitive: the proportion of the asset supply that each investor holds is equal to his risk bearing capacity $(1/A_m)$ as a proportion of the sum total of all investors risk bearing capacity. It might be more appropriate to label the risk sharing among investor as *constrained* efficient. The reason that the efficiency is constrained is because the risks of the assets in the market segment are only shared by a tiny fraction of the economy's investors. If instead, investors could freely enter the market, they would drive the price of nonmarket risk to 0. It is the limited participation assumption is what causes the non-market risk to be priced.

If there is imperfect competition in asset markets, then if large investors asset holdings are not efficient, then they will only slowly trade towards efficient asset holdings in order to minimize the price impact of their trades. As a result, the deviation of each large investors asset holdings from his efficient asset holdings behaves as if it is a factor that is priced in financial markets. The pattern of expected returns when there is imperfect competition is presented below.

[10] Stapleton and Subrahmanyam (1978) derive circumstances in which the CAPM holds dynamically through time when investors have CARA utility and trade risky assets whose dividend payments are normally distributed.

Asset Pricing with Imperfect Competition

The main results on asset pricing when there is imperfect competition are provided in the next proposition:

Proposition 3 *When investors holdings of the first set of risky assets are not efficient, then equilibrium excess expected returns for the first set of assets satisfy a linear factor model in which first factor is the market portfolio, the second factor is the "market portfolio" of nonmarket risk, and the remaining factors correspond to the deviation of large investors asset holdings from those associated with efficient sharing of the nonmarket risk:*

$$\bar{Z}^1(t) = \beta_{12}\bar{Z}^2(t) + \lambda_{[X^1]}\Omega_e X^1 + \sum_{m=2}^{M} \lambda(m,t)\Omega_e(Q^1_m(t) - Q^{1W}_m) \tag{19}$$

Proof: See section D of the appendix.

The proposition shows that if investors asset holdings are the same as in the competitive version of the model, then assets returns will also be the same as in the competitive model. However, if a large investors asset holdings deviate, then the deviation in sharing of nonmarket risk, measured as $[Q^1_m(t) - Q^{1W}_m(t)]'e(t)$ for large investor m, behaves like a priced factor, and assets expected returns depend on their covariances with these factors. In equation (19), $\lambda(m,t)$ represents prices of risk for these additional factors at time t. These prices of risk are negative because if a large investor holds more than his efficient amount of risky assets, then because he will only sell it slowly through time, the marginal investor, in this case the small investors, expect to hold less and hence require a smaller premium for holding the nonmarket risk.

The theoretical results on asset pricing generate potential explanations for why IPO's underperform the market. The results can also be used to predict which IPO's will not underperform, and might overperform. These topics are further elaborated on below.

Potential Explanations for Underperformance

An assets return underperforms the market when its expected excess return is less than its market beta times the expected return on the market. Examination of equation (19) shows that an assets excess return can underperform the market when the sum of the second and third terms on the right hand side of the equation is less than zero. This suggests that the imperfect competition model provides two potential channnels that could cause excess returns on the new issue to underperform the market. To begin, note that even if there is efficient sharing of nonmarket risk (which makes the third term 0), if the new issue's nonmarket returns are negatively correlated with the portfolio of nonmarket returns in its segment, then this negative covariance will cause the new issue to underperform the market. This channel for underperformance is not as far-fetched as it might seem to be. For example, if the firm that does the IPO competes with other firms in its segment, then good news for it might mean bad news for its competitors.[11]

The second channel in the model that can generate underperformance comes from the third term in equation (19), which represents inefficient risk sharing among the investors. I want to study whether there are initial inefficient asset holdings that then casue the returns of the IPO asset to underperform.To analyze this question, pretend for a moment that all investors holdings of all assets in the segment are efficient. Without loss of generality, assume the first asset in the segment is the new issue. If I perturb asset holdings away from efficiency by increasing investor 2's holdings of the new issue, while holding the supply of the risky assets constant, I need to change the asset holdings of another investor; because the holdings of the other large investors are constant, it is the holdings of the small investors that are being changed. The change in small investors asset holdings is implicit in equation (19). Because $\lambda_{m,t}$ is nonzero, it is clear from the equation that for any target amount of return underperformance at time t, there is a perturbation of investor 2's holdings of the IPO firm away from efficient asset holdings such that the model generates that amount of underperformance. In other words, in theory the imperfect competitition model makes very large amounts of underperformance theoretically possible over the next period. A corollary of proposition 3 shows that inefficient risk sharing at period t affects equilibrium excess returns at future time periods as well:

Corollary 1 *When asset holdings at time t are not efficient, then the expected value of τ periods ahead 1-period excess returns follow a factor model in which the market portfolio, the portfolio of nonmarket risk, and the deviation of large investors time t asset holdings from efficient asset holdings are factors:*

$$\mathrm{E}_t[P^1(t+\tau+1) + D^1 - rP^1(t+\tau)] = \beta_{12}\bar{Z}^2 + \lambda_{[X^1]}\Omega_e X^1 + \sum_{m=2}^{M} \lambda_m(t,\tau)\Omega_e(Q_m^1(t) - Q_m^{1W})$$

Proof: See section D of the appendix.

Provided that the risk prices $\lambda_m(t,\tau)$ are nonzero for all τ, then using the same reasoning as for 1-period returns, the corollary shows that there are initial asset allocations in the imperfect competition model that can generate arbitrary amounts of return underperformance over the time horizon from periods 1 to T.

To provide intuition for the result in the proposition, pretend for a moment that there is only 1 large investor and a continuum of small investors and that the large investor has a very large long position and the small investors have a very large short position. In the appendix I show that because all risky assets are liquid from the perspective of

[11] More specifically, good news about the nonmarket component of the IPO firm's business might be associated with bad news for the nonmarket component of its competitors businesses.

each small investor, small investors demand for risky assets only depend on the assets 1 period return and variance-covariance matrix. As a result, when small investors take a short position on assets in a segment, they require the expected return on the assets to underperform the market. Standard intuition suggests that return underperformance cannot represent an equilibrium because the large investor who has a long position should sell and this will cause returns to equilibrate. This intuition is largely correct; with one addendum: because of imperfect competition the large investor sells slowly to reduce his price impact, and thus the equilibration takes time. As a result, asset holdings and trades follow an equilibrium adjustment path; and along large portions of the adjustment path, asset returns can underperform the market return.

Although I have shown that in theory the model of asset returns can generate amounts of underpricing of arbitrary magnitude, whether such returns are actually generated following an IPO depends on the competitiveness of the aftermarket.[12] If the aftermarket is sufficiently competitive, then large investors trades will not have much price impact, and they will be able to trade more quickly towards efficient risk sharing. Therefore, when the aftermarket is very competitive, the asset allocations that are needed to generate large amounts of underpricing will be very extreme. If the market is sufficiently competitive, the asset allocations are not extreme, and the asset is the only one traded in its segment, then the assets return will be dominated by the second term on the right hand side of equation (19). Because this term must be positive if there is only a single asset in the segment, the model predicts that such an asset will outperform the market. Therefore, the model can be used to predict both overperformance and underperformance.

Conversely, if the aftermarket is not very competitive, whether there is underpricing due to imperfect risk sharing will also depend on which investors receive the assets at the IPO, and it will depend on the quantities they receive. To see why it matters which investors receive the assets, note that the $\lambda_m(t,\tau)$ coefficients that determine how imperfect risk sharing today affects future excess returns varies by large investor. In particular, $\lambda_m(t,\tau)$ is greater in magnitude for those large investors who have more market power, where market power measures an investors ability to influence asset prices. In the model, large investors have more market power the greater is their risk tolerance as a share of all investors risk tolerances.[13] Therefore, the question that needs to be answered is whether the asset allocations at the IPO are sufficiently distorted away from efficient risk sharing and towards those investors that have market power, that the result is market underperformance. This question is discussed

[12]There are many possible methods to measure the competitiveness of the aftermarket. In the empirical analysis I use the Herfindahl index, which is a measure of the concentration of risk bearing capacity among the investors.

[13]As intuition for why large investors who are more risk tolerant have more market power suppose that a syndicate of M investors with CARA utility who differ in their absolute risk aversion bid the syndicate's reservation price for a pool of assets that have a 1-period nonmarket risky expected payoff $\bar{D}$ with variance σ^2. If all syndicate members participate the reservation price is $\frac{\bar{D}/r}{\sigma^2\sum_{m=1}^{M}(1/A_m)}$ and if investor j does not participate the reservation price is $\frac{\bar{D}/r}{\sigma^2[\sum_{m=1}^{M}(1/A_m)-(1/A_j)]}$. It is straightforward to show that syndicate members with greater risk tolerance have more ability to influence the syndicate's reservation price by not participating.

in the next section, when I describe how the IPO offer price is set, and how the assets are allocated.

4 The IPO Problem and IPO Underpricing

The motivation for the analysis in this section is based on Pritsker (2004). Pritsker studies a situation in which a distressed seller has a given number of shares to sell into an imperfectly competitive market. Because the seller is essentially selling to the equivalent of an oligopoly in financial markets, it is not surprising that the seller receives a price that is worse than the competitive price. The size of the price discount depends on the intensity of competition for the distressed sellers orderflow; and it depends on the amount of impatience that the distressed seller has when selling his shares. Regarding the intensity of competition, it turns out that it depends on cross-sectional dispersion of large investors risk tolerances. If one large investor is far more risk tolerant than the others, then that large investor has significant market power because if he purchases a smaller amount in the distressed sale, then the asset sales will have to be absorbed by investors with greater risk aversion who will require a large price discount in order to hold the assets. By contrast, if the risk tolerances are spread more evenly among large investors, then the competition for the distressed sales is more intense and the drop in price due to the distressed sales is consequently smaller.[14]

The distressed seller may be able to sell at better prices if he is more patient and breaks up his trades through time instead of selling all at once. This forces the large investors to compete for the distressed sales through time and dilutes their market power.

Pritsker's distressed seller analysis is applicable to the IPO setting. In the IPO, the distressed seller is the issuing firm. For simplicity, in this version of the paper I abstract away from how the size of the issue is chosen, and simply assume the issuer needs to sell X^{IPO} shares. The underwriter acts on his behalf by lining up investors to buy the issue, and by supporting the issue in the aftermarket, as described above. For his efforts, the underwriter receives a fee. I assume that the investors that seek shares in the IPO are the same investors that are modeled as trading in the IPO aftermarket. The IPO process resembles bookbuilding as practiced in the United States. The underwriter gathers demand information on the issue; in the model this consists of gathering information on the other risky assets that are traded in the market segment; the investors holdings of these assets; the investors risk preferences; and he learns about the market power of the investors that trade in the aftermarket. Based on this information, the underwriter proposes an IPO offer price

[14]In Pritsker (2004) large investors can be interpreted as trading on behalf of identical small investors. Under this interpretation, large investors are agents who purchase risk and then spread it to their base of small investors. The large investors absolute risk tolerance is equal to the small investors risk tolerance multiplied by the mass of small investors that the large investor represents. This result is intuitive because the large investor should be more risk tolerant if he can spread a given amount of risk that he purchases among a larger base of investors.

P^{IPO} and take it or leave it share allocations $X_m^{IPO}, m = 2, \ldots M$ to the large investors, and he proposes identical share allocations $X_1^{IPO} ds$ to a fraction ϕ of the small investors.

The relevance of the distressed seller analysis is that if a large or small investor turns down the share allocation that he is offered, then I assume that the unallocated shares are sold immediately by the underwriter in the IPO aftermarket. The possibility that a large investor can force distressed sales in the IPO aftermarket serves as a threat that constrains how the issuer allocates shares and chooses the IPO offer price.[15] In particular, for any large investors that receive shares, the allocation must be set so that it cannot be in the interest of any of the investors to dump their shares and instead force them to be sold by the underwriter in the IPO aftermarket. Of course, it is possible in theory that the underwriter might find it optimal to sell some shares in the IPO secondary market; denote these shares as X_U^{IPO} and the aftermarket price on the first day of trading as P_A^{IPO}.[16] This suggests that the underwriter chooses share allocations to maximize:

$$P^{IPO} \times (\phi X_1^{IPO} + \sum_{m=2}^{M} X_m^{IPO}) + P_A^{IPO} X_U^{IPO}, \tag{20}$$

where the first term measures revenues raised at the IPO, and the second represents revenues raised by distressed sales in the IPO aftermarket.

