Diversification and Correlation Ambiguity

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ABSTRACT

One of the most important insights of modern finance is the diversification of the optimal portfolio that usually contains all available risky assets. In this paper, we show that correlation ambiguity can generate anti-diversification in the sense that the optimal portfolio has exactly one risky asset even though there are $N > 1$ available risky assets. Generally, correlation ambiguity leads to under-diversification in the sense that the optimal portfolio contains only a portion of the available risky assets. With 100 stocks randomly selected from S&P 500, on average, approximately 20 stocks will be held in the optimal portfolio when the sets of ambiguous correlations are given by 95% confidence intervals. Our results suggest that aversion to correlation ambiguity may provide an explanation for the under-diversification documented in empirical studies.

JEL classification: G11

Diversification and Correlation Ambiguity

WE STUDY THE PORTFOLIO CHOICE PROBLEM of an agent who is averse to ambiguity in correlations. We find that the aversion to correlation ambiguity may lead to *anti-diversification*[1], that is, the optimal portfolio contains one sole risky asset. In general, the aversion to correlation ambiguity generates *under-diversification*, in the sense that the optimal portfolio contains only a portion of the available risky assets. In contrast, one of the most important insights of modern finance theory is diversification in that the optimal portfolio should contain all available risky assets. This insight is true for expected utility theories including the Markowitz's static portfolio choice theory and the Merton's dynamic portfolio choice theory. Our paper suggests that diversification may not hold in the presence of correlation ambiguity.

We show that anti-diversification may arise with correlation ambiguity. We characterize the sufficient and necessary condition for occurrence of anti-diversification and disclose that anti-diversification occurs when correlations are sufficiently ambiguous. Intuitively, when correlations are totally ambiguous, an optimal portfolio for an agent who is averse to this ambiguity should be insensitive to correlations. Only portfolios that consist solely of one asset are insensitive to correlations; hence in the mean-variance framework the optimal strategy for the agent with ambiguity aversion is to hold the asset that has the greatest Sharpe ratio.

Generally, when correlations are not completely ambiguous, that is, when correlations can take values in strict subsets of $[-1, 1]$, we have under-diversification in the sense that the optimal portfolio does not contain all risky assets. The number of risky assets in the optimal portfolio can be substantially smaller than the total number of the available risky assets. For example, given 100 randomly-selected U.S. stocks with ambiguous sets being 95% confidence intervals of correlation estimations, the optimal portfolio has approximately 20 stocks. As the degree of correlation ambiguity increases, stocks with a lower Sharpe ratio

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[1]Goldman (1979) coined this term for holding one risky asset in the optimal portfolio.

will tend to be omitted from the optimal portfolio until the one with the greatest Sharpe ratio remains. In contrast, without ambiguity aversion all 100 stocks are held under the mean-variance framework.

Under-diversification is documented in many empirical studies. For example, Campbell (2006) suggests that the financial portfolios of households contain a few risky assets. Goetzmann and Kumar (2008) report that the majority of individual investors hold a single digit number of assets in a sample data set from 1991-1996. Among many other empirical findings regarding under-diversification from various data sets, we refer to Mitton and Vorkink (2007), Calvet, et al. (2008), and Ivković, et al. (2008). Our result suggests that correlation ambiguity may be an explanation of these findings.

There are other explanations of under-diversification. Brennan (1975) finds that the optimal number of risky assets in a portfolio is small when there are fixed transaction costs. Goldman (1979) shows that, for buy-and-hold strategies, an infinite time horizon leads to holding one risky asset. Liu (2014) proposes a model in which under-diversification may be caused by solvency requirements in the presence of committed consumption. Roche et al. (2013) suggest that financial constraints can lead to under-diversification. Boyle, et al. (2012) can produce under-diversification with ambiguous expected returns.

Among related literature, Uppal and Wang (2003) study dynamic portfolio choice and show that the ambiguity of the return distribution can cause biased positions relative to the standard mean-variance portfolio. Guidlin and Liu (2014) examine asset allocation decisions under ambiguity aversion. The researchers find that the ambiguity aversion can generate a strong *home bias*. The term *under-diversification* used in both papers refers to bias in certain assets, whereas, in our paper, under-diversification means zero holdings of certain risky assets. Goldfarb and Iyengar (2003) and Tütüncü and Koenig (2004) study the (robust) portfolio choice problem under uncertain expected return and covariance; however, none of these discuss anti-diversification or under-diversification. Izhakian (2012) suggests that holding a fully diversified portfolio is not necessarily optimal in her framework

of ambiguity.

The present paper studies ambiguity in correlations. We assume that correlations are unable to be estimated accurately and the agent is averse to the uncertainty of the estimations. Jagannathan and Ma (2003) note that covariances (and correlations) are imprecisely estimated particularly when the number of assets is large. Moreover, empirical studies, for example, Longin and Solnik (2001) and Cappiello et al. (2006) [2], show that correlations are dynamic; therefore, they are more difficult to estimate than constant correlations in a model. We refer to Engle and Sheppard (2001) and Engle (2002) for estimation of dynamic correlations.

As in the literature, we use the terms *ambiguity* (*ambiguous*) and *uncertainty* (*uncertain*) indifferently. Both terms are different from *risk*, which has known probability. We refer to Knight (1921), Ellsberg (1961), Maehout (2004) and Hansen and Sargent (2001) for more discussions on ambiguity.

The paper is organized as follows. In Section I, we formulate the framework of portfolio choice with an aversion to correlation ambiguity. In Section II, we present anti-diversification results for a case of two risky assets. In Section III we study anti-diversification for a general case of N risky assets. Under-diversification is studied in Section IV. The results regarding empirical calibration are also reported in this section. Our conclusions are presented in Section V. Proofs for certain propositions and a technique note are collected in the Appendix.

I. Correlation Ambiguity

In this section, we present the formulation of portfolio choice under correlation ambiguity.

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[2] There are many other empirical studies on correlations, for example, Ball and Torous (2000), Ang and Chen (2002), and Driessen, et al. (2007).

A. Objective Function

We assume that there is a risk-free asset with a constant return r_f, and there are N risky assets with random returns $r_1, ..., r_N$. Let $\mu = (\mu_1, ..., \mu_N)^\top$ denote the expected excess return vector of the risky assets, where the convention $^\top$ denotes the transpose, and let Σ denote the variance-covariance matrix of the excess returns. Let ϕ_n, $n = 1, ..., N$, denote the dollar amounts that are invested in risky asset n and denote the portfolio by $\phi = (\phi_1, ..., \phi_N)^\top$. We consider the following objective

$$\max_{\phi} \mu^\top \phi - \frac{A}{2} \phi^\top \Sigma \phi, \tag{1}$$

where A is the absolute risk aversion coefficient. This objective function can be justified as the utility equivalent of an expected utility with normally distributed returns and a constant absolute risk aversion utility function.

The optimal portfolio without ambiguity of μ and Σ is the solution to the optimization problem (1) given by

$$\phi^* = \frac{1}{A} \Sigma^{-1} \mu.$$

Let σ_n denote the standard deviation of excess return n, $n = 1, ..., N$ and let σ be the diagonal matrix with diagonal entries $\sigma_1, ..., \sigma_N$ in order. Let $\rho = (\rho_{ij})_{1 \leq i,j \leq N}$ be the correlation matrix of the excess returns, where $\rho_{ij} = 1$ if $i = j$, and ρ_{ij} is the correlation of risky asset i and risky asset j if $i \neq j$. Define $s = (s_1, ..., s_N)^\top$, where $s_n = \mu_n / \sigma_n$ is the Sharpe ratio of risky asset n. Without loss of generality, we assume that Σ is non-singular.

By the change of variable $\psi = \sigma \phi$, the objective problem (1) can be re-written as

$$\max_{\psi} s^\top \psi - \frac{A}{2} \psi^\top \rho \psi. \tag{2}$$

The optimal solution to (2) is $\psi^* = \frac{1}{A}\rho^{-1}s$ and the optimal portfolio ϕ^* is re-represented by

$$\phi^* = \sigma^{-1}\psi^* = \frac{1}{A}\sigma^{-1}\rho^{-1}s.$$

The value function is then obtained by substituting the optimal portfolio into (1) and is given by

$$V = \frac{1}{2A}s^\top \rho^{-1}s.$$

Hence the value function depends two sets of parameters: ρ and s. While previous studies focus on s (mostly on μ), the present paper investigates the role of ρ in portfolio choice when the parameters are ambiguous. From the above expressions, volatility ambiguity can be treated the same as expected return ambiguity or Sharpe ratio ambiguity. However, correlation ambiguity is different from Shapre ratio ambiguity.

As studied in Gilboa and Schmeidler (1989) or Garlappi, et al. (2007), an agent with ambiguity aversion takes the following max-min objective, where the minimization reflects the agent's aversion to the ambiguity.

$$J = \max_\phi \min_\rho \mu^\top \phi - \frac{A}{2}\phi^\top \Sigma \phi.$$

By applying a version of the minimax theorem (see e.g., Theorem 45.8, Strasser (1985)), we can simplify the objective as follows.

$$\max_\phi \min_\rho \mu^\top \phi - \frac{A}{2}\phi^\top \Sigma \phi = \min_\rho \max_\phi \mu^\top \phi - \frac{A}{2}\phi^\top \Sigma \phi = \min_\rho \frac{1}{2A}s^\top \rho^{-1}s. \tag{3}$$

If $\rho = I_N$, the $N \times N$ identity matrix, $s^\top \rho^{-1}s$ is the sum of squared Sharpe ratios. If $\rho \neq I_N$, $s^\top \rho^{-1}s$ is the sum of squared Sharpe ratios of independent risk, as we will show later.

B. Ambiguous Set

We need to specify ambiguous sets in which the minimization in (3) is implemented. It is standard to use confidence intervals as the ambiguous sets in the case of expected return ambiguity. We will also use confidence intervals as ambiguity sets of correlations in this paper. However, most of our results do not rely on this specification.

We obtain confidence intervals for point estimations of correlations by a standard method in statistics. Let $R_p = \frac{\sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y})}{\sqrt{\sum_{i=1}^{n}(X_i - \bar{X})^2(Y_i - \bar{Y})^2}}$ for a paired sample $(X_1, Y_1), ..., (X_n, Y_n)$, with sample mean $(\bar{X}, \bar{Y})$. The Fisher transform $F(R_p) = \frac{1}{2}\ln(\frac{1+R_p}{1-R_p})$ is approximately normally distributed with mean $\frac{1}{2}\ln(\frac{1+p}{1-p})$ and variance $\frac{1}{n-3}$, where p is the population correlation. The confidence bounds are based on the asymptotic normal distribution. These bounds are accurate for large samples when variables have a multivariate normal distribution.

It is well-known that correlations satisfy constraints $|\rho_{ij}| \leq 1$. When $N \geq 3$, there are additional constraints on $\{\rho_{ij}\}_{i<j}$ due to the requirement that ρ must be positive definite. For example, $N = 3$, the three pairs of correlation coefficients must satisfy

$$\rho_{12}^2 + \rho_{13}^2 + \rho_{23}^2 - 2\rho_{12}\rho_{13}\rho_{23} < 1. \tag{4}$$

When confidence intervals are sufficiently small, $\rho_{ij}, 1 \leq i < j \leq N$, inside the confidence intervals should satisfy the constraints. When confidence intervals are large, only those ρ_{ij}'s that satisfy the positive definite constraints are chosen from the intervals. Such $\{\rho_{ij}\}$'s will be referred to as *admissible*. One can also specify ambiguous sets of correlations by an elliptical set or a sphere $\sum_{i<j} |\rho_{ij} - \hat{\rho}_{ij}|^2 < \delta$, where $\hat{\rho}_{ij}$ are estimations. Our general results hold true for such a setting. For our theoretical results, one can verify that the optimal ρ satisfies these constraints. When we solve the optimization problem numerically, we use an algorithm under which the positive definite constraint is always binding.

Our formulation is based on a mean-variance static portfolio choice framework. Assuming constant expected returns and variance-covariance matrix, our results also apply to Merton's

dynamic portfolio choice framework.

Note that when we consider Sharpe ratio ambiguity, the objective is similar to (3) as follows.

$$\max_{\phi} \min_{s} \mu^{\top}\phi - \frac{A}{2}\phi^{\top}\Sigma\phi = \min_{s} \max_{\phi} \mu^{\top}\phi - \frac{A}{2}\phi^{\top}\Sigma\phi = \min_{s} \frac{1}{2A}s^{\top}\rho^{-1}s. \tag{5}$$

The Sharpe ratio ambiguity nests the expected return ambiguity, which has been studied in the literature extensively. In an Internet appendix, we derive general results on Sharpe ratio ambiguity, which are extensions of the existing literature. In the remainder of this paper, we focus on correlation ambiguity.

II.   Two Risky Assets

When there are two risky assets ($N = 2$), the optimal portfolio with aversion to correlation ambiguity can be solved in closed form. This case is also interesting by itself; it is relevant to the situation if an investor considers inclusion of a new risky asset in her existing portfolio. The proposition below shows that the agent with ambiguity aversion may hold only one of the two risky assets.

PROPOSITION 1: *Assume $s_1 > s_2 \geq 0$. If the correlation is completely ambiguous, that is, the ambiguous set of the correlation ρ_{12} is $[-1, 1]$, the agent will solely hold asset 1.*

Proof. When $N = 2$,
$$s^{\top}\rho^{-1}s = s_1^2 + \frac{(s_2 - \rho_{12}s_1)^2}{(1 - \rho_{12}^2)}.$$

It follows that

$$\min_{\rho} \frac{1}{2A}s^{\top}\rho^{-1}s = \min_{\rho} \frac{1}{2A}\left(s_1^2 + \frac{(s_2 - \rho_{12}s_1)^2}{(1 - \rho_{12}^2)}\right),$$

where the first term in the bracket is the squared Sharpe ratio of asset 1, the second term

is the squared Sharpe ratio of a portfolio given by $(-\frac{\sigma_2}{\sigma_1}\rho_{12}, 1)$. Note that asset 1 and the portfolio are uncorrelated.

The first order condition of the above minimization problem leads to $\rho_{12}^* = s_2/s_1 \in [0, 1]$, given that the range of ρ_{12} is $[-1, 1]$, and $s_1 > s_2 \geq 0$. Then the optimal portfolio is

$$\phi^* = \frac{1}{A}\sigma^{-1}(\rho^*)^{-1}s = \frac{1}{A}\left(\frac{s_1}{\sigma_1}, 0\right)^\top. \tag{6}$$

Note that the second component is exactly zero, and thus we have anti-diversification. $\square$

Intuitively, when the correlation is completely ambiguous, an agent who is ambiguity averse will hold a portfolio that is insensitive to correlation.[3] Such portfolios are portfolios with only one risky asset. When there are two risky assets, there are two such portfolios. The one with a higher Sharpe ratio will be chosen.

The above result can be intuitively understood in the following way. Suppose that the risky asset returns are given by

$$r_1 = r_f + \mu_1 + \sigma_1\epsilon_1,$$

$$r_2 = r_f + \mu_2 + \sigma_2\epsilon_2,$$

where ϵ_1, ϵ_2 are two sources of shocks with correlation ρ_{12}. For a given correlation ρ_{12}, we can decompose the returns as follows

$$r_1 = r_f + \mu_1 + \sigma_1\epsilon_1,$$

$$r_2 = r_f + \mu_2 + \sigma_2(\rho_{12}\epsilon_1 + \sqrt{1 - \rho_{12}^2}\hat{\epsilon}_2), \tag{7}$$

where $\hat{\epsilon}_2$ is a shock independent of ϵ_1. Both ϵ_1 and $\hat{\epsilon}_2$ are standard normal random variables.

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[3]We thank Michael Brennan for pointing this out to us.

Then the return of asset 2 can be expressed as

$$r_2 = r_f + \beta(\mu_1 + \sigma_1\epsilon_1) + \alpha + \sqrt{1 - \rho_{12}^2}\sigma_2\hat{\epsilon}_2.$$

where $\beta = \rho_{12}\frac{\sigma_2}{\sigma_1}$ is the regression (in population) coefficient of return 2 on return 1 and $\alpha = (\mu_2 - \rho_{12}\frac{\sigma_2}{\sigma_1}\mu_1)$.

If $\alpha = 0$, there is no compensation for $\hat{\epsilon}_2$ risk, and return 2 is just return 1 plus a pure noise. In this case, $\hat{\epsilon}_2$ is similar to an idiosyncratic risk in the market model with r_1 as the market. If $\alpha \neq 0$, it is the compensation for $\hat{\epsilon}_2$.

The optimal portfolio which solves the optimization problem (1) is given by

$$\begin{aligned}
\phi_1^* &= \frac{\mu_1}{A\sigma_1^2} - \beta\phi_2^*, \\
\phi_2^* &= \frac{\alpha}{A(1-\rho_{12}^2)\sigma_2^2}.
\end{aligned} \tag{8}$$

Hence asset 2 will not be held if and only if $\alpha = 0$, which is equivalent to the condition $\rho_{12} = s_2/s_1$. Note that without loss of generality, we assume that the two Sharpe ratios are sorted in descending order $s_1 > s_2 \geq 0$. So $0 \leq s_2/s_1 < 1$.

Given the above optimal portfolio $\phi^* = (\phi_1^*, \phi_2^*)^\top$, the value function in the optimization problem (3) is

$$J = \min_\rho \mu^\top \phi^* - \frac{A}{2}(\phi^*)^\top \Sigma \phi^* = \min_\rho \frac{1}{2A}\left(\frac{\mu_1^2}{\sigma_1^2} + \frac{\alpha^2}{(1-\rho_{12}^2)\sigma_2^2}\right) = \min_\rho \frac{1}{2A}\left(s_1^2 + \frac{(s_2 - \rho_{12}s_1)^2}{(1-\rho_{12}^2)}\right). \tag{9}$$

Therefore, the value function is determined by the sum of squared Sharpe ratios of independent risks. In the above decomposition, the independent risks are ϵ_1 and $\hat{\epsilon}_2$, with Sharpe ratios s_1 and $\sqrt{\alpha^2/(1-\rho_{12}^2)\sigma_2^2} = \sqrt{(s_2 - \rho_{12}s_2)^2/(1-\rho_{12}^2)}$ respectively. Such an uncorrelated/independent decomposition will be exploited again in the case of N risky assets.

Note that the value function is non-monotonic in ρ_{12}. In general, $\rho_{12} = 1$ is not the worst

case. In fact, the utility level is unbounded as $\rho_{12} \to 1$ as long as $s_2 < s_1$. When $\rho_{12} = 1$, the two assets are perfectly substitutable as long as risk is concerned, but asset 1 is better when risk-return tradeoff is taken into account because it has a higher Sharpe ratio. Thus the agent views this as an arbitrage opportunity and would take infinite positions (infinite long position on asset 1 and infinite short position on asset 2).[4]

Proposition 1 can be extended to the case that the correlation is in a subinterval of $[-1, 1]$, that is, $\rho_{12} \in [\underline{\rho}_{12}, \bar{\rho}_{12}] \subset [-1, 1]$. For this case, we will see a more general portfolio choice under correlation ambiguity.

Note that

$$\frac{\partial}{\partial \rho_{12}} \left(s_1^2 + \frac{(s_2 - \rho_{12}s_1)^2}{1 - \rho_{12}^2} \right) = 2s_1^2 \frac{(\frac{s_2}{s_1} - \rho_{12})(-1 + \rho_{12}\frac{s_2}{s_1})}{(1 - \rho_{12}^2)^2}.$$

The right hand side is a function of ρ_{12} and its only root in $[-1, 1]$ is s_2/s_1. Thus, the function $(s_1^2 + \frac{(s_2 - \rho_{12}s_1)^2}{1 - \rho_{12}^2})$ decreases when ρ_{12} is in $[-1, s_2/s_1]$ and increases in $[s_2/s_1, 1]$. From this property one can determine that ρ_{12}^* where

$$\rho_{12}^* = \text{argmin}_{\rho_{12} \in [\underline{\rho}_{12}, \bar{\rho}_{12}]} \frac{1}{2A} \left(s_1^2 + \frac{(s_2 - \rho_{12}s_1)^2}{1 - \rho_{12}^2} \right)$$

is the correlation coefficient chosen by the ambiguity averse agent as follows:

If s_2/s_1 is in the range of ambiguity, then $\rho_{12}^* = s_2/s_1$ and only asset 1 will be held. Anti-diversification occurs. This includes the complete ambiguity of Proposition 1 as a special case.