This maximization takes place subject to the constraints that the total issue is allocated:

$$\phi X_1^{IPO} + \sum_{m=2}^{M} X_m^{IPO} + X_U^{IPO} = X^{IPO}, \tag{21}$$

that there are no short-sales[17]:

$$X_U^{IPO} \geq 0, \text{ and } X_m^{IPO} \geq 0, m = 1, \ldots M, \tag{22}$$

and subject to incentive compatibility constraints that those who receive allocations in the IPO will accept the allocations. For small investors who receive allocations this condition takes the form that the value associated with participating in the IPO is greater than the value from not participating:

$$V_s[Q_s^{IPO}; Q^{IPO}, t^{IPO} + 1] \geq V_s[Q_s; Q^{IPO}, t^{IPO} + 1], \tag{23}$$

where the IPO occurs at time t^{IPO} and investors decide whether or not to participate based on the effect that the IPO has on their time $t^{IPO} + 1$ value functions. The value functions that large and small investors use to evaluate whether to participate in the IPO are value

[15] There are many other possible ways to model the threats that available to the large investors and the threats that are available to the underwriter.

[16] The aftermarket price on the first day of trading is equal to the equilibrium price function for time period 1 in the aftermarket.

[17] This constraint will be eventually relaxed to examine how underwriter short-selling affects the results.

functions that were derived for time 1 post-IPO trading in the aftermarket. For convenience, I have suppressed most, but not all, of the notation in the value functions. Specifically, small investors that participate have post IPO risky asset holdings Q_s^{IPO}. The post-IPO risky asset holdings of all investors is denoted Q^{IPO}. If a small investor chooses not to participate in the IPO, his post IPO risky asset holdings are Q_s. Note: that the above expression is for a multiple-asset context in which I assume that the shares of one of the assets is an IPO and the others are not. Note also that whether or not a small investor participates in the IPO has no effect on the state vector Q^{IPO} because each small investor is infinitesimal.

For large investors who receive share allocations, the incentive compatibility constraints take the form:

For every $m > 1$ such that $Q_m^{IPO} > 0$

$$V_m[Q^{IPO}, t^{IPO} + 1] \geq V_m[Q_{-m}^{IPO}, t^{IPO} + 1] \tag{24}$$

where large investor m's share allocation in the IPO is Q_m^{IPO} and Q_{-m}^{IPO} is the post-IPO share allocation if large investor m chooses not to accept his allocation.[18]

The assumption that the distressed sales occur immediately following the IPO is very strong. A more reasonable assumption is that any shares that the underwriter fails to sell at the IPO will instead be sold over τ_S periods following the IPO. This modeling assumption is consistent with empirical evidence, reported in Ellis et. al. (2000), that IPO underwriters engage in price support activities in the IPO aftermarket, and with evidence reprted by Ellis et. al. (2002) which shows that underwriters are active participants in the IPO aftermarket for long periods of time.[19].

I assume that when the underwriter sells shares over τ_s time periods he will sell them optimally. By optimality I mean that the underwriter buys shares at the IPO offer price, and then trades his shares over the following τ_S time periods in order to maximize his own utility subject to the constraint that by time τ_S the underwriter holds no shares of the issue. It is assumed that the certainty equivalent value of the underwriters utility from buying and trading the shares is turned over to the issuing firm at the time of the IPO. For tractability I assume that the underwriter has CARA utility like the other large investors. Let $CE_U(Q^{IPO}, \tau_s)$ denote the underwriters certainty equivalent. Then, under the less restrictive assumption, the underwriter maximizes:

$$P^{IPO} \times (\phi X_1^{IPO} + \sum_{m=2}^{M} X_m^{IPO}) + CE_U(Q^{IPO}, \tau_s), \tag{25}$$

[18]When there is the possibility of distressed sales, as there is here, the equilibrium value functions and equilibrium price that that is associated with entering period $t + 1$ have a similar form to those given in equations (13), (14), and (15), but with the state vector supplemented by an additional argument, which is the amount of distressed sales.

[19]In Ellis et. al.ś (2002) sample of 313 NASDAQ IPOs, the lead underwriter participated in an average of more than 90 percent of post IPO NASDAQ trades during the first day of the IPO; this amount tapers down over the next 140 days, but remained above 40 percent on average on the 140th day

subject to the constraints that the total issue is allocated [equation (21)], that there are no short sales [equation (22)], and subject to a new set of participation constraints that account for the new behavior of the underwriter:

$$V_s[Q_s^{IPO}; Q^{IPO}, U(\tau_s), t^{IPO}+1] \geq V_s[Q_s; Q^{IPO}, U(\tau_s), t^{IPO}+1], \tag{26}$$

and

$$V_m[Q^{IPO}, U(\tau_s), t^{IPO}+1] \geq V_m[Q_{-m}^{IPO}, , U(\tau_s), t^{IPO}+1]. \tag{27}$$

The addition of the argument $U(\tau_s)$, which denotes the possibility that an underwriter optimally liquidates over τ_s periods differentiates the incentive compatibility constraints in equations (26) and (27) from those when the underwriter must sell his holdings immediately after the IPO (equations (23) and (24)). Because the underwriter is modeled as selling any unallocated shares over a longer amount of time, it alters large investors market power after the IPO. I expect that this will raise the IPO offer price and revenues raised through the IPO. Below I investigate whether it actually does so in the simulations that follow.[20]

5 Simulation Analysis

To study whether imperfect competition in the aftermarket can help explain underpricing and underperformance, I studied the behavior of the model when only a single risky asset, the new issue, is traded in the aftermarket. Liquidity in the aftermarket depends on two state-variables. The first is the distribution of risk tolerances across investors, which was alluded to above, and the second is the number of post-IPO trading periods. When the number of post IPO trading periods is small, there is little opportunity to spread risks across investors through time. Consequently, investors who take on positions require more compensation for doing so and the market becomes more illiquid. The market is most illiquid when no trading periods remain. Conversely, as the number of remaining time periods gets large, the market becomes increasingly liquid and in the limit becomes perfectly competitive.[21] I believe that in reality financial markets are not perfectly competitive; the only way to accomodate this

[20]The solution for the model with distressed sales is closely based on Pritsker (2004). To save space, it is not presented in the appendix.

[21]Recall that in my model, the concepts of illiquidity is that trades move prices, which is the same as the concept of market power. Additional intuition for the relationship between illiquidity and the number of post-IPO trading periods is based on Coasian analysis of the market power of a durable goods monopolist. Coase argues that the monopolist can get a higher price if he can commit to selling over a single time period; the possibility that he will sell over several periods erodes his market power. Kihlstrom argues that the Coasian analysis applies to stocks because stocks are durable goods; and he too shows that additional periods of retrade erode the monopolists market power. In the model of aftermarket trading, I suspect that the Coasian argument also holds; and that a larger number of periods of retrade erodes the oligopolists (large investors) market power.

within the present model is through a finite number of trading periods. I study the behavior of the model when the number of trading periods after the IPO ranges from a high of 2000 trading periods, to a low of 200. Each trading period is interpreted as 1-business day. To date, I have solved the model for 4 configurations of investors. I am also currently working on calibrating the model, but have not done so yet. In all configurations, large investors are labelled as "Institutional Investors" and the small investors as "Retail Investors". Results are presented when there is a continuum of small investors and 5 large investors who differ in their risk aversion. Recall that when risk-sharing is efficient, the proportion of each risky asset's supply that should be held by each large investor is equal to his risk tolerance as a fraction of the sum total of all investor's risk tolerances. I refer to this quantity as the investor's share of risk bearing capacity. Intuitively, an investor with a higher share of risk bearing capacity has more market power. One gauge of the competitiveness of trading in this segment is the concentration of risk bearing capacity among the investors. The concentration of risk bearing is measured by using the Herfindahl index from the Industrial Organization literature. The Herfindahl index is equal to 10,000 times the sum of the squares of each investors share of risk bearing capacity.[22] The maximum size of the index is 10,000 which corresponds to all of the risk bearing capacity held by one investor; the minimum size of the index is 0 which corresponds to perfect competition which formally requires that all investors are infinitesimal.[23]

Before presenting the simulation results, it is important to note that the parameters of investors value functions are solved backwards for thousands of periods using a system of nonlinear Riccati difference equations; and each step backward in the solution involves a matrix inversion. The parameters of investors value functions are then used as inputs to solve the pricing and allocation problem in the IPO. The constraints in the IPO allocation and pricing problem are themselves nonlinear; and it is not certain that my optimization routines are finding global maxima. Given the numerical difficulties, the simulation results should be treated as preliminary.

The results from the simulations are provided in Tables 1 through 4; and are sorted by Herfindahl indices with the results for the least competitive cases presented first. The simulations shed light on five questions. First, how does imperfect competition in the IPO aftermarket affect asset allocations at the IPO.

Asset Allocations

The distorting influence of imperfect competition on allocations at the IPO is measured in terms percentage deviations from each investors efficient asset holdings. For example, Table 1, Panel B, shows that under retail investors should receive 10 percent of shares in the IPO if the assets at the IPO are allocated to ensure efficient risk-sharing. Panel C, shows that retail investors were distorted by -100 percent from their optimal holdings; which means

[22] Each small investors ideal percentage share of the market is 0.

[23] When each investor is infinitesimal, and indexed on $s \in [0, 1]$ then his risk bearing capacity is $1/A_s$ ds; and the Herfindahl index is 0 because the integral of the investors squared risk bearing capacities is 0.

that in the optimal IPO allocations in the simulations, retail investors receive nothing and institutional investors receive everything. This pattern of allocations that are distortion away from retail investors is repeated in all of the results (Tables 1 - 4, panel C) and is consistent with stylized empirical evidence that retail investors perceive that they are cut out of IPO allocations, and that institutional investors benefit at their expense.

The fact that retail investors are cut out and institutional investors receive more allocations raises the question of which institutional investors receive the allocations, and how does this depend on the institutions' risk bearing capacities. The simulations suggest that the relationship is complicated. Intuition suggests that asset holdings should be distorted towards those investors with the greatest risk bearing capacity because risk bearing capacity is a proxy for market power. The results are partially consistent with this intuition: for a given Herfindahl index, when the number of Post-IPO trading periods is small enough, then asset holdings are distorted towards those institutional investors with the greatest amounts of risk bearing capacity (Tables 1-3, panel C). However, the simulations show that the intuition is incomplete because when there are a large enough number of post-IPO trading periods, asset holdings can be distorted away from large investors with the most risk bearing capacity and towards large investors that have less risk bearing capacity (Table 2 and 3, panel C).

Aftermarket trading

The second question is can the model rationalize the large amounts of trading volume after the IPO? The results on asset allocation distortions provide one potential explanation. Recall, that if the assets were allocated to those investors who valued them most, and if there was no private information, then there should be no trade. However, if asset allocations at the IPO are distorted away from efficient allocations and towards investors with market power, then trading volume will be generated the aftermarket as investors adjust their asset holdings towards those assoicated with optimal risk sharing. I plan to present more detailed results on whether the model matches the time series pattern of post-IPO trading volume in future revisions.

Underperformance

The third question is can imperfect competition lead to return underperformance after the IPO. The intuition that was provided earlier suggested that when asset allocations after the IPO are sufficiently distorted toward large investors, the returns would underperform. There are 2 notions of return underperformance: the first is relative to the market porfolio, and the second is short-term underperformance, which occurs if he returns on the asset following the IPO are lower than the returns will be in the long run. The tables report results on the expected component of nonmarket returns. Therefore, return undeperformance relative to the market portfolio occurs if the expected return on the nonmarket component of returns is negative. Examination of Tables 1 through 3 shows that return underperformance relative to the market occurs when the Herfindahl index is high, or the number of trading periods following the IPO is sufficiently small. This result confirms that the model can generate underperformance relative to the market. Additionally, the second type of underperformance is present in all market configurations (Tables 1 - 4, panel C). An important question that is not answered by these simulation results is how long does the underperformance persist. Unfortunately, for these configurations, I did not compute the answer to this question. However, although I have not yet computed a full set of results, experiments with other market configurations have generated underperformance relative to the market that persists for periods of more than one year. I view these results on underperformance as encouraging.

Underpricing

The fourth question is whether imperfect competition generates underpricing in the IPO. The answer is a qualified yes: when the Herfindahl index is high enough, and the number of remaining trading periods is small enough, then underpricing does result (Panel A of Tables 1 - 3); and the amount of underpricing increases when the number of post-IPO trading periods is small. As the market becomes more competitive, the underpricing vanishes, but overpricing does not result. Hence, when averaging across different market configurations, it is clear that the model produces underpricing on average.

Can the underwriters fees be rationalized?