If the range of ρ_{12} is given by $\bar{\rho}_{12} < s_2/s_1$, then $\rho_{12}^* = \bar{\rho}_{12}$ and both assets will be held in long position.

If the range of ρ_{12} is given by $\underline{\rho}_{12} > s_2/s_1$, then $\rho_{12}^* = \underline{\rho}_{12}$ and both assets will be held. Asset 1 will be held in long position while asset 2 will be held in short position.

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[4]If further the condition $s_1 = s_2$ holds along with $\rho_{12} = 1$, the two assets are completely substitutable in both risk and risk-return tradeoff. The optimal portfolio is arbitrary as long as it satisfies $\phi_1^*/\sigma_1 + \phi_2^*/\sigma_2 = s_1/A$.

The above analysis is summarized as a proposition below.

PROPOSITION 2: *Suppose that there are only two risky assets. Assume $s_1 > s_2 \geq 0$, and $\rho_{12} \in [\underline{\rho}_{12}, \bar{\rho}_{12}]$ which is a subinterval of $[-1, 1]$. Depending on the values of $\underline{\rho}_{12}$ and $\bar{\rho}_{12}$, the optimal portfolio under correlation ambiguity is given by*

$$\phi^* = \begin{cases} (\frac{\mu_1}{A\sigma_1^2}, 0)^\top, & if \ \underline{\rho}_{12} < \frac{s_2}{s_1} < \bar{\rho}_{12}, \\ \frac{1}{A}\Sigma^{-1}(\bar{\rho}_{12})\mu, & if \ \bar{\rho}_{12} < \frac{s_2}{s_1}, \\ \frac{1}{A}\Sigma^{-1}(\underline{\rho}_{12})\mu, & if \ \underline{\rho}_{12} > \frac{s_2}{s_1}, \end{cases} \tag{10}$$

where $\Sigma^{-1}(\bar{\rho}_{12})$ and $\Sigma^{-1}(\underline{\rho}_{12})$ are the inverse matrices of Σ with ρ_{12} replaced by $\bar{\rho}_{12}$ and $\underline{\rho}_{12}$ respectively.

We have shown that anti-diversification may occur under correlation ambiguity by Proposition 1 and Proposition 2. We next discuss more features of the optimal portfolio choice under correlation ambiguity.

One might expect that the aversion to correlation ambiguity leads to an optimal portfolio with more risk-free asset holding than the standard mean-variance portfolio. It turns out that this is not generally true. The following numerical example illustrates that the total allocation on risky assets may be increased under ambiguous correlations. Therefore, the allocation on the risk-free asset may be lower under the aversion to correlation ambiguity.

Example 1: Suppose $\mu = (0.3, 0.5)^\top, \sigma = diag(0.4, 0.8)$. Then $s = (0.75, 0.625)^\top$. Assume $\rho_{12} = 0.82$ and its ambiguous set $[0.6, 0.85]$. Let the risk aversion A be 1. Then the optimal portfolio under the ambiguity is $\phi^* = (1.8750, 0)^\top$, while the optimal portfolio without the ambiguity is $(1.8124, 0.0382)^\top$.

It follows that the total risky position (1.8506) of the latter portfolio is less than 1.8750, the total risky position of the optimal portfolio under the ambiguity. The intuitive reason for the result is that by (8), without the ambiguity we can adjust β (or ρ) to make the two

risky positions small, while the positions under the ambiguity really depend on the ambiguity level, and they may be large.

We next examine the relative absolute weight $\frac{\|\phi_1^*\|}{\|\phi_1^*\|+\|\phi_2^*\|}$ of asset 1 in the optimal portfolio. For the anti-diversification case, this weight is 1, greater than the weight of asset 1 in the mean-variance portfolio. In general, we define the function of the relative absolute weight as follows.

$$F(\rho_{12}) = \frac{\frac{1}{\sigma_1}(s_1 - \rho_{12}s_2)}{\frac{1}{\sigma_1}(s_1 - \rho_{12}s_2) + \frac{1}{\sigma_2}|s_2 - \rho_{12}s_1|}.$$

Then

$$\frac{dF(\rho_{12})}{d\rho_{12}} = \begin{cases} \frac{1}{(...)^2}\frac{\sigma_1}{\sigma_2}(s_1^2 - s_2^2) > 0, & \text{for } \rho_{12} < s_2/s_1, \\ \frac{1}{(...)^2}\frac{\sigma_1}{\sigma_2}(s_2^2 - s_1^2) < 0, & \text{for } \rho_{12} > s_2/s_1. \end{cases}$$

Therefore, for all cases the relative absolute weight of asset 1 in the optimal portfolio is greater than that in the mean-variance portfolio, which equals $F(\hat{\rho}_{12})$ for some $\hat{\rho}_{12} \in [\underline{\rho}_{12}, \bar{\rho}_{12}]$.

Hence, in the sense of the relative absolute weight, the optimal portfolio under correlation ambiguity biases toward asset 1, and the portfolio is less "balanced" compared to the mean-variance portfolio.

III.   Anti-Diversification

In this section, we show that, correlations with sufficient ambiguity lead to anti-diversification in the sense that the optimal portfolio consists of exactly one risky assets, even though there are $N > 1$ risky assets available. A sufficient and necessary condition for occurrence of anti-diversification is presented.

We index the risky asset with the greatest Sharpe ratio as asset 1 as before. The optimal portfolio with aversion to correlation ambiguity should produce a value function not less

than the portfolio of investing only any one of the risky assets. If only investing asset i, the optimal portfolio is $\frac{1}{A}\frac{\mu_i}{\sigma_i^2}$ and the value function is $\frac{1}{2A}s_i^2$. So the optimal value function J satisfies $J \geq \frac{1}{2A}s_i^2, i = 1, 2, ...N$. Thus, we have $J \geq \frac{1}{2A}\max_i s_i^2 = \frac{1}{2A}s_1^2$. That is, investing in asset 1 reaches the lower bound of the value function and holding one of other assets can not be optimal. We write this result and its consequence as a proposition below.

PROPOSITION 3: *Suppose $s_1 > \max\{s_2, ..., s_N\} \geq 0$. Then the value function is greater than or equal to $\frac{1}{2A}s_1^2$. When anti-diversification occurs, only the risky asset with the greatest Sharpe ratio among all available risky assets is held.*

To study the N risky assets case, let us consider the following change of variables.

$$
\begin{aligned}
\varphi_1 &= \phi_1 + \beta_2\phi_2 + ... + \beta_N\phi_N, \\
\varphi_i &= \phi_i, \ i = 2, 3, ..., N.
\end{aligned}
$$

In matrix notation, we can write

$$
\varphi = \begin{pmatrix} 1 & \beta^\top \\ 0 & I_{N-1} \end{pmatrix} \phi, \tag{11}
$$

with $\varphi = (\varphi_1, ..., \varphi_N)^\top, \phi = (\phi_1, ..., \phi_N)^\top, \beta = (\beta_2, ..., \beta_N)^\top$, and for each $i = 2, ..., N$,

$$
\beta_i = \frac{\rho_{1i}\sigma_i}{\sigma_1}
$$

is the beta coefficient of regression (in population) coefficient of r_i on r_1.

We can express Σ matrix in the following block-diagonal form

$$
\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_1^2\beta^\top \\ \beta\sigma_1^2 & \Sigma^\perp \end{pmatrix}
$$

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where $\Sigma^{\perp}$ is the $(N-1) \times (N-1)$ variance-covariance matrix for assets $2, \ldots, N$. Note that

$$\begin{pmatrix} 1 & 0 \\ -\beta & I_{N-1} \end{pmatrix} \begin{pmatrix} \sigma_1^2 & \beta^{\top}\sigma_1^2 \\ \beta\sigma_1^2 & \Sigma^{\perp} \end{pmatrix} \begin{pmatrix} 1 & -\beta^{\top} \\ 0 & I_{N-1} \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \Sigma^{\perp} - \sigma_1^2\beta\beta^{\top} \end{pmatrix}. \tag{12}$$

We then have

$$\phi^{\top}\Sigma\phi = \varphi_1^2\sigma_1^2 + \hat{\varphi}^{\top}(\Sigma^{\perp} - \sigma_1^2\beta\beta^{\top})\hat{\varphi},$$

where $\hat{\varphi} = (\varphi_2, ..., \varphi_N)^{\top}$, and

$$\mu^{\top}\phi = (\mu_1, \hat{\mu}^{\top} - \beta^{\top}\mu_1)\varphi,$$

where $\hat{\mu} = (\mu_2, ..., \mu_N)^{\top}$. Thus, the agent's objective function without ambiguity becomes

$$\max_{\phi} \mu^{\top}\phi - \frac{A}{2}\phi^{\top}\Sigma\phi = \max_{\varphi} \mu_1\varphi_1 - \frac{A}{2}\sigma_1^2\varphi_1^2 + (\hat{\mu} - \beta\mu_1)^{\top}\hat{\varphi} - \frac{A}{2}\hat{\varphi}^{\top}(\Sigma^{\perp} - \sigma_1^2\beta\beta^{\top})\hat{\varphi},$$

and the maximizer can be separated into two components as follow

$$\varphi_1^* = \frac{\mu_1}{A\sigma_1^2} \quad \text{and} \quad \hat{\varphi}^* = \frac{1}{A}(\Sigma^{\perp} - \sigma_1^2\beta\beta^{\top})^{-1}\alpha, \tag{13}$$

where

$$\alpha = \hat{\mu} - \beta\mu_1.$$

Note that we use the same notations of α and β as in Section II because they are identical when $N = 2$.