Finally, the fifth question is whether the fees that are received by underwriters can be rationalized. I have attempted to provide an answer by simulating equilibrium offer prices when the underwriter can trade over many periods in the aftermarket, but I have encountered some numerical difficulties. Nevertheless, I but do have some very preliminary results. The first set of results were computed for the configurations in tables 1 through 4. In Table 1, results were computed for the case of 1000 Post-IPO trading periods. For this case, the presence of an underwriter who sells over 200 trading periods has essentially no effect on the revenues of the issuer; and little effect on the allocations or underpricing in the IPO. For the results in Table 2, the underwriter generates more revenue, but the allocations are little changed. By contrast, for the results in Tables 3 and 4, the optimum involves the underwriter allocating none of the issue in the IPO; instead he sells it over 200 periods in the aftermarket — and this increases the revenues received by the issuer. Although these preliminary findings are discouraging, in a set of additional recent simulations that are not fully reported here, I sometimes found circumstances when the underwriter keeps a large (33 percent) but not 100 percent stake in the issue and then sells it through time. By doing so, he increases the proceeds that the issuer receives by 25%. This suggests that the underwriter can sometimes provide very significant value to the seller by trading in the aftermarket. It is important to reiterate that this finding is a result of a purely strategic setting that does not contain any informational asymmetries. Although some of the most recent results on the underwriter are very encouraging, it is important to stress again that the results on the underwriter are preliminary and the numerical optimizations need to be carefully checked.

Cautions in Interpretation and Calibration

The results presented in this section are qualitatively very useful because they show that the channels identified by the theory can generate underpricing and underperformance. Caution should be exercised in attempting to interpret the results quantitatively. The main issue is difficulties with the scale of the results. More specifically, the empirical literature reports percentage returns and percentage underpricing; but they are not reported this way in the model. Moreover, the results in this model probably should not be reported in percentage terms because altering the level of expected dividends would allow the model to match nearly any level of percentage underpricing, and nearly any level of percentage underperformance. Nevertheless, I am currently working on calibrating the parameters in the model so that I can study percentage returns and underpricing. But, it remains an open question whether such a calibration exercise is feasible or meaningful.

This completes all of the simulation results. In sum, I think the results are encouraging because the model qualitatively matches many of the puzzling stylized facts about the inefficiency of IPOs. The next section discusses some of the empirical literature in light of the theoretical results.

6 Is After Market Illiquidity Empirically Relevant

What is the state of the empirical evidence on whether after-market liquidity is a determinant of underpricing and underperformance? These questions have only been addressed by a few papers; and the answers remain unsettled. The relationship between IPO underpricing and illiquidity has been empirically studied by Booth and Chua (1996), Hahn and Ligon (2004), and Ellul and Pagano (2003).[24] Although the Booth and Chua model makes predictions about the relationship between underpricing and aftermarket liquidity, they don't test this implication of their model; instead their tests focus on underpricing as compensation for costs of information gathering. Because such costs could generate underpricing irrespective of illiquidity, the implications of their tests for the relationship between underpricing and aftermarket liquidity are unclear. Hahn and Ligon attempt to directly test the Booth and Chua hypothesis that underpricing is used to increase liquidity by running OLS regressions of market microstructure measures of aftermarket liquidity on IPO underpricing. In regressions that account for other determinants of illiquidity, their results are mixed; with coefficients

[24] In related research that does address asset pricing per se, Corwin, etálśtudy the evolution of market microstructure measures of liquidity through time follwing an IPO. A special aspect of their research is that they observe the limit order book, and hence can study the evolution of liquidity measures such as the depth of the limit order book, and the depth of the book relative to trading volume.

on underpricing sometimes statistically significant and positive, sometimes statistically signifcant and negative, and sometimes not statistically significant at all. A potential difficulty with the Hahn and Ligon regressions is that causality may run from underpricing to illiquidity (as in Booth and Chua) as well as from illiquidity to underpricing (as in Ellul and Pagano). The possibility that causality runs in both directions suggests that an instrumental variable approach is needed. In Ellul and Pagano, they regress underpricing on a set of determinants for underpricing, including measures of aftermarket liquidity. Additionally, they recognize the potential for simultaneity bias and instrument for it in some of their regressions.[25] In all of Ellul and Pagano's regressions they find that more aftermarket illiquidity increases the amount of IPO underpricing. This finding is consistent with both their theory and my theory of IPO underpricing.

Although Ellul and Pagano's findings are favorable for liquidity-based theories of IPO underpricing, there is reason for caution in interpreting their results. One reason for caution is if underpricing is a risk premium for aftermarket illiquidity, then the logical extension of Ellul and Pagano's theory would suggest that in the aftermarket, IPO's should earn a positive and significant risk premium for aftermarket liquidity. The fact that IPO returns underperform in the aftermarket, suggests that the mechanism driving aftermarket returns is more complicated than the theory of illiquidity considered by Ellul and Pagano.[26] Eckbo and Norli (2002) take this argument one step further; they claim that newly issued stocks are more liquid than other stocks with similar risk characteristics; and thus their returns should underperform. To establish this point empirically, Eckbo and Norli compare the returns of a rolling portfolio of newly issued stocks that are held for up to five years against the returns a portfolio of more seasoned issues that are matched on size and book to market. They find that after adjusting for these factors, and controlling for differences in liquidity, new issues do not underperform.

The Eckbo and Norli analysis highlights an important issue: what is the appropriate risk-adjustment to apply when determining whether the returns on new issues underperform. The theory in this paper suggests that adjusting returns for book-to-market is problematic. The reason is that the theory shows that when share allocations at the IPO are biased towards large investors; this creates an allocation effect in aftermarket prices that makes them higher in the short-run, just after the IPO, than they will be in the long-run. The temporarily high stock price will cause new issues to initially have a low book to market. At the same time the theory also predicts there will be return underperformance following the IPO. Because the theory's predictions of low returns and low book-to-market are consistent with the empirical evidence on how the Fama-French "book-to-market" factor affects returns, tests that adjust for book-to-market will remove the predictions of my liquidity/imperfect competition theory from the data being analyzed. Such tests will then have low power to detect underperformance due to illiquidity even when such underperformance is present. Therefore, I believe it remains an open empirical question whether imperfect competition

[25]They do not report any results on tests for the strength of the instruments, nor do they report any results of tests for instrument validity.

[26]This critique does not rule out my theory that underperformance is caused by how imperfect competition in the aftermarket distorts share allocations at the IPO.

and illiquidity play a significant role in explaining IPO underpricing and underperformance.

7 Conclusions

In this paper I have presented a fully-rational symmetric information model of IPO book-building that is followed by imperfect competition and illiquidity in a dynamic post-IPO trading environment. For some parameter values the model generates IPO allocations and offer prices that are consistent with underpricing at the IPO, return underperformance following the IPO, and a tilt in share allocations toward institutional investors and away from retail investors. I have also begun a highly preliminary analysis of the behavior of the underwriter in the IPO aftermarket; and have found that for some model parameterizations the underwriter, by trading in the IPO aftermarket, can dilute other investors market power and substantially increase the revenues raised by the issuer.

An important question going forward is determining the percentages of underpricing, underperformance, and underwriter fees, that can plausibly be attributed to imperfect competition and illiquidity in aftermarket trading. A partial answer to this question can be provided through model calibration; I am currently involved in a calibration effort. A better way to answer the question is through empirical analysis that studies the relationship between aftermarket competitiveness and the inefficiencies that are associated with the IPO process. Hopefully the results in this paper will stimulate these types of empirical research.

Appendix

A Notation

There are M investors and $N = N_1 + N_2$ risky assets. The first N_1 assets are illiquid. The next N_2 assets are perfectly liquid. The risky asset holdings of investor m at time t are denoted by

$$Q_m(t) = \begin{pmatrix} Q_m^1(t) \\ Q_m^2(t) \end{pmatrix}$$

where $Q_m^1(t)$ and $Q_m^2(t)$ are investor $m's$ holdings of illiquid and liquid risky assets respectively. $Q^1(t)$ denote the $N_1 M \times 1$ vector of all investors illiquid asset holdings at time t where

$$Q^1(t) = \begin{pmatrix} Q_1^1(t) \\ \vdots \\ Q_M^1(t) \end{pmatrix}.$$

$Q_1^1(t)$ represents the net asset holdings of a continuum of infinitesimal small investors indexed by s:

$$Q_1^1(t) = \int_0^1 Q_s^1(t)\mu(s)ds.$$

The small investors are often collectively referred to as the competitive fringe. $Q_2^1(t)$ through $Q_M^1(t)$ denotes the net illiquid risky asset holdings of large investors, and is denoted by the $N_1 \times (M-1)$ vector $Q_B^1(t)$. The change in investors illiquid risky asset holdings from the beginning of time period t to the beginning of time period $t+1$ is denoted by the $N_1 M \times 1$ vector $\Delta Q^1(t)$. Similarly, $\Delta Q_1^1(t)$ and $\Delta Q_B^1(t)$ denote the change in the competitive fringe's illiquid asset holdings, and the change in the illiquid asset holdings of the large investors.

The algebra which follows requires many matrix summations and the use of selection matrices. Rather than write summations explicitly, I use the matrix $S = \iota_M' \otimes I_N$ to perform summations where ι_M is an M by 1 vector of ones, and I_N is the $N \times N$ identity matrix.[27] In some cases, the matrix S may have different dimensions to conform to the vector whose elements are being added. In all such cases, S will always have N, or N_1 rows. The matrix S_i is used for selecting submatrices of a larger matrix. S_i has form

$$S_i = \iota_{i,M}' \otimes I_N,$$

where $\iota_{i,M}$ is an M vector has a 1 in its i'th element, and has zeros elsewhere.[28] As above S_i will sometimes have different dimensions to conform with the matrices being summed, but it will always have N or N_1 rows.

In the rest of the exposition, I will occasionally suppress time subscripts to save space.

[27] For example, $SQ(t) = \sum_{m=1}^{M} Q_m(t)$

[28] To illustrate the use of the selection matrix, $Q_m(t) = S_m Q(t)$.

B Proof of Proposition 1

Proposition 1: *Small investors value functions for entering period t with liquid wealth W_s, when investors' state vector of illiquid asset holdings is given by Q^1 is given by:*

$$
\begin{aligned}
V_s(W_s, Q^1, t) &= -K_1(t)\ F(Q^1, t)\ e^{-A_s(t)W_s}, \\
\textit{where } F(Q^1, t) &= e^{-Q^1(t)'\bar{v}_s(t) - Q^1(t)'\theta_s(t)Q^1(t)}.
\end{aligned}
\tag{13}
$$

Large investor m's value function for entering period t when the state vector of illiquid asset holdings is Q and his holdings of liquid wealth is W_m is given by:

$$
V_m(W_m, Q^1, t) = -K_m(t)e^{-A_m(t)W_m - A_m(t)Q^{1\prime}\Lambda_m(t) + .5A_m(t)^2 Q^{1\prime}\Xi_m(t)Q^1} \quad m = 2, \ldots M, \tag{14}
$$

and the price function for illiquid assets has the functional form:

$$
P^1(t) = \frac{1}{r}(\alpha(t) - \Gamma(t)Q^1) \tag{A1}
$$

Proof: The proof is by induction. Part I of the proof establishes that if the value function has this form at time t, then it has the same form at time $t-1$. Part II of the proof establishes the result for time T, the first period in which trade cannot occur.

B.1 Part I:

Suppose the form of the value function is correct for time t. Then, to establish the form of the value function at time $t-1$, I first solve for the competitive fringe's demand curve for absorbing the net order flow of the large investors. I then solve the large investors and competitive fringe's equilbrium portfolio and consumption choices, and then solve for the value function at time $t-1$.

The competitive fringe's demand curve

The competitive fringe represents a continuum of infinitesimal investors that are distributed uniformly on the unit interval with total measure 1, i.e. $\mu(s) = 1$ for $s \in [0, 1]$. At time $t-1$, each participant s of the competitive fringe solves:

$$\max_{\substack{C_s(t-1),\\ Q_s,\\ q_s}} \quad -e^{-A_s C_s(t-1)} - \delta\,\mathrm{E}[K_s(t)F(Q^1,t)e^{-A_s(t)W_s(t)}] \tag{A2}$$

where, Q_s is the stacked vector of small investor s's holdings of illiquid (Q_s^1) and perfectly liquid (Q_s^2) risky assets:

$$Q_s = \begin{pmatrix} Q_s^1 \\ Q_s^2 \end{pmatrix};$$

$Z(t)$ is the stacked vector of excess returns for the illiquid and liquid assets:

$$Z(t) = \begin{pmatrix} Z^1(t) \\ Z^2(t) \end{pmatrix} = \begin{pmatrix} P^1(t) + D^1(t) - rP^1(t) \\ P^2(t) + D^2(t) - rP^2(t) \end{pmatrix}; \tag{A3}$$

and small investors liquid wealth is given by

$$W_s(t) = Q_s' Z(t) + r[W_s(t-1) - C_s(t-1)].$$

Note: Although I refer to the first set of assets as illiquid, they are only illiquid for large investors whose trades have price impact. Because each small investor is infinitesimal, their trades do not have price impact and hence both assets are perfectly liquid from their perspective.

In equation (A3),

$$\mathrm{E}\,Z(t) \equiv \bar{Z}(t) \equiv \begin{pmatrix} \bar{Z}_1(t) \\ \bar{Z}_2(t) \end{pmatrix},$$

and

$$\mathrm{Var}\,Z(t) \equiv \Omega \equiv \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix}.$$

Substituting the expression for W_s in (A2) and taking expectations shows that small investors maximization becomes:

$$\max_{\substack{C_s(t-1),\\ Q_s}} \quad -e^{-A_s C_s(t-1)} - \delta F(Q^1,t)e^{-A_s(t)r[W_s(t-1)-C_s(t-1)]-A_s(t)Q_s'\bar{Z}(t)+.5A_s(t)^2 Q_s'\Omega Q_s} \tag{A4}$$

In solving the model, it is useful to break small investors maximization into pieces by first solving for optimal Q_s^2 as a function of Q_s^1, and then solving for optimal Q_s^1. For given

Q_s^1, the first order condition for optimal Q_s^2 shows that optimal Q_s^2 is given by

$$Q_s^2 = \frac{1}{A_s(t)} \Omega_{22}^{-1} \bar{Z}_2(t) - \beta_{12}' Q_s^1, \tag{A5}$$

where $\beta_{12} = \Omega_{12}\Omega_{22}^{-1}$.

Plugging the solution for Q_s^2 into the small investors value function and simplifying then shows that the small investors maximization problem reduces to:

$$\max_{\substack{C_s(t-1),\\ Q_s^1}} \quad -e^{-A_s C_s(t-1)} - \delta F(Q^1,t) K_s(t) \operatorname{Exp}\left\{-.5\bar{Z}_2'\Omega_{22}^{-1}\bar{Z}_2 - A_s(t) r[W_s(t-1) - C_s(t-1)]\right\}$$
$$\times \operatorname{Exp}\left\{-A_s(t) Q_s^{1\prime}[\bar{Z}_1(t) - \beta_{12}\bar{Z}_2(t)] + .5 A_s(t)^2 Q_s^{1\prime} \Omega_e Q_s^1\right\} \tag{A6}$$

where Ω_e is given by

$$\Omega_e = \Omega_{11} - \Omega_{12}\Omega_{22}^{-1}\Omega_{21}.$$

To gain intuition for the above expression, note that the excess return on each illiquid asset can be decomposed into a component that is correlated with the liquid assets and into a second idiosyncratic component.

$$Z_1(t) = \beta_{12} Z_2(t) + \epsilon_1(t)$$

$\bar{Z}_1 - \beta_{12}\bar{Z}_2(t)$ is the vector of expected returns on the idiosyncratic components at time t and Ω_e is the variance covariance matrix of the idiosyncratic returns. The expression shows that small investors portfolio maximization problem can equivalently be written in terms of choosing an exposure to the returns of the liquid assets, and to the idiosyncratic component of returns of the illiquid assets.

Solving for optimal $Q_s^1(t)$ then shows

$$Q_s^1(t) = \frac{1}{A_s(t)} \Omega_e^{-1} [\bar{Z}_1(t) - \beta_{12}\bar{Z}_2(t)] \tag{A7}$$

The aggregate demand for Q^1 at time t by all small investors can be found by integrating both sides of equation (A7) with respect to μ_s, the density of small investors, yielding:

$$\begin{aligned} Q_1^1(t) &= \int_0^1 Q_s^1(t)\mu_s ds \\ &= \left[\int_0^1 \tfrac{1}{A_s(t)}\mu_s ds\right] \Omega_e^{-1}[\bar{Z}_1(t) - \beta_{12}\bar{Z}_2(t)] \\ &= \tfrac{1}{A_1(t)}\Omega_e^{-1}[\bar{Z}_1(t) - \beta_{12}\bar{Z}_2(t)] \end{aligned} \tag{A8}$$

The Price Schedule Faced by Large Investors

The price schedule faced by large investors at time $t-1$ maps large investors desired orderflow of the illiquid assets into the time $t-1$ prices at which the competitive fringe is willing to absorb the net orderflow. To solve for the price schedule, I solve for prices $P(.,t-1)$ in equation (A8) such that when the large investors choose trade $\Delta Q_B^1(t-1)$ at time t-1, then the competitive fringe chooses trade $-S\Delta Q_B^1(t-1)$.

Rearranging, equation (A8) while making the substitutions

$$\begin{aligned} Q^1(t) &= Q^1(t-1) + \Delta Q^1(t-1), \\ Q_1^1(t) &= S_1[Q^1(t-1) + \Delta Q^1(t-1)], \\ \Delta Q^1(t-1) &= \begin{pmatrix} -S\Delta Q_B^1(t-1) \\ I\Delta Q_B^1(t-1) \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} \bar{Z}^1(t) &= P^1(t) + \bar{D}^1 - rP^1(t-1,.) \\ P^1(t) &= \frac{1}{r}\left(\alpha(t) - \Gamma(t)[Q^1(t-1) + \Delta Q^1(t-1)]\right) \end{aligned}$$

then produces the price schedule faced by large investors at time $t-1$:

$$P^1(.,t-1) = \frac{1}{r}\left(\beta_0(t-1) - \beta_{Q^1}(t-1)Q^1(t-1) - \beta_{Q_B^1}(t-1)\Delta Q_B^1(t-1)\right), \tag{A9}$$

where,

$$\begin{aligned} \beta_0(t-1) &= \bar{D}^1 + (1/r)\alpha(t) - \beta_{12}\bar{Z}^2 && \text{(A10)} \\ \beta_{Q^1}(t-1) &= (1/r)(\Gamma(t) + rA_1(t)\Omega_e S_1) && \text{(A11)} \\ \beta_{Q_B^1}(t-1) &= (1/r)\Gamma(t)\begin{pmatrix} -S \\ I \end{pmatrix} - A_1(t)\Omega_e S && \text{(A12)} \end{aligned}$$

Given the price schedule in equation (A9), large investors at time $t-1$ solve the maximization problem:

Large Investors Maximization Problem

$$\max_{\substack{C_m(t-1),\\ Q_m}} \quad -e^{-A_m C_m(t-1)} - \mathrm{E}\left\{\delta K_m(t)\,\mathrm{Exp}\left(-A_m(t)W_m - A_m(t)Q^{1\prime}\Lambda_m(t) + .5A_m(t)^2 Q^{1\prime}\Xi_m(t)Q^1\right)\right\} \tag{A13}$$

where, substituting in the budget constraint, liquid wealth at the beginning of time t is given by

$$\begin{aligned} W_m(t) = & Q_m^1(t)'D^1(t) + Q_m^2(t)'Z^2(t) \\ & + r(W_m(t-1) - \Delta Q_m^1(t-1)'P^1(t-1,.) - C_m(t-1)) \end{aligned} \tag{A14}$$

Note: Because dividends are paid in cash, the dividend payments received for holdings of illiquid asset are counted as part of liquid wealth even though the illiquid assets themselves are not counted.

Note that in equation (A13), $\Lambda_m(t)$ and $\Xi_m(t)$ are deterministic functions of time that are parameters of the value function. Keeping this in mind, large investors holdings of the liquid assets are solved in the same way as for small investors. Taking expectations in equation (A13), solving for optimal Q_m^2 given Q^1, and substituting the optimal choice back into the large investor's value function, transforms the large investors maximization problem so that it has the following form:

$$\begin{aligned} \max_{\substack{C_m(t-1), \\ Q_m^1}} \quad & - e^{-A_m C_m(t-1)} \\ & - \delta K_m(t) \left\{ \text{Exp}(-.5\bar{Z}^{2\prime}\Omega_{22}^{-1}\bar{Z}_2 - A_m(t) r[W_m(t-1) - \Delta Q_m(t-1)'P^1(t-1,.) - C_m(t-1)] \right. \\ & \left. \times \text{Exp}(-A_m(t)Q^{1\prime}\bar{v}_m(t) + .5A_m(t)^2 Q^{1\prime}\theta_m(t)Q^1) \right\} \end{aligned} \tag{A15}$$

where,

$$\bar{v}_m(t) = S_m'(\bar{D}_1 - \beta_{12}\bar{Z}_2) + \Lambda_m(t) \tag{A16}$$

$$\theta_m(t) = S_m'\Omega_e S_m + \Xi_m(t) \tag{A17}$$

The large investors play a Cournot game in which each choose his time $t-1$ trade $\Delta Q_m(t-1)$ in the illiquid assets to solve the maximization problem in (A15) while taking the trades of the other large investors as given, but while taking into account the effect that his own trades have on the prices of the illiquid assets. Recall the price impact function for the illiquid assets at time $t-1$ is given by equation (A9).

The first order condition for large investors illiquid asset choices is given by:

$$\begin{aligned} 0 = & -A_m(t)[(-S_1+S_m)\bar{v}_m(t)] + A_m(t)^2(-S_1+S_m)[(\theta_m(t)+\theta_m(t)')/2](Q^1+\Delta Q^1) \\ & + A_m(t)\left[rP^1(.,t-1) - S_m\beta_{Q_B^1}(t-1)'S_m\Delta Q_B^1\right], \end{aligned} \tag{A18}$$

After substituting for $P^1(.,t-1)$ from equation (A9), writing $Q^1+\Delta Q^1$ as $Q^1+\begin{pmatrix} -S\Delta Q_B^1 \\ \Delta Q_B^1 \end{pmatrix}$ and simplifying, this produces the following reaction function for large investor m:

$$\pi_m(t-1)\Delta Q_B^1 = \chi_m(t-1) + \xi_m(t-1)Q^1, \tag{A19}$$

where,

$$\begin{aligned}\pi_m(t-1) =& A_m(t)(-S_1+S_m)[(\theta_m(t)+\theta_m(t)')/2]\begin{pmatrix} -S \\ I \end{pmatrix} \\ & - \beta_{Q_B^1}(t-1) - S_m\beta_{Q_B^1}(t-1)'S_m\end{aligned} \tag{A20}$$

$$\chi_m(t-1) \;=\; (-S_1+S_m)\bar{v}_m(t) - \beta_0(t-1) \tag{A21}$$

$$\xi_m(t-1) \;=\; \beta_{Q^1}(t-1) - A_m(t)(-S_1+S_m)[(\theta_m(t)+\theta_m(t)')/2] \tag{A22}$$

Stacking the (M-1) reaction functions produces a system of $(M-1)N$ linear equations in $(M-1)N$ unknowns:

$$\Pi(t-1)\Delta Q_B^1(t-1) = \chi(t-1) + \xi(t-1)Q^1(t-1) \tag{A23}$$

Assume that $\Pi(t-1)$ is invertible. Then the solution for $\Delta Q_B^1(t-1)$ is unique, and given by

$$\Delta Q_B^1(t-1) = \Pi(t-1)^{-1}\chi(t-1) + \Pi(t-1)^{-1}\xi(t-1)Q^1(t-1) \tag{A24}$$

Equilibrium Asset Holdings

The solution for $\Delta Q_1^1(t-1)$ is $-S\Delta Q_B^1(t-1)$. Therefore, the solution for $\Delta Q^1(t-1) = (\Delta Q_1^1(t-1)', \Delta Q_B^1(t-1)')'$ can be written as:

$$\Delta Q^1(t-1) = H_0(t-1) + H_1(t-1)Q^1(t-1). \tag{A25}$$

where,

$$H_0(t-1) = \begin{pmatrix} -S\Pi(t-1)^{-1}\chi(t-1) \\ \Pi(t-1)^{-1}\chi(t-1) \end{pmatrix}, \quad \text{and} \quad H_1(t-1) = \begin{pmatrix} -S\Pi(t-1)^{-1}\xi(t-1) \\ \Pi(t-1)^{-1}\xi(t-1) \end{pmatrix}. \tag{A26}$$

With the above notation, the equilibrium purchases by large participant m in period $t-1$ are given by

$$\Delta Q_m^1(t-1) = S_m[H_0(t-1) + H_1(t-1)Q^1(t-1)] \tag{A27}$$

Additionally, the equilibrium transition dynamics for beginning of period illiquid risky asset holdings are given by:

$$Q^1(t) = G_0(t-1) + G_1(t-1)Q^1(t-1) \tag{A28}$$

where $G_0(t-1) = H_0(t-1)$ and $G_1(t-1) = H_1(t-1) + I$.