Next, by the relation (11), we have

$$\phi_1^* + \sum_{n=2}^{N} \beta_n\phi_n^* = \frac{\mu_1}{A\sigma_1^2}$$

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and

$$(\phi_2^*, ..., \phi_N^*)^\top = \frac{1}{A}(\Sigma^\perp - \sigma_1^2 \beta\beta^\top)^{-1}\alpha.$$

Therefore,

$$\max_\phi \mu^\top \phi - \frac{A}{2}\phi^\top \Sigma\phi = \frac{1}{2A}(\frac{\mu_1^2}{\sigma_1^2} + \alpha^\top(\Sigma^\perp - \sigma_1^2\beta\beta^\top)^{-1}\alpha),$$

Then the objective function with aversion to correlation ambiguity becomes

$$\min_\rho \frac{1}{2A}\left(\frac{\mu_1^2}{\sigma_1^2} + \alpha^\top(\Sigma^\perp - \sigma_1^2\beta\beta^\top)^{-1}\alpha\right), \tag{14}$$

which is the N risky assets version of (9).

From (14), because $(\Sigma^\perp - \sigma_1^2\beta\beta^\top)^{-1}$ is positive definite, $\frac{1}{2A}\frac{\mu_1^2}{\sigma_1^2}$ is the minimum if and only if $\alpha = 0$. In this case, the optimal portfolio has only one risky asset which is asset 1 and we have anti-diversification.

Note that $\alpha = 0$ is equivalent to $\rho_{1i}^* = \frac{s_i}{s_1}, i = 2, ..., N$. This is just the N-dimensional extension of the two-risky-asset case we study in Section II. The intuition for this condition is the same as given for the case of $N = 2$.

Note that $\rho_{1i}, i = 2, ..., N$, can be independently specified, as long as $|\rho_{1i}| < 1$. [5] However, after $\rho_{1i}, i = 2, ..., N$, are given, $\{\rho_{ij}\}_{2 \le i < j}$ can not be specified independently. They have to satisfy additional constraints for ρ to be positive definite. For example, when $N = 3$, ρ_{12} and ρ_{13} can be independently specified to take any value between -1 and 1. But given ρ_{12} and ρ_{13}, we can no longer specify ρ_{23} to take any value between -1 and 1, due to the constraint (4).

The above analysis yields that the condition $\alpha = 0$ is a sufficient and necessary condition for occurrence of anti-diversification if $\rho_{1i}^* = s_i/s_1$ is admissible.

PROPOSITION 4: *Anti-diversification occurs if and only if* $\rho_{1i}^* = s_i/s_1, i = 2, ..., N, $ *is ad-*

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[5]This is because for any values of $\rho_{1i}, i = 2, ..., N$, we can find values between -1 and 1 for other ρ_{ij}, such that ρ is positive definite. For example, we can let $\rho_{ij} = \rho_{1i}\rho_{1j}$, for $1 < i \ne j \le N$, then the correlation matrix ρ is positive definite for any values of ρ_{1i}, ρ_{1j} between -1 and 1.

missible.

An alternative way to understand this proposition is as follows. The decomposition (14) implies that we can construct two groups of assets: one denoted by X consisting of asset 1 only, another group denoted by Y with return vector

$$(-\beta, I_{N-1})(r_1, r_2, ..., r_N)^\top = (r_2 - \beta_2 r_1, ..., r_N - \beta_N r_1)^\top.$$

Then Y has expected return vector

$$(-\beta, I_{N-1})\mu = \hat{\mu} - \beta \mu_1 = \alpha$$

and variance-covariance matrix

$$(-\beta, I_{N-1})^\top \Sigma (-\beta, I_{N-1}) = \Sigma^\perp - \sigma_1^2 \beta \beta^\top.$$

One can check that X is uncorrelated with each member of Y, and hence uncorrected with any portfolio over Y. When correlations are ambiguous and the agent is averse to ambiguity, she will consider the worst case in which the second term of (14) is minimized over the ambiguous sets of correlations. The value function is the sum of the squared Sharpe ratio of X and the optimal Sharpe ratio of Y divided by $2A$. Only the latter Sharpe ratio contains ρ; in the worst scenario ($\alpha = 0$) this Sharpe ratio is zero and asset $2, ..., N$, are not held. Note that α depends on ρ_{1n} only, not on all ρ_{ij}'s. Hence, it induces anti-diversification that $\alpha = 0$, given $\rho_{1i}^* = s_i/s_1, i = 2, ..., N$, is admissible.

When anti-diversification occurs, only asset 1 with the greatest Sharpe ratio is held. Any portfolio formed with the rest of the risky assets should have a lower Sharpe ratio than asset 1. Here is an example for the three risky assets case. When anti-diversification occurs, by Proposition 4, it holds that $\rho_{12}^* = s_2/s_1, \rho_{13}^* = s_3/s_1$, and the correlation matrix shall be

positive definite. Substituting $\rho_{12} = \rho_{12}^*, \rho_{13} = \rho_{13}^*$ into (4), it follows that

$$s_2^2 + s_3^2 - 2\rho_{23}s_2s_3 < (1 - \rho_{23}^2)s_1^2. \tag{15}$$

Meanwhile, note that the Sharpe ratio of the optimal portfolio formed by using asset 2 and 3 is

$$(s_2, s_3) \begin{pmatrix} 1 & \rho_{23} \\ \rho_{23} & 1 \end{pmatrix}^{-1} \begin{pmatrix} s_2 \\ s_3 \end{pmatrix} = \frac{s_2^2 + s_3^2 - 2\rho_{23}s_2s_3}{1 - \rho_{23}^2}.$$

This is smaller than s_1 if and only if

$$s_2^2 + s_3^2 - 2\rho_{23}s_2s_3 < (1 - \rho_{23}^2)s_1^2,$$

which is the same as (15). So any portfolio of asset 2 and asset 3 has a lower Sharpe ratio than asset 1 if anti-diversification occurs. However, the converse is not true. One may consider a case without correlation ambiguity.

When $\rho_{1i}^* = s_i/s_1, i = 2, ..., N$, and $\rho_{ij}^* = s_is_j/s_1^2, 1 < i < j \leq N$, the correlation matrix $\rho^* = (\rho_{ij}^*)$ is positive definite. This gives us a simple sufficient condition for occurrence of anti-diversification by Proposition 4.

PROPOSITION 5: *If s_i/s_1 is contained in the ambiguous sets of ρ_{1i} for $i = 2, ..., N$, and s_is_j/s_1^2 is contained in the ambiguous sets of ρ_{ij} for $1 < i < j \leq N$, anti-diversification occurs.*

The sufficient condition in Proposition 5 is automatically satisfied when the correlations are completely ambiguous, that is, the ambiguous sets are $[-1, 1]$. Hence we obtain a general result for the case of N risky assets, in a line with Proposition 1, where only two risky assets are assumed.

COROLLARY 1: *Assume $s_1 > \max\{s_2, s_3, ..., s_N\} \geq 0$. If the correlations are completely*

ambiguous, that is, the ambiguous set for each correlation coefficient ρ_{ij} is $[-1, 1]$, then the agent will hold asset 1 only.

When anti-diversification occurs, only the asset with the greatest Sharpe ratio will be held. However, when anti-diversification does not occur, this asset may not be even held in the optimal portfolio while the one with the smallest Sharpe ratio may be held, depending on correlations and their ambiguous levels. The reason is that other assets except asset 1 may combine to achieve a higher Sharpe ratio than asset 1.

Under the mean-variance expected utility, the portfolio is given by $\phi^* = \frac{1}{A}\Sigma^{-1}\mu$. The probability of $\phi_i^* = 0$ for some i is zero. Alternatively, if Σ is non-singular, then each asset i has an independent risk and an risk premium α_i associated with the risk. Holding of asset i is zero if α_i is zero. Obviously, $\alpha_i = 0$ is a zero probability event in a space of all possible α_i. Thus, under the expected utility, all available risky assets will be held and there is exactly diversification. In contrast, under aversion to correlation ambiguity, minimizing over ρ identifies $\alpha_i = 0$. In this case, $\alpha_i = 0$ and anti-diversification occurs for sure.

It seems that the optimal portfolio containing only one risky asset is riskier than the standard mean-variance portfolio which contains all risky assets. For example, suppose $s_1 > s_i, i = 2, ..., N$ and $\frac{\mu_1}{\sigma_1^2} = ... = \frac{\mu_N}{\sigma_N^2}$. Then the optimal portfolio under correlation ambiguity is given by $\phi^* = \frac{\mu_1}{A\sigma_1^2}(1, 0, ..., 0)^\top$ while note that under the expected utility, the optimal portfolio is given by $\phi_{MV}^* = \frac{1}{A}\left(\frac{\mu_1}{\sigma_1^2}, ..., \frac{\mu_N}{\sigma_N^2}\right)^\top = \frac{\mu_1}{A\sigma_1^2}(1, 1, ..., 1)^\top$, assuming $\rho = I_N$. A portfolio with $\phi^* = \frac{1}{A}\frac{\mu_1}{\sigma_1^2}(1, 0, ..., 0)$ seems to be much more "imbalanced" thus riskier than a portfolio with $\phi_{MV}^* = \frac{1}{A}\frac{\mu_1}{\sigma_1^2}(1, ..., 1)^\top$. However, the variance of the portfolio ϕ^* is

$$(\phi^*)^\top \Sigma \phi^* = \frac{1}{A^2}s_1^2$$

while the variance of the mean-variance portfolio ϕ^*_{MV} is

$$(\phi^*_{MV})^\top \Sigma \phi^*_{MV} = \frac{1}{A^2} \sum_i s_i^2 > \frac{1}{A^2} s_1^2.$$

Thus, the portfolio ϕ^* actually has a lower variance and thus is less risky. In this example, we assume $\rho = I_N$ to get the mean-variance portfolio. In fact, for any admissible ρ, the same conclusion holds. Note $\phi^* = \frac{1}{A}\sigma^{-1}\rho^{-1}s$ and

$$\min_\rho \max_\phi \phi^\top \mu - \frac{A}{2}\phi^\top \Sigma \phi = \frac{1}{2A} \min_\rho s^\top \rho^{-1} s = \frac{A}{2} \min_\rho (\phi^*)^\top \sigma \rho \sigma \phi^*.$$

The term $(\phi^*)^\top \sigma \rho \sigma \phi^*$ is in deed the variance of the return regarding portfolio ϕ^*. Hence the optimal portfolio under the ambiguity has the minimum variance among all portfolios in the form $\frac{1}{c}\sigma^{-1}\rho^{-1}s$ for any non-zero constant scale c. In this sense, the optimal portfolio under the ambiguity aversion is more conservative and the optimal portfolio is less risky than the standard mean-variance portfolio.