Equilibrium Price Function

Recall that the equilibrium price function in each time period maps investors beginning of period holdings of risky assets to an equilibrium price after trade. The equilibrium price function for period $t-1$ is found by plugging the solution for large investors equilibrium trades from equation (A24) into the price schedule faced by large investors (equation (A9)). The resulting price function for illiquid asset in period $t-1$ has form:

$$P^1(t-1, Q^1) = \frac{1}{r}\left(\alpha(t-1) - \Gamma(t-1)Q^1\right) \tag{A29}$$

where,

$$\begin{aligned} \alpha(t-1) &= \beta_0(t-1) - \beta_{Q_B^1}(t-1)\pi(t-1)^{-1}\chi(t-1) && \text{(A30)} \\ \Gamma(t-1) &= \beta_Q(t-1) + \beta_{Q_B^1}(t-1)\pi(t-1)^{-1}\xi(t-1) && \text{(A31)} \end{aligned}$$

Large Investors Consumption

Large investors optimal time $t-1$ consumption depends on optimal time $t-1$ trades. After plugging the expressions for equilibrium prices, and equilibrium trades [equations (A28), (A29), and (A25)] into equation (A15), large investors consumption choice problem has form:

$$\max_{C_m(t-1)} -e^{-A_m C_m(t-1)} - \delta k_m(t) e^{rA_m(t)C_m(t-1)} \times \psi_m(Q^1(t-1), W_m(t-1), D(t-1), t-1), \tag{A32}$$

where

$$\begin{aligned} \psi_m(Q^1, W_m(t-1), t-1) = & e^{-.5\bar{Z}^{2\prime}\Omega_{22}^{-1}\bar{Z}^2 - A_m(t)rW_m(t-1)} \\ & \times e^{+A_m(t)r[S_m(H_0(t-1)+H_1(t-1)Q^1(t-1)]'(\alpha(t-1)-\Gamma(t-1)Q^1(t-1))/r} \\ & \times e^{-A_m(t)(G_0(t-1)+G_1(t-1)Q^1(t-1))'\bar{v}_m(t)} \\ & \times e^{.5A_m(t)^2[G_0(t-1)+G_1(t-1)Q^1(t-1)]'\theta_m(t)[G_0(t-1)+G_1(t-1)Q^1(t-1)]} \end{aligned} \tag{A33}$$

The first order condition for choice of consumption implies that optimal consumption is given by:

$$C_m(t-1) = \frac{-1}{A_m(t)r + A_m} \ln\left(\frac{\delta k_m(t)A_m(t)r\psi_m(Q^1(t-1), W_m(t-1), t-1)}{A_m}\right) \tag{A34}$$

Large investors value function at time $t-1$

Define $V_m(t-1, Q^1, W_m(t-1))$ as the value function to large investor m from entering period $t-1$ when the vector of illiquid risky asset holdings is Q^1, and his liquid asset holdings are $W_m(t-1)$. After substituting the optimal consumption choice in (A34) into equation (A32), this value function is given by:

$$V_m(W_m(t-1), Q^1, t-1) = -\left[\frac{1+r_m^*(t)}{r_m^*(t)}\right]\left[\delta k_m(t) r_m^*(t)\psi_m(Q^1, W_m(t-1), t-1)\right]^{\frac{1}{1+r_m^*(t)}} \tag{A35}$$

where,

$$r_m^*(t) = A_m(t)r/A_m \tag{A36}$$

Tedious algebra then shows that large investor m's value function at time $t-1$ has form:

$$V_m(t-1, Q^1, W_m(t-1)) = -k_m(t-1) \times e^{-A_m(t-1)W_m(t-1) - A_m(t-1)Q^{1\prime}\Lambda_m(t-1) + .5A_m(t-1)^2 Q^{1\prime}\Xi_m(t-1)Q^1} \tag{A37}$$

where the parameters of the value function at time $t-1$ are given by the following Riccati difference equations.

$$A_m(t-1) = A_m(t)r/(1+r_m^*(t)) \tag{A38}$$

$$\begin{aligned} k_m(t-1) = & \left[\frac{r_m^*(t)+1}{r_m^*(t)}\right]\left[\delta k_m(t) r_m^*(t)\right]^{\frac{1}{1+r_m^*(t)}} \\ & \times e^{\frac{-.5\bar{Z}^{2\prime}\Omega_{22}^{-1}\bar{Z}^2}{1+r_m^*(t)}} \\ & \times e^{A_m(t-1)H_0(t-1)'S_m'\alpha(t-1)/r - A_m(t-1)G_0(t-1)'\bar{v}_m(t)/r + .5A_m(t-1)^2((1+r_m^*(t))/r^2)(G_0(t-1)'\theta_m(t)G_0(t-1))} \end{aligned} \tag{A39}$$

$$\begin{aligned} \Lambda_m(t-1) = & -H_1(t-1)'S_m'\alpha(t-1)/r + \Gamma(t-1)'S_m H_0(t-1)/r + G_1(t-1)'\bar{v}_m(t)/r \\ & - A_m(t-1)(1+r_m^*(t))G_1(t-1)'\left(\frac{\theta_m(t)+\theta_m(t)'}{2}\right)G_0(t-1)/r^2 \end{aligned} \tag{A40}$$

$$\Xi_m(t-1) = \frac{-2H_1(t-1)'S_m'\Gamma(t-1)}{rA_m(t-1)} + (1+r_m^*(t))G_1(t-1)'\theta_m(t)G_1(t-1)/r^2 \tag{A41}$$

Small investors optimal consumption

The solution for each small investors consumption depends on small investors optimal trades. To solve for optimal consumptions, I first use equation (A7) to substitute out for Q_s^1 in equation (A6). I then substitute out for $\bar{Z}^1(t) - \beta_{12}\bar{Z}^2(t)$ with the expression:

$$\bar{Z}^1(t) - \beta_{12}\bar{Z}^2(t) = a_0(t-1) + a_1(t-1)Q^1(t-1), \tag{A42}$$

where,

$$\begin{aligned} a_0(t-1) &= \frac{\alpha(t)}{r} - \alpha(t-1) + \bar{D}^1 - \beta_{12}\bar{Z}^2(t) - \frac{\Gamma(t)G_0(t-1)}{r} && \text{(A43)} \\ a_1(t-1) &= \Gamma(t-1) - \frac{\Gamma(t)G_1(t-1)}{r}. && \text{(A44)} \end{aligned}$$

Finally I substitute out $Q^1(t)$ with $[G_0(t-1) + G_1(t-1)Q(t-1)]$. With these substitutions, small investors choice of optimal consumptions simplifies to:

$$\max_{C_s(t-1)} -e^{-A_sC_s(t-1)} - \delta k_s(t)e^{rA_s(t)C_s(t-1)} \times \psi_s(Q^1(t-1), W_s(t-1), t-1), \tag{A45}$$

where,

$$\begin{aligned} \psi_s(Q^1(t-1), W_s(t-1), t-1) = {} & e^{-A_s(t)rW_s(t-1) - .5\bar{Z}_2'\Omega_{22}^{-1}\bar{Z}_2} \\ & \times e^{-.5[a_0(t-1)+a_1(t-1)Q^1(t-1)]'\Omega_e^{-1}[a_0(t-1)+a_1(t-1)Q^1(t-1)]} \\ & \times e^{-[G_0(t-1)+G_1(t-1)Q^1(t-1)]'\bar{v}_s(t)} \\ & \times e^{-[G_0(t-1)+G_1(t-1)Q^1(t-1)]'\theta_s(t)[G_0(t-1)+G_1(t-1)Q^1(t-1)]} \end{aligned} \tag{A46}$$

The first order condition for choice of optimal consumption implies that optimal consumption is given by:

$$C_s(t-1) = \frac{-1}{A_s(t)r + A_s}\ln\left(\frac{\delta k_s(t)A_s(t)r\psi_s(Q^1(t-1), W_s(t-1), t-1)}{A_s}\right) \tag{A47}$$

Small investors value function at time $t-1$

Define $V_s(W_s(t-1), Q^1(t-1), t-1)$ as the value function to small investor s from entering period $t-1$ when the vector of illiquid risky asset holdings is $Q^1(t-1)$, and his liquid wealth is $W_s(t-1)$. After substituting the optimal consumption choice in (A47) into equation (A45), this value function is given by:

$$V_s(W_s(t-1), Q^1(t-1), t-1) = \\ -\left[\frac{1 + r_s^*(t)}{r_s^*(t)}\right]\left[\delta k_s(t)r_s^*(t)\psi_s(Q^1(t-1), W_s(t-1), t-1)\right]^{\frac{1}{1+r_s^*(t)}} \tag{A48}$$

where,

$$r_s^*(t) = A_s(t)r/A_s \tag{A49}$$

Simplification then shows that the value function has form:

$$\begin{aligned} V_s(W_s(t-1), Q^1(t-1), t-1) &= -K_s(t-1)\ F(Q^1, t-1)\ e^{-A_s(t-1)W_s(t-1)}, \\ \text{where } F(Q^1(t-1), t-1) &= e^{-Q^1(t-1)'\bar{v}_s(t-1) - Q^1(t-1)'\theta_s(t-1)Q^1(t-1)} \end{aligned} \tag{A50}$$

The parameters in the small investors value functions at time $t-1$ are a function of time t parameters as expressed in the following Riccati equations:

$$A_s(t-1) = \frac{rA_s(t)}{1 + r_s^*(t)} \tag{A51}$$

$$\begin{aligned} k_s(t-1) = &\left[\frac{r_s^*(t)+1}{r_s^*(t)}\right] \left[\delta k_s(t-1) r_s^*(t) e^{-.5\bar{Z}^{2\prime}\Omega_{22}^{-1}\bar{Z}^2}\right]^{\frac{1}{1+r_s^*(t)}} \\ &\times \text{Exp}\left\{\frac{-a_0(t-1)'\Omega_e^{-1}a_0(t-1) - G_0(t-1)'\bar{v}_s(t) - G_0(t-1)'\theta_s(t)G_0(t-1)}{1+r_s^*(t)}\right\}, \end{aligned} \tag{A52}$$

$$\bar{v}_s(t-1) = \frac{a_1(t-1)'\Omega_e^{-1}a_0(t-1) + G_1(t-1)'\bar{v}_s(t) + G_1(t-1)'(\theta_s(t) + \theta_s(t)')G_0(t-1)}{1 + r_s^*(t)}, \tag{A53}$$

$$\theta_s(t-1) = \frac{.5a_1(t-1)'\Omega_e^{-1}a_1(t-1) + G_1(t-1)'\theta_s(t)G_1(t-1)}{1 + r_s^*(t)} \tag{A54}$$

This completes part I of the proof because equations (A37) and (A50) verify that the value functions at time $t-1$ have the same form as at time t.

B.2 Part II

To establish part II of the proof, I need to show that investors value functions for entering entering period T, the last period of trade, has the same functional form as given in the proposition. To establish this result, I first need to solve for investors value function at time $T+1$, the first period when investors cannot trade the illiquid assets (recall they can continue to trade the riskless asset and the liquid assets indefinitely). Then, given this value function, I use backwards induction to solve for investors value function at time T.

Investors Value Functions at Time T+1

Recall that investors are infinitely lived but that from time T onwards they cannot alter their holdings of illiquid assets, but they can continue to alter their consumption, and their holdings of liquid and riskless assets. Because investors cannot trade in period $T+1$ and after, the distinction between small and large investors after this period is irrelevant. Hence, the index m used below could be for either a large or small investor. Using the Bellman principle, the value function $V_m(.)$ of entering period $t+1$ ($t \geq T$) with illiquid asset holdings Q_m^1 and liquid wealth W_m satisfies the functional equation:

$$V_m(W_m(t+1), Q_m^1) = \max_{\substack{C_m(t+1) \\ Q_m^2(t+2)}} - \exp^{-A_m C_m(t+1)} + \delta \, \mathrm{E}\{V_m(W_m(t+2), Q_m^1)\}, \quad t \geq T, \tag{A55}$$

where,

$$W_m(t+2) = Q_m^1{}' D^1(t+2) + Q_m^2{}' Z^2(t+2) + r[W_m(t+1) - C_m(t+1)], \tag{A56}$$

and,

$$Z^2(t+2) = P^2(t+2) + D^2(t+2) - rP^2(t+1).$$

Inspection shows that the function

$$V_m(W_m, Q_m^1) = -K_m \exp^{-A_m[1-(1/r)]W_m - A_m[1-(1/r)]Q_m^1{}' \frac{(1/r)[\bar{D}^1 - \beta_{12}\bar{Z}^2]}{1-(1/r)} + \frac{1}{2}A_m^2[1-(1/r)]^2 Q_m^1{}' \frac{(1/r)\Omega_e}{1-(1/r)} Q_m^1} \tag{A57}$$

with

$$K_m = \frac{r}{r-1} \times (\delta r)^{\frac{1}{r-1}} \times \exp^{-.5 \frac{\bar{Z}^2{}' \Omega_{22}^{-1} \bar{Z}^2}{r-1}},$$

satisfies the Bellman equation (A55) for all time periods $\geq T+1$.

Given the value function at time $T+1$, to solve for investors value functions at time T, I follow the same steps as in equations (A2) through equation (A54). Therefore, substituting in from equation (A57), small investors maximization problem at time T has form:

$$\max_{\substack{C_s(T), \\ Q_s}} \; -e^{-A_s C_s(T)} - \delta \, \mathrm{E}\left\{K_s(T+1) e^{-A_s(T+1)W_s(T+1) - A_s(T+1)Q_s^1{}'\Lambda_s(T+1) + \frac{1}{2}A_s(T+1)^2 Q_s^1{}' \Xi_{se}(T+1) Q_s^1}\right\} \tag{A58}$$

such that,

$$W_s(T+1) = Q_s^1{}' Z^1(T+1) + Q_s^2{}' Z^2(T+1) + r[W_s(T) - C_s(T)], \tag{A59}$$

where,

$$
\begin{aligned}
K_s(T+1) &= \frac{r}{r-1} \times (\delta r)^{\frac{1}{r-1}} \times \exp^{-.5\frac{\bar{Z}^{2\prime}\Omega_{22}^{-1}\bar{Z}^2}{r-1}}, && \text{(A60)}\\
A_s(T+1) &= A_s[1-(1/r)], && \text{(A61)}\\
\Lambda_s(T+1) &= \frac{(1/r)[\bar{D}^1 - \beta_{12}\bar{Z}^2]}{1-(1/r)}, && \text{(A62)}\\
\Xi_{se}(T+1) &= \frac{(1/r)\Omega_e}{1-(1/r)}, && \text{(A63)}\\
Z^1(T+1) &= D^1(T+1) - rP^1(T), && \text{(A64)}\\
Z^2(T+1) &= P^2(T+1) + D^2(T+1) - rP^2(T). && \text{(A65)}
\end{aligned}
$$

Substituting the expression for $W_s(T+1)$ into the value function, taking expectations, and then solving for optimal Q_s^2 given Q_s^1, and substituting that into the value function shows that small investors optimal choice of Q_s^1 and $C_s(T)$ problem has form:

$$
\begin{aligned}
\max_{\substack{C_s(T),\\ Q_s^1}} \quad & -e^{-A_s C_s(T)} - \delta K_s(T+1)\,\mathrm{Exp}\left\{-.5\bar{Z}_2'\Omega_{22}^{-1}\bar{Z}_2 - A_s(T)r[W_s(T) - C_s(T)]\right\}\\
& \times \mathrm{Exp}\left\{-A_s(T+1)Q_s^{1\prime}[\bar{v}_s(T+1) - rP^1(T)] + .5A_s(T+1)^2 Q_s^{1\prime}\Omega_e(T+1)Q_s^1\right\}
\end{aligned} \tag{A66}
$$

where

$$
\begin{aligned}
\bar{v}_s(T+1) &= \left[\frac{\bar{D}^1(T+1) - \beta_{12}\bar{Z}_2(T+1)}{1-(1/r)}\right] && \text{(A67)}\\
\Omega_e(T+1) &= \left[\frac{\Omega_e}{1-(1/r)}\right] && \text{(A68)}
\end{aligned}
$$

Integrating the solution for optimal Q_s^1 over the set of small investors then reveals that the net demand for the illiquid assets by the competitive fringe is:

$$
Q_1^1(T+1) = \frac{1}{A_1(T+1)}[\Omega_e(t+1)]^{-1}[\bar{v}_s(T+1) - rP(t)] \tag{A69}
$$

Following the approach that was used earlier to solve for the price schedule faced by large investors in equation (A9), inverting the small investors demand schedule for the illiquid assets reveals that the price schedule faced by large investors has the form:

$$
P^1(.,T) = \frac{1}{r}\left(\beta_0(T) - \beta_{Q^1}(T)Q^1(T) - \beta_{Q_B^1}(T)\Delta Q_B^1(T)\right), \tag{A70}
$$

$$\beta_0(T) = \bar{v}_s(T+1) \tag{A71}$$
$$\beta_{Q^1}(T) = A_1(T+1)\Omega_e(T+1)S_1) \tag{A72}$$
$$\beta_{Q_B^1}(T) = -A_1(T+1)\Omega_e(T+1)S \tag{A73}$$

Given the price schedule at time T, and the value function in equation (A57), large investors maximization problem at time T can be written in the form:

$$\max_{\substack{C_m(T),\\ Q_m}} \; - e^{-A_m C_m(T)} - \mathrm{E}\left\{\delta K_m(T+1)e^{-A_m(T+1)W_m(T+1)-A_m(T+1)Q^{1\prime}\Lambda_m(T+1)+.5A_m(t)^2Q^{1\prime}\Xi_m(T+1)Q^1}\right\} \tag{A74}$$

where,

$$A_m(T+1) = A_m[1-(1/r)] \tag{A75}$$
$$\Lambda_m(T+1) = S_m'\left[\frac{(1/r)[\bar{D}^1-\beta_{12}\bar{Z}^2]}{1-(1/r)}\right], \tag{A76}$$
$$\Xi_m(T+1) = S_m'\left(\frac{(1/r)\Omega_e}{1-(1/r)}\right)S_m. \tag{A77}$$
$$K_m(T+1) = \frac{r}{r-1}\times(\delta r)^{\frac{1}{r-1}}\times \exp^{-.5\frac{\bar{Z}^{2\prime}\Omega_{22}^{-1}\bar{Z}^2}{r-1}} \tag{A78}$$

Substituting in the budget constraint, liquid wealth at the beginning of time $T+1$ is given by

$$\begin{aligned} W_m(T+1) =& Q_m^1(T+1)'D^1(T+1) + Q_m^2(T+1)'Z^2(T+1) \\ &+ r(W_m(T) - \Delta Q_m^1(T)'P^1(T,.) - C_m(T)) \end{aligned} \tag{A79}$$

Large investors maximization problem at time T has exactly the same form as given in equation (A13). Therefore, the optimal trades and consumption of large investors follow precisely the same equations as given in Part I of the proof. Large investors value function at time T also has the same functional form as in part I. The equilibrium price function at time T also has the same functional form as in part I. Therefore, to complete the proof, it sufficies to solve for small investors consumption and then value function and verify that the value function has the appropriate functional form.

To do so, note that from equation (A66), it is straightforward to show that the optimal choice of $Q_s^1(T+1)$ is

$$Q_s^1(T+1) = \frac{1}{A_s(T+1)}[\Omega_e(T+1)]^{-1}\times[\bar{v}_s(T+1) - rP^1(T)],$$

and that after substituting this expression back in the value function, and making the substitution $P^1(T) = \frac{1}{r}(\alpha(t) - \Gamma(t)Q^1(t))$, then the maximization in equation (A66) simplifies to have the form:

$$\max_{C_s(T)} -e^{-A_s C_s(T)} - \delta K_s(T+1)\,\mathrm{Exp}\{A_s(T) r C_s(T)\} \times \Psi_s(T, Q^1) \tag{A80}$$

where,

$$\begin{aligned}\Psi_s(T, Q^1) = &\,\mathrm{Exp}\left\{-.5\bar{Z}_2'\Omega_{22}^{-1}\bar{Z}_2 - A_s(T) r W_s(T)\right\} \\ &\times \mathrm{Exp}\left\{-.5[\bar{v}_s(T+1) - \alpha(T)]'[\Omega_e(T+1)]^{-1}[\bar{v}_s(T+1) - \alpha(T)]\right\} \\ &\times \mathrm{Exp}\left\{-Q^1(T)'\Gamma(T)'[\Omega_e(T+1)]^{-1}[\bar{v}_s(T+1) - \alpha(T)]\right\} \\ &\times \mathrm{Exp}\left\{-.5Q^1(T)'\Gamma(T)'[\Omega_e(T+1)]^{-1}\Gamma(T)Q^1(T)\right\}\end{aligned} \tag{A81}$$

Using the same approach that was used to solve for large investors optimal consumption and then value function in part I of the proof, tedious algebra shows that small investors value function at time T has form

$$-F(Q^1, T) K_s(T)\,\mathrm{Exp}(-A_s(T) W_s(T))$$

where, $F(Q^1, T) = e^{-Q^1(T)'\bar{v}_s(T) - Q^1(T)'\theta_s(T)Q^1(T)}$,

$$r_s^*(T+1) = A_s(T+1) r / A_s, \tag{A82}$$

$$A_s(T) = A_s(T+1) r / (1 + r_s^*(T+1)), \tag{A83}$$

$$\begin{aligned}K_s(T) = &\left[\frac{r_s^*(T+1)+1}{r_s^*(T+1)}\right][\delta K_s(T+1) r_s^*(T+1)]^{\frac{1}{1+r_s^*(T+1)}} \\ &\times \mathrm{Exp}\left(\frac{-.5\bar{Z}^{2\prime}\Omega_{22}^{-1}\bar{Z}^2 - .5[\bar{v}_s(T+1) - \alpha(T)]'[\Omega_e(T+1)]^{-1}[\bar{v}_s(T+1) - \alpha(T)]}{1 + r_s^*(T+1)}\right),\end{aligned} \tag{A84}$$

$$\bar{v}_s(T) = \frac{\Gamma(T)'[\Omega_e(T+1)]^{-1}[\bar{v}_s(T+1) - \alpha(T)]}{1 + r_s^*(T+1)}, \tag{A85}$$

$$\theta_s(T) = \frac{\Gamma(T)'[\Omega_e(T+1)]^{-1}\Gamma(T)}{1 + r_s^*(T+1)}. \tag{A86}$$

This completes the proof by establishing that large and small investors value functions take the hypothesized form in all periods that involve trade. □

C Solutions for Value Function Parameters

Proposition 4 *For all time periods $t = 1, \ldots, T$, and for large investors $m = 2, \ldots M$:*

$$\bar{v}_m(t) = \frac{S_m'(\bar{D}^1 - \beta_{12}\bar{Z}^2)}{1 - (1/r)} \quad \text{(A87)}$$

$$\alpha(t) = (\bar{D}^1 - \beta_{12}\bar{Z}^2) \quad \text{(A88)}$$

$$A_m(t) = A_m[1 - (1/r)] \quad \text{(A89)}$$

$$r^*(t) = r - 1 \quad \text{(A90)}$$

$$k_m(t) = \left(\frac{r}{r-1}\right) \times (\delta r)^{\frac{1}{r-1}} e^{-.5\frac{\bar{Z}^{2\prime}\Omega_{22}^{-1}\bar{Z}^2}{r-1}} \quad \text{(A91)}$$

Proof:

For $\bar{v}_m(t)$ and $\alpha(t)$:

The proof is by induction. First, suppose that the results for $\bar{v}_m(t)$ and $\alpha(t)$ are true at time t. Then, from equation (A10), $\beta_0(t-1) = \alpha(t)$. This implies that from equation (A21) that$(-S_1 + S_m)\bar{v}_m(t) - \beta_0(t-1) = 0$. As a result $\chi(t-1) = 0$, which implies from equation (A30) that $\alpha(t-1) = \beta_0(t-1)$ and from equations (A26) and (A28) that $H_0(t-1) = G_0(t-1) = 0$. Substituting for $H_0(t-1)$ and $G_0(t-1)$ in equation (A40) and simplifying then shows:

$$\Lambda_m(t-1) = S_m'\alpha(t)/r. \quad \text{(A92)}$$

Finally, substituting this result in equation (A16) proves the result for $\bar{v}_m(t-1)$. To complete the induction, I use equations (A76) and (A16) to solve for $\bar{v}_m(T+1)$; I then substitute the resulting expression as well as the one for $\beta_0(T)$ (equation (A71)) in equation (A21) and use it to show that $\chi(T) = 0$, which implies $G_0(T) = H_0(T) = 0$. Substituting into equation (A30), then shows that $\alpha(T) = \beta_0(T) = S_m'(\bar{D}^1 - \beta_{12}\bar{Z}^2)/[1 - (1/r)]$, which confirms the result for $\alpha(T)$. Finally, given the solutions for $\alpha(T)$ and $\bar{v}_m(T+1)$, substitution in equations (A76) and (A16) confirms the result for $\bar{v}_m(T)$ and completes the induction.