Goldman (1979) shows that in an infinite time horizon, buy-and-hold strategy will result in anti-diversification. In his paper, the asset with the highest risk and risk aversion adjusted expected return will be held. In our paper, it is the asset with the highest Sharpe ratio.

IV.   Underdiversification

In this section, we show that the optimal portfolio with aversion to correlation ambiguity typically does not contain all available risky assets. Thus we have under-diversification. As a calibration exercise, with 100 stocks randomly drawn from S&P500 stocks, approximately 80 stocks will not be held in the optimal portfolio, if the correlations can take any values within 95% confidence intervals.

We first study conditions under which an asset is not held. For convenience, we use asset

N as an example. We write the variance-covariance matrix in a form of blocks as follows.

$$\Sigma = \begin{pmatrix} \tilde{\Sigma}^\perp & \sigma_N^2 \tilde{\beta} \\ \sigma_N^2 \tilde{\beta}^\top & \sigma_N^2 \end{pmatrix},$$

where $\tilde{\Sigma}^\perp$ is the $(N-1) \times (N-1)$ variance-covariance matrix of asset 1,..., asset $N-1$, and

$$\tilde{\beta} = (\rho_{1N} \frac{\sigma_1}{\sigma_N}, ..., \rho_{N-1,N} \frac{\sigma_{N-1}}{\sigma_N})^\top.$$

Note that $\tilde{\beta}$ is the population regression coefficient of r_N on $(r_1, ..., r_{N-1})$.

Define

$$\tilde{\alpha}_N = \mu_N - \sigma_N^2 \tilde{\mu}^\top (\tilde{\Sigma}^\perp)^{-1} \tilde{\beta}, \quad \tilde{\sigma}_N^2 = \sigma_N^2 - \sigma_N^4 \tilde{\beta}^\top (\tilde{\Sigma}^\perp)^{-1} \tilde{\beta}, \quad (16)$$

where $\tilde{\mu} = (\mu_1, ..., \mu_{N-1})^\top$. Note that $\tilde{\alpha}_N$ is treated as a function of $\tilde{\Sigma}^\perp$ and $\tilde{\beta}$, whose values depend on the ambiguous sets of $\{\rho_{ij}\}_{1 \leq i < j < N}$ and $\{\rho_{iN}\}_{1 \leq i < N}$ respectively.

The following proposition presents a necessary condition and a sufficient condition for not holding asset N.

PROPOSITION 6: *If asset N is not held in the optimal portfolio under correlation ambiguity then $\min_{\tilde{\beta}} \tilde{\alpha}_N^2 / \tilde{\sigma}_N^2 = 0$. Conversely, if for all $\tilde{\Sigma}^\perp$, $\tilde{\alpha}_N = 0$, then asset N is not held.*

Proof. Note that the objective function can be written as follows.

$$\frac{1}{2A} \min_\rho \left(\frac{\tilde{\alpha}_N^2}{\tilde{\sigma}_N^2} + \tilde{\mu}^\top (\tilde{\Sigma}^\perp)^{-1} \tilde{\mu} \right) = \frac{1}{2A} \min_{\tilde{\Sigma}^\perp} \left(\min_{\tilde{\beta}} (\frac{\tilde{\alpha}_N^2}{\tilde{\sigma}_N^2}) + \tilde{\mu}^\top (\tilde{\Sigma}^\perp)^{-1} \tilde{\mu} \right). \quad (17)$$

So if asset N is not held, we must have $\min_{\tilde{\beta}} \tilde{\alpha}_N^2 / \tilde{\sigma}_N^2 = 0$. Conversely, if $\tilde{\alpha}_N = 0$ holds for all $\tilde{\Sigma}^\perp$, there is no reason to hold asset N which has an zero Sharpe ratio in the worst case. $\square$

As an example of $N = 2$, $\tilde{\alpha}_2 = \mu_2 - \frac{\sigma_2}{\sigma_1} \rho_{12} \mu_1$, $\tilde{\sigma}_2^2 = \sigma_2^2 (1 - \rho_{12}^2)$, and $\tilde{\Sigma}^\perp$ is the constant σ_1^2. The condition $\tilde{\alpha}_2 = 0$ becomes a sufficient and necessary condition for not holding asset

2. It turns out that the condition reduces to Proposition 2 of the case of two risky assets.

Note that $\tilde{\alpha}_N$ is an analog of α defined in the preceding sections, and the condition $\tilde{\alpha}_N = 0$ is an analog of $\alpha = 0$ condition. The above proposition provides a necessary and a sufficient condition for not holding an asset. In the following proposition, we prove that as one of correlations of asset N approaches to 1, $\tilde{\alpha}_N$ goes to zero. Then by Proposition 6, an risky asset with one large correlation is not likely held in the optimal portfolio under correlation ambiguity. Note that for any asset i, we can define $\tilde{\alpha}_i$ and $\tilde{\sigma}_i$ by a similar way of (16).

PROPOSITION 7: *For any one of correlations $\rho_{iN}, i = 1, ..., N - 1$, we have $\tilde{\alpha}_N \to 0$ and $\tilde{\alpha}_i \to 0$ as $\rho_{iN} \to 1$. Furthermore, if a pair of risky assets have a sufficiently large correlation, then one of the risky assets is not held in the optimal portfolio.*

We leave the proof in the appendix. Intuitively, as the correlation of a pair of risky assets is close to 1, the two assets shall have very close Sharpe ratios in order to avoid arbitrage. As a result, the ratio of their Sharpe ratios must be very close to 1, hence more likely falls in the ambiguous set of the correlation. By Proposition 2 of the two assets case, one of the pair will not be held.

We note that $\tilde{\alpha}_N = 0$ is not easy to check. Instead, the following result is useful in empirical calibration exercises, relating our problem to vast literature of semi-definite programming in operations research.

PROPOSITION 8: *The optimal portfolio under correlation ambiguity is given by*

$$\phi^* = \frac{1}{A}\sigma^{-1}(\rho^*)^{-1}s,$$

where ρ^* is given by

$$\rho^* = \arg\min_{\rho} s^\top \rho^{-1} s. \tag{18}$$

Proposition 8 is actually from (3). We list it as a proposition here because it provides us a quick way to find the optimal portfolio from a large set of risky assets under correlation ambiguity. Given s, the optimization problem (18) can be transformed to a semi-definite programming (SDP) problem. This is used in our calibration exercise. It's worth mentioning that the positive definite constraint is binding when we solve the problem numerically. We leave the detailed transforming steps in the appendix.

The following proposition suggests that under-diversification is associated with interior solutions to the minimization problem. This result also suggests that under-diversified portfolios may occur quite generally under correlation ambiguity.

PROPOSITION 9: *If $\rho_{ij}^* \in (\underline{\rho}_{ij}, \bar{\rho}_{ij})$, then $\phi_i^* = 0$ or $\phi_j^* = 0$ for $1 \le i \ne j \le N$. In other words, if an optimal correlation is achieved at an interior point of the ambiguity set, then at least one of the two corresponding risky assets will not be held.*

We leave the proof in the appendix. The intuition behind the proposition is that if changing ρ_{ij} does not affect the utility then $\phi_i^* \phi_j^*$ must be zero, as the product is the coefficient of ρ_{ij}^* in the optimal utility function.

A direct consequence of the above proposition is anti-diversification: If all correlations are completely ambiguous ($\underline{\rho}_{ij} = -1, \bar{\rho}_{ij} = 1$ for all $i, j = 1, 2, ..., N$), then at most one ϕ_i is not zero, hence only one asset will be held and anti-diversification occurs. By Proposition 3, the risky asset with the greatest Sharpe ratio is held.

In our empirical calibration exercise, the optimal portfolio under correlation ambiguity is under-diversified even when we replace confidence intervals by a sphere or an ellipsoidal set as the ambiguity set of correlations. In contrast, under the expected return ambiguity, there is no under-diversification for an ellipsoidal or a sphere ambiguity set. This can be seen from the case of two risky assets discussed in the Internet Appendix of this paper, or from Proposition 2, Garlappi, et al. (2007). In fact, under-diversification (holding only part of risky assets) shown in Boyle, et al. (2012) may not occur if the ellipsoidal ambiguous set $\{\mu : (\mu - \hat{\mu})^\top \Sigma^{-1} (\mu - \hat{\mu}) < \delta\}$ is used as the range of expected returns there.

A. Empirical Calibration

We calibrate our model using U.S. stock market data and study the optimal portfolio under correlation ambiguity. We use the monthly data of S&P 500 adjusted for dividends. The data set spans from January, 1993 to December, 2012, for a total of 240 months. After filtering out those with incomplete data, there are 319 stocks that remain for our study. Then, we compute mean excess returns, variances and correlations of the excess returns of these stocks, using average monthly LIBOR as the riskless returns.

The brief statistical information regarding the data are reported in Table 1. Of a total of 50721 ($319 \times 318/2$) estimated correlations, 0.19% have a p-value greater than 5%; hence, most of the correlations are significant. The maximum of correlations is 0.8365, and the minimum is -0.6385. Neither is close to the singular value -1 or 1. The average length of 95% confidence intervals is 0.2357. The length rises to 0.3088 when 99% confidence intervals are adopted. Hence, a higher confidence interval level corresponds to a higher level of ambiguity. As is standard in the literature, confidence intervals are used as ambiguous sets of correlations.

Empirical studies document that investors usually hold much less risky assets than they could have. For example, Campbell (2006) suggests that the financial portfolios of households contain only a few assets. Goetzmann and Kumar (2008) report that the majority of individual investors hold a single digit number of assets in a sample data set during 1991-1996.

In preceding sections, we have shown that aversion to correlation ambiguity can generate under-diversification. To quantitatively study the phenomenon of under-diversification, we randomly choose a sample group of stocks from the S&P 500 stocks for $N = 10, 20, ..., 100$, and then we compute the optimal portfolio from this chosen group under ambiguous correlations for investors with ambiguity aversion. The ambiguous intervals of the estimated correlations are given by different levels of confidence subject to the positive definite constraint. We repeat this procedure 100 times, and obtain 100 optimal portfolios. The average

number of stocks in the optimal portfolios for each N is considered to be the typical size of the investors' optimal portfolios when they encounter N available stocks.

The result is reported in Figure 1. When there are 100 stocks, optimal portfolios consist of 22 stocks, on average, given the 95% confidence intervals as the ambiguous sets of the correlations. Moreover, optimal portfolios consist of approximately 18 and 24 stocks when investors select from the 100 stocks and use 90% and 99% confidence intervals, respectively, as the ambiguity sets of the correlations. Hence the aversion to correlation ambiguity generates under-diversification, as documented in the empirical studies.

We let the risk aversion $A = 1$ in our empirical tests. It is important to note that the risk aversion solely affects the magnitude of risky positions by a manner of scaling; it does not change the choices of risky assets in the optimal portfolio under correlation ambiguity. In fact, we may obtain a corollary from Proposition 9 as follows.

COROLLARY 2: *The set of risky assets in the optimal portfolio under correlation ambiguity is independent of the risk aversion.*

Proof. By Proposition 9, we observe that whether asset i has zero positions depends on whether ρ_{ij}^* falls into the ambiguous set; this, in addition to the objective function in (18), is independent of the risk aversion A. $\square$

Next, we focus on one randomly selected sample of 100 stocks. The results are reported in Table II. When the 95% confidence intervals are used as the ambiguous sets for the correlations, we obtain an optimal portfolio that consists of 20 stocks from these 100 candidates. The maximum, minimum and average of the Sharpe ratios of the (held) stocks in the optimal portfolio are 0.1674, -0.0489, and 0.0859, respectively, whereas the corresponding quantities are 0.1153, -0.0064, and 0.0600, respectively, for the (not-held) stocks with zero positions in the optimal portfolio. The former group has a higher average Sharpe ratio than the latter. The distributions of the Sharpe ratios in the optimal portfolio and in the entire sample are presented in Figure 2. Note that the stock with the greatest ratio is held, but not all the

held stocks have top Sharpe ratios. Point estimations of correlations that determine the positions of ambiguity sets, and ambiguity levels that determine the size of ambiguity sets, matter here as well as Sharpe ratios.

By Proposition 2, a stock will not be held in the optimal portfolio if the ratio of its Sharpe ratio to the maximal Sharpe ratio falls into the ambiguous interval of the correlation. Given the average correlation of 0.2220, and the maximal Sharpe ratio of 0.1674, we can expect that stocks with a Sharpe ratio of approximate $0.0372(= 0.2220 \times 0.1674)$ tend not to be held. This analysis is confirmed by Figure 2, which shows that all stocks with Sharpe ratios near the point 0.03 are not held.

When the level of ambiguity is greater, the average Sharpe ratio in the optimal portfolio is higher. This finding suggests that the stocks with low Sharpe ratios are more likely to be eliminated from the optimal portfolio as the ambiguity level increases, until the one with the greatest Sharpe ratio remains.

In Figure 3, we compare the optimal portfolio with the standard mean-variance portfolio constructed from the sample moments. Table III lists the exact values of the non-zero positions in the optimal portfolio and the corresponding positions in the mean-variance portfolio. The extreme (very positive large or very negative small) positions in the mean-variance portfolio are significantly reduced in the optimal portfolio under the ambiguity. For example, the largest position 6.1355 in the corresponding mean-variance portfolio is reduced to 0.1604 in the optimal portfolio, and the smallest -2.5868 is reduced to -0.0483.

The reason that the extreme allocations are largely reduced in the optimal portfolio under the ambiguity can be inferred from Figure 4. This figure plots the distributions of correlations of all 100 stocks, as well as those stocks in the optimal portfolio. As suggested by Proposition 7, many stocks with large correlations are eliminated from the optimal portfolio. Hence the stocks in the optimal portfolio tend to have small correlations. Consequently, extreme positions are generated with less possibility when the correlations are closer to zero, and the optimal portfolio under correlation ambiguity likely has less extreme positions than

the mean-variance portfolio.

Extreme positions are considered to be one of major reasons that cause poor out-of-sample performance.[6] The aversion to correlation ambiguity can reduce extreme positions, and hence, may improve the performance of optimal portfolios under correlation ambiguity. In fact, in our out-of-sample tests for various data sets not reported in this paper, the aversion to correlation ambiguity generates portfolios with a more stable and higher Sharpe ratio than the mean-variance portfolios. These findings regarding performance are similar to Garlappi, et al. (2007); they study portfolio choice with aversion to expected return ambiguity. However, here, by examining extreme positions, we provide one potential reason for why the performance of optimal portfolios under correlation ambiguity can be better than the mean-variance portfolios.

V. Conclusions

In this paper, we study optimal portfolios for an agent who is averse to correlation ambiguity. We prove that anti-diversification (the optimal portfolio has only one risky asset) occurs when correlations are sufficiently ambiguous. Generally, we have under-diversification in the sense that optimal portfolios contain only part of available risky assets. Given the plausible levels of ambiguity, the optimal portfolio contains approximately 20 stocks in a case with randomly-selected 100 stocks from S&P 500. Thus, correlation ambiguity may provide an explanation for under-diversification documented in the literature.

Anti-diversification or under-diversification implies that the optimal portfolios are less diversified and less "balanced". Furthermore, less risk-less assets may be held in the optimal portfolio than in the mean-variance portfolio. Thus one may be tempted to conclude that the optimal portfolio is riskier. In fact, the optimal portfolio is less risky because it has less variance. In addition, the optimal portfolio has less "extreme" positions because risky

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[6]For example, Jagannathan (2002) and DeMiguel et al (2009) show out of sample performance can be improved by constraining the weights(hence, reducing extreme positions) in optimal portfolios.

assets with high correlations are more likely to be omitted from the optimal portfolio under correlation ambiguity.

Appendix A.   Proof of Propositions

Proof of Proposition 7:

For notional continence and without loss of generality, we use the pair of asset 1 and asset 2 as an example. Asset 1 is not assumed to have the greatest Sharpe ratio among all risky assets in this proof. Note that $\tilde{\alpha}_1$ is defined accordingly as follows.

$$\tilde{\alpha}_1 = \mu_1 - \sigma_1^2 \hat{\mu}^\top (\Sigma^\perp)^{-1} \beta, \quad \tilde{\sigma}_1^2 = \sigma_1^2 - \sigma_1^4 \beta^\top (\Sigma^\perp)^{-1} \beta,$$

where $\hat{\mu} = (\mu_2, ..., \mu_N)^\top$, $\Sigma^\perp$ and β are defined the same as in Section III. We need to prove that as ρ_{12} goes to 1, $\tilde{\alpha}_1$ goes to zero. Write ρ in a form of blocks as follows.

$$\rho = \begin{pmatrix} 1 & \rho_{12} & \hat{\rho}_{13}^\top \\ \rho_{12} & 1 & \hat{\rho}_{23}^\top \\ \hat{\rho}_{13} & \hat{\rho}_{23} & \hat{\rho}_3 \end{pmatrix} \tag{A1}$$

where $\hat{\rho}_{13} = (\rho_{13}, ..., \rho_{1N})^\top$, $\hat{\rho}_{23} = (\rho_{23}, ..., \rho_{2N})^\top$, $\hat{\rho}_3$ is a $(N-2) \times (N-2)$ matrix. Denote

$$\hat{\rho}_2 = \begin{pmatrix} 1 & \hat{\rho}_{23}^\top \\ \hat{\rho}_{23} & \hat{\rho}_3 \end{pmatrix},$$

with inverse

$$\hat{\rho}_2^{-1} = \begin{pmatrix} (1 + \hat{\rho}_{23}^\top (\hat{\rho}_3 - \hat{\rho}_{23} \hat{\rho}_{23}^\top)^{-1} \hat{\rho}_{23}) & -\hat{\rho}_{23}^\top (\hat{\rho}_3 - \hat{\rho}_{23} \hat{\rho}_{23}^\top)^{-1} \\ -(\hat{\rho}_3 - \hat{\rho}_{23} \hat{\rho}_{23}^\top)^{-1} \hat{\rho}_{23} & (\hat{\rho}_3 - \hat{\rho}_{23} \hat{\rho}_{23}^\top)^{-1} \end{pmatrix}.$$

Denote $\hat{s}_3 = [s_3, ..., s_N]^\top$. Then

$$\tilde{\alpha}_1 = \sigma_1 \left(s_1 - (s_2, \hat{s}_3^\top)\hat{\rho}_2^{-1}(\rho_{12}, \hat{\rho}_{13}^\top)^\top \right)$$

$$= \sigma_1 s_1 - \sigma_1(s_2, \hat{s}_3^\top) \begin{pmatrix} \rho_{12} + (\hat{\rho}_{13} - \rho_{12}\hat{\rho}_{23})^\top(\hat{\rho}_3 - \hat{\rho}_{23}\hat{\rho}_{23}^\top)^{-1}\hat{\rho}_{23} \\ (\hat{\rho}_3 - \hat{\rho}_{23}\hat{\rho}_{23}^\top)^{-1}(\hat{\rho}_{13} - \rho_{12}\hat{\rho}_{23}) \end{pmatrix}.$$

Note that as ρ_{12} goes to 1, ρ_{13} should go to ρ_{23}. Furthermore, $s_1 \to s_2$ as $\rho_{12} \to 1$ because there is no arbitrage. Hence

$$\lim_{\rho_{12} \to 1} \tilde{\alpha}_1 = 0.$$

Moreover, similarly we find that

$$\tilde{\sigma}_1^2 = \sigma_1^2 - \sigma_1^2(\rho_{12}, \hat{\rho}_{13})^\top \begin{pmatrix} \rho_{12} + (\hat{\rho}_{13} - \rho_{12}\hat{\rho}_{23})^\top(\hat{\rho}_3 - \hat{\rho}_{23}\hat{\rho}_{23}^\top)^{-1}\hat{\rho}_{23} \\ (\hat{\rho}_3 - \hat{\rho}_{23}\hat{\rho}_{23}^\top)^{-1}(\hat{\rho}_{13} - \rho_{12}\hat{\rho}_{23}) \end{pmatrix}.$$