For $A_m(t)$ and $r^*(t)$:

The proof is by backwards induction. We know $A_m(T+1) = A_m[1-(1/r)]$ from equation (A75). Using this expression, and iterating on equations (A38) and (A36) proves the result for all times $t = 1, \ldots T$.

For $k_m(t)$:

The proof is by backwards induction. Equation (A78) establishes that it is true at time $T+1$. Plugging the solution for $K_m(T+1)$ into equation (A39) while using the solutions for $r_m^*(t)$ and the result $H_0(t-1) = G_0(t-1) = 0$ confirms the result for periods $1, \ldots T$. □

The next proposition provides information on the value functions of the small investors:

Proposition 5 *For all time periods $t = 1, \ldots, T$, and for each small investor s*

$$\begin{aligned} a_0(t) &= 0 && \text{(A93)} \\ \bar{v}_s(t) &= 0 && \text{(A94)} \\ A_s(t) &= A_s[1 - (1/r)] && \text{(A95)} \\ r_s^*(t) &= r - 1 && \text{(A96)} \\ k_s(t) &= \left(\frac{r}{r-1}\right) \times (\delta r)^{\frac{1}{r-1}} e^{-.5\frac{\bar{Z}^{2\prime}\Omega_{22}^{-1}\bar{Z}^2}{r-1}} && \text{(A97)} \end{aligned}$$

Proof:

For $a_0(t)$ and $\bar{v}_s(t)$: Plugging the solutions for $\alpha(t)$ and $G_0(t-1)$ from proposition 4 into equation (A43) shows that $a_0(t) = 0$ for all times t. Since $G_0(t-1) = 0$ for all times t, it then follows from equation (A53) that if $\bar{v}_s(t) = 0$, then so does $\bar{v}_s(t-1)$. To complete the induction, note that substituting the solutions for $\bar{v}_s(T+1)$ (equation (A67)) and $\alpha(T)$ (proposition 4) into equation (A85) confirms the result.

For $A_s(t)$, $r_s^*(t)$, and $k_s(t)$:

The form of the proof is identical to that given in proposition 4.□

Proposition 6 *Assume that for $t \leq T$, conditional on state variable $Q^1(t)$ the Nash Equilibrium trades of the large investors exists and is unique. Then for all $m = 2, \ldots, M$ and $t = 1, \ldots, T$, $\theta_m(t)$ has form:*

$$\vartheta_m(t) \otimes \Omega_e, \tag{A98}$$

where, $\vartheta_m(t)$ is $M \times M$; and

$$\Gamma(t) = \gamma(t) \otimes \Omega_e, \tag{A99}$$

where, $\gamma(t)$ is $1 \times M$.

Proof: The proof is by induction. First, assume that the theorem is true at time t. Then, from equations (A12) and (A11) $\beta_{Q_B}(t-1) = B_{Q_B}(t-1) \otimes \Omega_e$, and $\beta_Q(t-1) = B_Q(t-1) \otimes \Omega_e$, where $B_{Q_B}(t-1)$ is $1 \times M - 1$ and $\beta_Q(t-1)$ is $1 \times M$. Applying these substitutions in large investors reaction functions and then stacking the results reveals that in equation (A23), $\pi(t-1) = \mathcal{P}(t-1) \otimes \Omega_e$ and $\xi(t-1) = Z(t-1) \otimes \Omega_e$. The assumption that the Nash Equilibrium trades in each period are unique implies that $\mathcal{P}(t-1)$ is invertible. Solving for $H_0(t-1)$ and $H_1(t-1)$ then shows that $H_0(t-1) = 0$ and

$$\begin{aligned} H_1(t-1) &= \begin{pmatrix} -S[P(t-1)^{-1}Z(t-1)] \otimes I_{N_1} \\ (P(t-1)^{-1}Z(t-1)) \otimes I_{N_1} \end{pmatrix} && \text{(A100)} \\ &= \begin{pmatrix} [-\iota_M'\mathcal{P}(t-1)^{-1}Z(t-1)] \otimes I_{N_1} \\ (\mathcal{P}(t-1)^{-1}Z(t-1)) \otimes I_{N_1} \end{pmatrix} && \text{(A101)} \\ &= \mathcal{H}_1(t-1) \otimes I_{N_1} && \text{(A102)} \end{aligned}$$

where ι_M is a $1\times M$ vector of ones, and $\mathcal{H}_1(t-1)$ is $M\times M$. Since $G_1(t-1) = H_1(t-1)+I_{N_1 M}$, it follows that $G_1(t-1) = \mathcal{G}_1(t-1)\otimes I_{N_1}$ for $\mathcal{G}_1(t-1) = \mathcal{H}_1(t-1)+I_M$. From here, substitution in equation (A31) shows that $\Gamma(t-1) = \gamma(t-1)\otimes\Omega$ and substitution in equation (A41) and (A17) shows that $\theta_m(t-1) = \vartheta_m(t-1)\otimes\Omega$. To complete the induction, I substitute the expression for $\xi_m(T+1)$ (equation (A77)) into equation (A17) and show that the result is true for $\theta_m(T+1)$. Then, following steps similar to those in the first part of the induction, it is straightfoward to show that the result holds for $\Gamma(T)$ and $\theta_m(T)$, which completes the induction. $\Box$.

Corollary 2 *For each small investors, and for each time period* $t = 1, \ldots T$*,*

$$\theta_s(t) = \vartheta_s \otimes \Omega_e,$$

where ϑ_s *is* $M \times M$*.*

Proof: Straightforward induction involving application of the results from proposition 6.

D Proofs of Asset Pricing Propositions

Proposition 7 *When asset markets are imperfectly competitive as specified in section 2 of the text, then if market participants initial asset holdings are* Q^{1W}*, then investors will hold* Q^{1W} *forever, and asset prices and expected returns will be the same as when there is perfect competition.*

Proof: When investors risky asset holdings are Q^{1W}, then investors asset holdings are identical to those associated with a competitive equilibrium and complete markets in which trading is restricted to the set of market participants that has been modelled. Hence, when trade in the first set of assets is restricted to be among the market participants, asset holdings are pareto optimal in all time periods; and investors asset holdings will remain at Q^{1W} because investors have no basis to trade away from asset holdings that are associated with perfect risk sharing. Because Q^{1W} is the vector of asset holdings from a competitive equilibrium, the resulting prices and expected returns which support Q^W are the same as in the competitive equilibrium. $\Box$

Corollary 3 *For all* $t \geq T$*,*

$$[\Gamma(t) - \frac{1}{r}\Gamma(t+1)G_1(t)]Q^{1W} = \lambda_{[X^1]}\Omega_e X^1.$$

Proof: Algebra shows that when asset holdings of asset 1 at time t are Q^{1W}, then excess returns of asset 1 are equal to:

$$P^1(t+1) + \bar{D}^1 - rP^1(t) = \beta_{12}\bar{Z}^2 + [\Gamma(t) - \frac{1}{r}\Gamma(t+1)G_1(t)]Q^{1W}.$$

Proposition 7 shows that when asset holdings are Q^{1W} then the excess returns of asset 1 are $\beta_{12}\bar{Z}^2 + \lambda_{[X^1]}\Omega_e X^1$. Equating the two expressions confirms the claim in the corollary. □.

Proposition 3: *When investors asset holdings of the first asset are not Q^{1W}, then equilibrium expected asset returns satisfy a linear factor model in which one factor is the returns on asset 2, another factor corresponds to perfect risk-sharing, but imperfect diversification of the idiosyncratic risk of asset 1, and the remaining factors correspond to the deviation of large investors asset holdings from those associated with the large investors perfectly sharing the idiosyncratic risk of asset 1.*

Proof: Let Q^{1W} denote the vector of asset holdings of asset 1 that is associated with perfect risk sharing among the investors that trade in asset 1. Manipulation of the equation for equilibrium prices given in proposition 1, and substitution of $G_0(t) + G(t)Q(t)$ for $Q(t+1)$ shows:

$$P^1(t+1)+\bar{D}^1-rP^1(t) = [\frac{1}{r}\alpha(t+1)+\bar{D}^1-\alpha(t)]-[\frac{1}{r}\Gamma(t+1)G_0(t)]+[\Gamma(t)-\frac{1}{r}\Gamma(t+1)G_1(t)]Q^1(t)$$

Plugging in the solution for $\alpha(t) = \alpha(t-1) = [\bar{D}^1 - \beta_{12}\bar{Z}^2]/[1-(1/r)]$ shows the first term in braces on the right hand side of the equation is equal to $\beta_{12}\bar{Z}^2$. The second term in braces is zero since proposition 4 shows that $G_0(t) = 0$. Adding and subtracting Q^{1W} to $Q^1(t)$, the above equation can be rewritten as:

$$P^1(t+1) + \bar{D}^1 - rP^1(t) = \beta_{12}\bar{Z}^2 + [\Gamma(t) - \frac{1}{r}\Gamma(t+1)G_1(t)](Q(t) - Q^W) + [\Gamma(t) - \frac{1}{r}\Gamma(t+1)G_1(t)]Q^{1W} \tag{A103}$$

Using the fact that $Q^1_1 = X^1 - SQ^1_B$, the vector $Q^1(t) - Q^{1W}$ can be expressed in terms of the deviations of large investors asset holdings from pareto optimal asset holdings:

$$\begin{aligned} Q^1(t) - Q^{1W} &= \begin{bmatrix} (X^1 - SQ^1_B) - (X^1 - SQ^{1W}_B) \\ Q^1_B - Q^{1W}_B \end{bmatrix} \\ &= \begin{bmatrix} -S \\ I \end{bmatrix} (Q^1_B - Q^{1W}_B) \end{aligned}$$

Applying the substitution for $Q^1(t) - Q^{1W}$, and the result of corollary 3 in equation (A103) shows

$$P^1(t+1)+\bar{D}^1-rP^1(t) = \beta_{12}\bar{Z}^2+\lambda_{[X^1]}\Omega_e X^1+[\Gamma(t)-\frac{1}{r}\Gamma(t+1)G_1(t)]\begin{pmatrix} -S \\ I \end{pmatrix}(Q^1_B(t)-Q^{1W}_B)$$

Finally, applying the algebra used in the derivation of proposition 6 shows

$$[\Gamma(t) - \frac{1}{r}\Gamma(t+1)G_1(t)]\begin{pmatrix} -S \\ I \end{pmatrix} = \lambda(t) \otimes \Omega_e \tag{A104}$$

where $\lambda(t)$ is $1 \times M-1$. Making this substitution then shows:

$$\begin{aligned} P^1(t+1)+\bar{D}^1-rP^1(t) &= \beta_{12}\bar{Z}^2+\lambda_{[X^1]}\Omega_e X^1+[\lambda(t)\otimes\Omega_e](Q_B^1(t)-Q_B^{1W}) \\ &= \beta_{12}\bar{Z}^2+\lambda_{[X^1]}\Omega_e X^1+\textstyle\sum_{m=2}^{M}\lambda(m,t)\Omega_e(Q_m^1(t)-Q_m^{1W}) \end{aligned} \tag{A105}$$

where $\lambda(m,t)=\lambda(t)s'_{m-1}$. $\square$.