Then applying L'Hospital's rule in limiting, we can prove that

$$\lim_{\rho_{12} \to 1} \frac{\tilde{\alpha}_1^2}{\tilde{\sigma}_1^2} = 0.$$

Since $\lim_{\rho_{12} \to 1} \tilde{\alpha}_1^2/\tilde{\sigma}_2^2 = 0$, we can find a sufficiently large ρ_{12}, such that $\tilde{\alpha}_1^2/\tilde{\sigma}_2^2$ is close to zero so that $\min_\beta \tilde{\alpha}_1^2/\tilde{\sigma}_2^2 = 0$. Note that we do not assume asset 1 has the greatest Sharpe ratio in this proof; the similar results are applied to asset 2 as well. We have $\lim_{\rho_{12} \to 1} \tilde{\alpha}_2 = 0$, $\lim_{\rho_{12} \to 1} \tilde{\alpha}_2^2/\tilde{\sigma}_2^2 = 0$, and when ρ_{12} is sufficiently large, $\min_\beta \tilde{\alpha}_2^2/\tilde{\sigma}_2^2 = 0$. This yields that for a sufficiently large ρ_{12}, one of the two risky assets is not held in the optimal portfolio, according to Proposition 6. $\square$

Proof of Proposition 8 and 9:

Exploiting the variable transformation $\psi = \sigma\phi$ and the minimax theorem, we obtain

$$J = \max_{\psi} \min_{\rho_{ij} \in [\underline{\rho}_{ij}, \bar{\rho}_{ij}]} s^\top \psi - \frac{A}{2} \psi^\top \rho \psi = \min_{\rho_{ij} \in [\underline{\rho}_{ij}, \bar{\rho}_{ij}]} \max_{\psi} s^\top \psi - \frac{A}{2} \psi^\top \rho \psi.$$

It is trivial to solve the inner maximization problem in the right hand side above. We have $\psi^* = \frac{1}{A}\rho^{-1}s$, and $J = \min_\rho \frac{1}{2A} s^\top \rho^{-1} s$. Let $f = \frac{1}{2A} s^\top \rho^{-1} s$. The first order condition of f w.r.t. ρ_{ij} is

$$\frac{\partial f}{\partial \rho_{ij}} = \frac{-1}{2A} s^\top \rho^{-1} I_{ij} \rho^{-1} s = -\frac{A}{2}(\psi^\top I_{ij} \psi) = -A \psi_i^* \psi_j^*,$$

where I_{ij} is a $N \times N$ matrix with all zero entries except 1 at the entry (i, j) and (j, i). If the minimization is achieved at an interior point of $[\underline{\rho}_{ij}, \bar{\rho}_{ij}]$, then $\partial f / \partial \rho_{ij} = 0$. As a result, $\psi_i^* \psi_j^* = 0$. This completes the proof. $\square$

Appendix B.   Transforming to SDP

Note that

$$\min_\rho \max_\phi \phi^\top s - \frac{A}{2} \phi^\top \rho \phi = \min_\rho \frac{1}{2A} s^\top \rho^{-1} s.$$

We need to solve the minimization problem $\min_\rho s^\top \rho^{-1} s$. This minimization problem is not a standard semi-definite programming problem yet. We take a transformation as follows. Consider

$$(P1): \quad \min s^\top \rho^{-1} s,$$

$$s.t. \quad \rho \in [\underline{\rho}, \bar{\rho}], \quad \rho > 0,$$

where $\rho > 0$ denotes the positive definite constraint. The problem (P1) can be rewritten as follows.

$$(P1'): \quad \min t,$$

$$s.t. \qquad s^\top \rho^{-1} s \leq t, \quad \rho \in [\underline{\rho}, \overline{\rho}] \quad \rho > 0.$$

We claim that the constraint $s^\top \rho^{-1} s \leq t$ and $\rho \geq 0$ is equivalent to $\begin{bmatrix} \rho & s \\ s^\top & t \end{bmatrix} \geq 0$ because

$$\begin{bmatrix} \rho & s \\ s^\top & t \end{bmatrix} \geq 0 \iff \begin{bmatrix} \rho & 0 \\ 0 & t - s^\top \rho^{-1} s \end{bmatrix} \geq 0.$$

So, (P1) can be transformed to:

$$(P2): \quad \min t,$$

$$s.t. \qquad \begin{bmatrix} \rho & s \\ s^\top & t \end{bmatrix} \geq 0, \quad \rho \in [\underline{\rho}, \overline{\rho}], \quad \rho > 0.$$

(P2) is a standard SDP problem.

Semidefinite programming (SDP) is a subfield of convex optimization for a linear objective function over the intersection of the cone of positive semidefinite matrices with an affine space. SDP is a special case of cone programming and can be solved by interior point methods. There are many free codes available in various programming languages, for example, C, C++, Matlab, Python. In the paper, we use Yalmip toolbox[7] with DSDP solver, developed by Steve Benson, Yinyu Ye, and Xiong Zhang. For a complete description of the algorithm and a proof of convergence of DSDP, see "Solving Large-Scale Sparse Semidefinite Programs for Combinatorial Optimization", SIAM Journal on Optimization, 10(2), 2000, pp. 443-461.

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[7]The toolbox is available at http://users.isy.liu.se/johanl/yalmip/pmwiki.php?n=Main.WhatIsYALMIP

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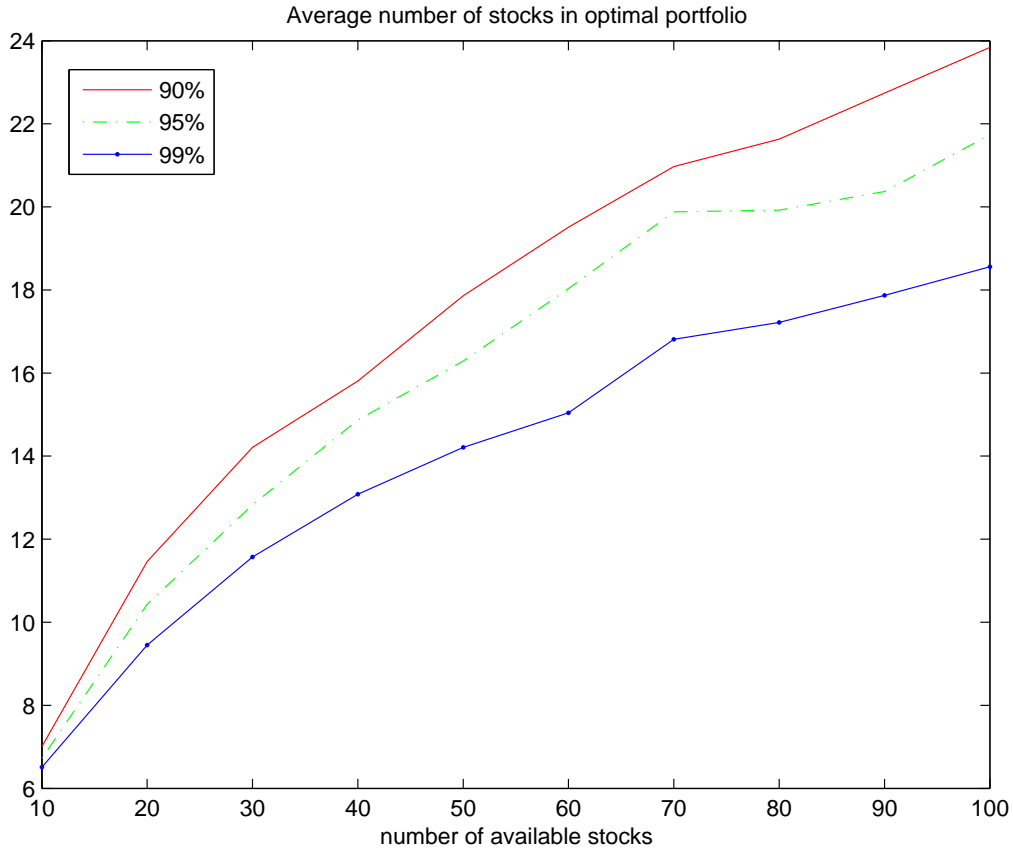
Figures and Tables