Corollary 1: *When asset holdings at time t are not efficient, then asset returns at time $t+\tau$ follow a factor model in which the market portfolio, the portfolio of nonmarket risk, and the deviation of large investors time t asset holdings from efficient asset holdings are factors.*

Proof: Iterating equation (A103), by τ periods shows:

$$\begin{aligned} P^1(t+\tau+1)+\bar{D}^1-rP^1(t+\tau) = \beta_{12}\bar{Z}^2+[\Gamma(t+\tau)-\frac{1}{r}\Gamma(t+1+\tau)G_1(t+\tau)](Q^1(t+\tau)-Q^{1W}) \\ +[\Gamma(t+\tau)-\frac{1}{r}\Gamma(t+\tau+1)G_1(t+\tau)]Q^{1W}. \end{aligned} \tag{A106}$$

Iterating the equation for equilibrium trades in each period shows

$$Q^1(t+\tau)=[\prod_{j=0}^{\tau-1}G_1(t+j)]Q^1(t).$$

Additionally, because the investors will not trade away from efficient asset holdings, it also follows that

$$[\prod_{j=0}^{\tau-1}G_1(t+j)]Q^{1W}=Q^{1W}.$$

Making both of these substitutions in equation (A106) shows that:

$$\begin{aligned} P^1(t+\tau+1)+\bar{D}^1-rP^1(t+\tau) &= \beta_{12}\bar{Z}^2+\lambda_{[X^1]}\Omega_e X^1+[\lambda(t,\tau)\otimes\Omega_e](Q^1(t)-Q^{1W}) \\ &= \beta_{12}\bar{Z}^2+\lambda_{[X^1]}\Omega_e X^1+\sum_{m=2}^{M}\lambda_m(t,\tau)\Omega_e(Q_m^1(t)-Q_m^{1W}) \end{aligned}$$

where,

$$\lambda(t,\tau)\otimes\Omega_e=[\Gamma(t+\tau)-\frac{1}{r}\Gamma(t+1+\tau)G_1(t+\tau)]\prod_{j=0}^{\tau-1}G_1(t+j),$$

and $\lambda_m(t,\tau)=\lambda(t,\tau)S'_{m-1}$ $\square$.

D.1 Competitive Benchmark Model

It is useful to contrast the behavior in the multi-market model with large investors with the behavior of asset prices and trades in the same model when all investors are price takers and can trade forever.

In this infinite period set-up with competitive markets, the equilibrium risk-premium should be time invariant. Denote this risk premium by ρ, where,

$$\rho = \begin{pmatrix} \rho^1 \\ \rho^2 \end{pmatrix} = \begin{pmatrix} \bar{Z}^1 \\ \bar{Z}^2 \end{pmatrix} = \begin{pmatrix} P^1(t+1) + \bar{D}^1 - rP^1(t) \\ P^2(t+1) + \bar{D}^2 - rP^2(t) \end{pmatrix} \tag{A107}$$

Note that $\bar{Z}^2$ is taken as exogenous. The goal is to solve for $\bar{Z}^1$ and P^1 that makes the prices of the first group of assets (the ones that are illiquid in the imperfect competition model) consistent with equilibrium in all time periods.

Solving the equation for $P^1(t)$ forward while imposing the transversality condition $\lim_{t\to\infty} r^{-t}P^1(t) = 0$, shows that

$$P^1(t) = \frac{\bar{D}^1 - \rho^1}{r-1}$$

for all time periods t.

Given the hypothesized behavior of prices, it remains to solve for ρ^1 and then to show that the hypothesized behavior of prices is consistent with equilibrium.

The function,

$$V_m(W,t) = -\frac{r}{r-1}(r\ \delta)^{\frac{-1}{r-1}} \exp^{-A_m(1-(1/r))W - \frac{.5\bar{Z}^{2\prime}\Omega_{22}^{-1}\bar{Z}^2}{r-1} - \frac{.5\rho^{1\prime}\Omega_e^{-1}\rho^1}{r-1}}$$

and the risk premium solution

$$\rho^1 = \bar{Z}^1 = \beta_{12}\bar{Z}^2 + \lambda_{[X^1]}\Omega_e X^1, \tag{A108}$$

where,

$$\lambda_{[X^1]} = \frac{(1-(1/r))}{\sum_{m=1}^{M}(1/A_m)} \tag{A109}$$

satisfies the Bellman equation,

$$V_m(W,t) = \max_{\substack{C_m(t), \\ Q_m^1(t), \\ Q_m^2(t)}} -e^{-A_m C_m(t)} + E_t\{\delta V_m(W(t+1), t+1)\},$$

such that,

$$W(t+1) = Q_m^1(t)'Z^1(t) + Q_m^2(t)'Z^2(t) + r[W(t) - C_m(t)].$$

In addition, in the competitive equilibrium, investors optimal choices of Q_m^1 are constant through time, and are market clearing for the hypothesized ρ^1. Investor $m's$ competitive equilibrium holdings of Q_m^1 is denoted by Q_m^{1W} and is equal to

$$Q_m^{1W} = \frac{(1/A_m)X^1}{\sum_{m=1}^{M}(1/A_m)}, \quad m = 1, \dots M. \tag{A110}$$

Substituting the hypothesized ρ^1 into the expression for equilibrium P^1, it follows that in a competitive equilibrium, the equilibrium price is given by

$$P^1(t) = \frac{\bar{D}^1 - \beta_{12}\bar{Z}^2}{r-1} - \frac{\Omega_e X^1}{r\sum_{m=1}^{M}\frac{1}{A_m}}, \quad t = 1, \dots \infty \tag{A111}$$

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Table 1: IPO Under-Pricing and Under-Performance by Competitiveness: I.

A. IPO Under-Pricing and Under-Performance

Herf.	Periods Liq	P_Offer	P_Open	Und_Price	S-T Return	L-T Return
3543.26	2000	43.94	44.61	-0.67	-0.01	0.13
3543.26	1800	43.93	44.70	-0.78	-0.01	0.13
3543.26	1600	43.92	44.80	-0.88	-0.01	0.13
3543.26	1400	43.91	44.90	-0.99	-0.01	0.13
3543.26	1200	43.90	45.00	-1.10	-0.01	0.13
3543.26	1000	43.89	45.10	-1.21	-0.01	0.13
3543.26	800	43.88	45.20	-1.33	-0.01	0.13
3543.26	600	43.87	45.31	-1.45	-0.01	0.13
3543.26	400	43.85	45.42	-1.56	-0.01	0.13

B. Investors Risk Bearing Capacity

Investor Number	Type	Risk Bearing Capacity
1	Retail	10.00
2	Institutional	54.56
3	Institutional	21.82
4	Institutional	8.73
5	Institutional	3.49
6	Institutional	1.40

C. IPO Allocation Distortions (Percent)

	Investor Number					
Post-IPO Trading Periods	1	2	3	4	5	6
2000	-100.00	83.29	-100.00	-100.00	-100.00	-100.00
1800	-100.00	83.29	-100.00	-100.00	-100.00	-100.00
1600	-100.00	83.29	-100.00	-100.00	-100.00	-100.00
1400	-100.00	83.29	-100.00	-100.00	-100.00	-100.00
1200	-100.00	83.29	-100.00	-100.00	-100.00	-100.00
1000	-100.00	83.29	-100.00	-100.00	-100.00	-100.00
800	-100.00	83.29	-100.00	-100.00	-100.00	-100.00
600	-100.00	83.29	-100.00	-100.00	-100.00	-100.00
400	-100.00	83.29	-100.00	-100.00	-100.00	-100.00

Table 2: IPO Under-Pricing and Under-Performance by Competitiveness II.

A. IPO Under-Pricing and Under-Performance

Herf.	Periods Liq	P_Offer	P_Open	Und_Price	S-T Return	L-T Return
2225.00	2000	44.00	44.00	0.00	0.12	0.13
2225.00	1800	44.00	44.00	0.00	0.12	0.13
2225.00	1600	44.00	44.03	-0.03	0.00	0.13
2225.00	1400	44.03	44.18	-0.16	-0.01	0.13
2225.00	1200	44.05	44.34	-0.29	-0.01	0.13
2225.00	1000	44.08	44.50	-0.42	-0.01	0.13
2225.00	800	44.11	44.67	-0.56	-0.01	0.13
2225.00	600	44.14	44.83	-0.70	-0.01	0.13
2225.00	400	44.16	45.00	-0.84	-0.01	0.13
2225.00	200	44.19	45.17	-0.98	-0.01	0.13

B. Investors Risk Bearing Capacity

Investor Number	Type	Risk Bearing Capacity
1	Retail	10.00
2	Institutional	40.00
3	Institutional	12.50
4	Institutional	12.50
5	Institutional	12.50
6	Institutional	12.50

C. IPO Allocation Distortions (Percent)

	Investor Number					
Post-IPO Trading Periods	1	2	3	4	5	6
2000	-100.00	-19.50	35.60	35.60	35.60	35.60
1800	-100.00	-19.44	35.55	35.55	35.55	35.55
1600	-100.00	150.00	-100.00	-100.00	-100.00	-100.00
1400	-100.00	150.00	-100.00	-100.00	-100.00	-100.00
1200	-100.00	150.00	-100.00	-100.00	-100.00	-100.00
1000	-100.00	150.00	-100.00	-100.00	-100.00	-100.00
800	-100.00	150.00	-100.00	-100.00	-100.00	-100.00
600	-100.00	150.00	-100.00	-100.00	-100.00	-100.00
400	-100.00	150.00	-100.00	-100.00	-100.00	-100.00
200	-100.00	150.00	-100.00	-100.00	-100.00	-100.00

Table 3: IPO Under-Pricing and Under-Performance by Market Competitiveness III.

A. IPO Under-Pricing and Under-Performance

Herf.	Periods Liq	P_Offer	P_Open	Und_Price	S-T Return	L-T Return
2100.00	2000	44.00	44.00	0.00	0.11	0.13
2100.00	1800	44.00	44.00	0.00	0.11	0.13
2100.00	1600	44.00	44.00	0.00	0.11	0.13
2100.00	1400	44.00	44.00	0.00	0.11	0.13
2100.00	1200	44.00	44.00	0.00	0.11	0.13
2100.00	1000	44.00	44.00	0.00	0.11	0.13
2100.00	800	44.00	44.00	0.00	0.11	0.13
2100.00	600	44.00	44.00	0.00	0.11	0.13
2100.00	400	44.00	44.00	0.00	0.04	0.13
2100.00	200	44.13	44.36	-0.23	-0.01	0.13

B. Investors Risk Bearing Capacity

Investor Number	Type	Risk Bearing Capacity
1	Retail	10.00
2	Institutional	30.00
3	Institutional	30.00
4	Institutional	10.00
5	Institutional	10.00
6	Institutional	10.00

C. IPO Allocation Distortions (Percent)

	Investor Number					
Post-IPO Trading Periods	1	2	3	4	5	6
2000.00	-100.00	0.46	0.46	32.42	32.42	32.42
1800.00	-100.00	0.46	0.46	32.42	32.42	32.42
1600.00	-100.00	-0.13	-0.13	33.58	33.58	33.58
1400.00	-100.00	0.46	0.46	32.42	32.42	32.42
1200.00	-100.00	0.46	0.46	32.42	32.42	32.42
1000.00	-99.99	0.46	0.46	32.42	32.42	32.42
800.00	-100.00	0.46	0.46	32.42	32.42	32.42
600.00	-100.00	0.46	0.46	32.42	32.42	32.42
400.00	-100.00	66.67	66.67	-100.00	-100.00	-100.00
200.00	-100.00	66.67	66.67	-100.00	-100.00	-100.00

Table 4: IPO Under-Pricing and Under-Performance by Market Competitiveness IV.

A. IPO Under-Pricing and Under-Performance

Herf.	Periods Liq	P_Offer	P_Open	Und_Price	S-T Return	L-T Return
1620.00	2000	44.00	44.00	0.00	0.11	0.13
1620.00	1800	44.00	44.00	0.00	0.11	0.13
1620.00	1600	44.00	44.00	0.00	0.11	0.13
1620.00	1400	44.00	44.00	0.00	0.11	0.13
1620.00	1200	44.00	44.00	0.00	0.11	0.13
1620.00	1000	44.00	44.00	0.00	0.11	0.13
1620.00	800	44.00	44.00	0.00	0.11	0.13
1620.00	600	44.00	44.00	0.00	0.11	0.13
1620.00	400	44.00	44.00	0.00	0.11	0.13
1620.00	200	44.00	44.00	0.00	0.11	0.13

B. Investors Risk Bearing Capacity

Investor Number	Type	Risk Bearing Capacity
1	Retail	10.00
2	Institutional	18.00
3	Institutional	18.00
4	Institutional	18.00
5	Institutional	18.00
6	Institutional	18.00

C. IPO Allocation Distortions (Percent)

	Investor Number					
Post-IPO Trading Periods	1	2	3	4	5	6
2000	-100.00	11.11	11.11	11.11	11.11	11.11
1800	-100.00	11.11	11.11	11.11	11.11	11.11
1600	-100.00	11.11	11.11	11.11	11.11	11.11
1400	-100.00	11.11	11.11	11.11	11.11	11.11
1200	-100.00	11.11	11.11	11.11	11.11	11.11
1000	-100.00	11.11	11.11	11.11	11.11	11.11
800	-100.00	11.11	11.11	11.11	11.11	11.11
600	-100.00	11.11	11.11	11.11	11.11	11.11
400	-100.00	11.11	11.11	11.11	11.11	11.11
200	-100.00	11.11	11.11	11.11	11.11	11.11