Figure 1. We randomly choose N stocks from S&P 500, and find the optimal portfolio under correlation ambiguity for these N stocks. The ambiguous sets are given by the 90%, 95%, and 99% confidence intervals respectively. We repeat the procedure 100 times, average number of stocks in optimal portfolios for each N and for each ambiguous level is calculated and reported in the figure. The X-axis denotes number of available stocks (N) varying from 10 to 100. The Y-axis denotes average number of stocks in optimal portfolios.
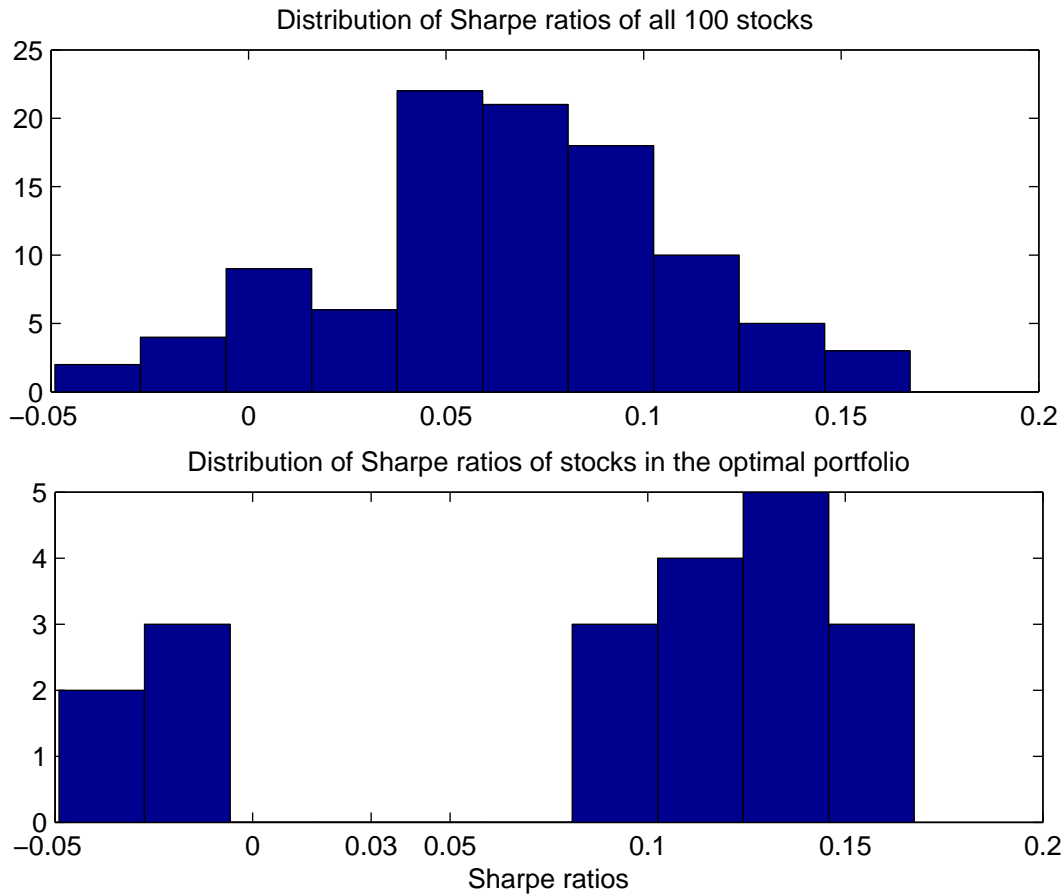
Figure 2. This figure shows the distributions of Sharpe ratios in the entire sample of randomly-selected 100 stocks, as well as in the optimal portfolio. Although many stocks with low Sharpe ratios are not held in the optimal portfolio, a few stocks with low Sharpe ratios are still held. Correlations and their ambiguous levels matter as well as Sharpe ratios. By Proposition 2, roughly speaking, a stock will not be held in the optimal portfolio if the ratio of its Sharpe ratio to the maximal Sharpe ratio falls into the ambiguous interval of the correlation. Hence given the average correlation of 0.2220 and the maximal Sharpe ratio of 0.1674, stocks with a Sharpe ratio of approximate $0.0372 (= 0.222 \times 0.1674)$ should tend not to be held. This analysis is consistent with the figure, which shows that stocks with a Sharpe ratio near 0.03 are not held.
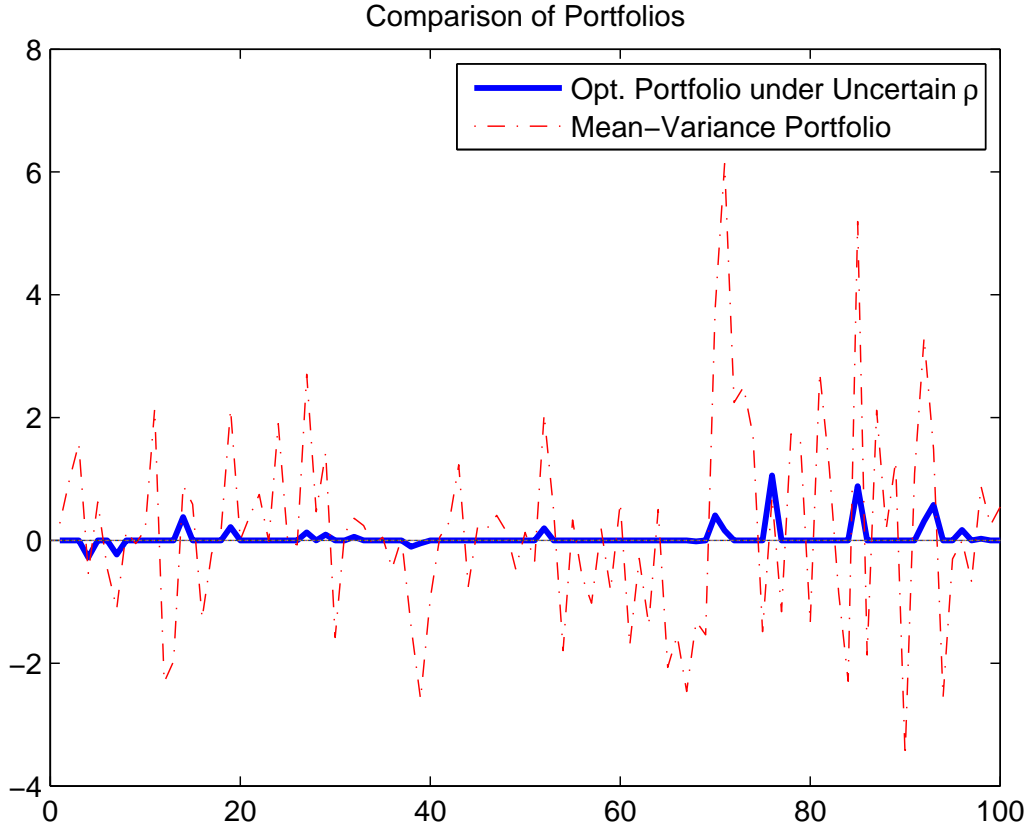
Figure 3. The magnitude of large positions are reduced greatly in the optimal portfolio compared with the mean-variance portfolio based on the estimation $\hat{\rho}$. A total of 100 stocks are in the sample. Every integer on the X-axis represents a stock. The Y-axis denotes allocations on stocks. The risk aversion coefficient is $A = 1$. The extreme positions in the mean-variance portfolio are significantly reduced in the optimal portfolio. Consequently, the optimal portfolio has less variance and is more conservative than the mean-variance portfolio in this sense.
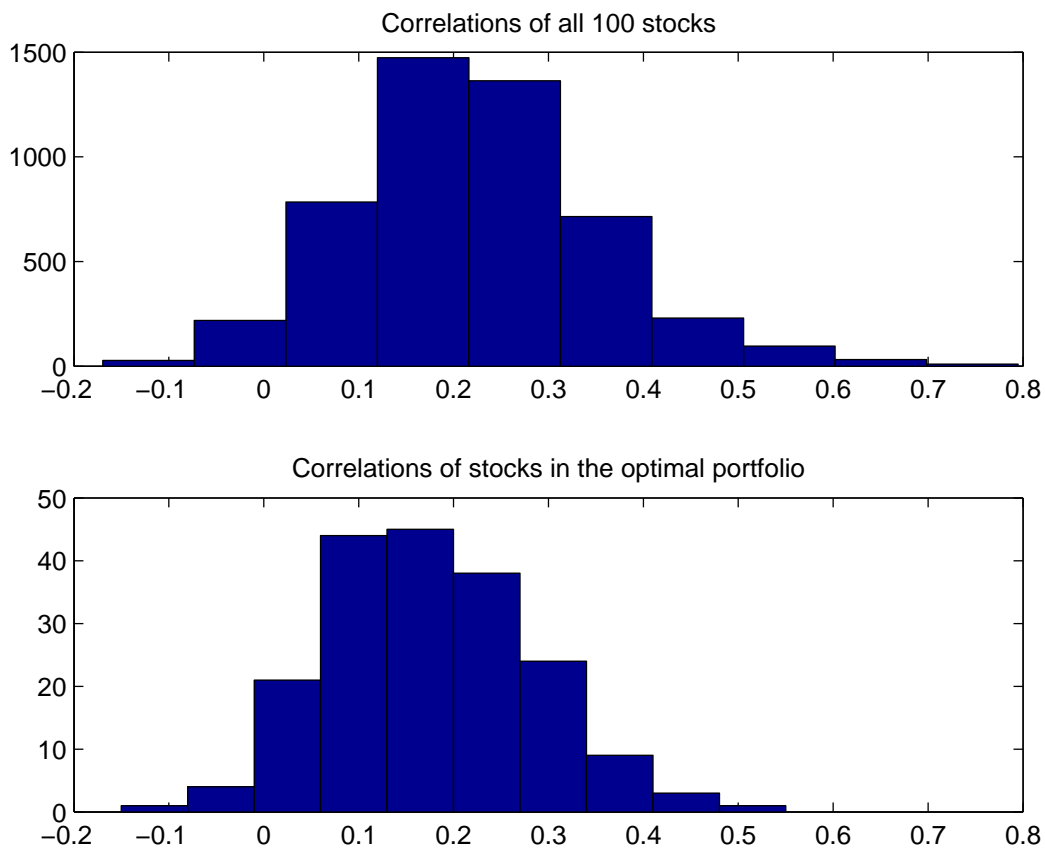
Figure 4. This figure shows the distributions of the correlations of all 100 stocks, as well as the stocks in the optimal portfolio. There are a total of $4950(= 100 \times 99/2)$ correlations in the top panel, whereas there are 190 $(= 20 \times 19/2)$ correlations in the bottom panel. The top correlation as well as many high correlations disappear in the optimal portfolio. Consequently, less extreme positions are found in the optimal portfolio.

Table I. Statistics of Correlations, Mean, and Sharpe Ratios of Excess Returns

level	correlations				length of C.I.			
of C.I.	mean	std	max	min	mean	std	max	min
99%	0.2369	0.1266	0.8365	-0.6385	0.3088	0.0227	0.3322	0.1016
95%					0.2357	0.0175	0.2537	0.0770
90%					0.1980	0.0148	0.2133	0.0645
	excess returns				Sharpe ratios			
	mean	std	max	min	mean	std	max	min
	0.0051	0.0042	0.0173	-0.009	0.060	0.0443	0.1829	-0.0762

The total number of stocks is 319. The percentage of p-values greater than 0.05 is 0.18%. Hence most correlations are significant. Average monthly LIBOR is used as the riskless returns.

Table II. Comparison of Sharpe Ratios between Held and Not-held Stocks

level of C.I.	95%		99%	
	Held	Not-held	Held	Not-held
Number of Stocks	20	80	16	84
max. Sharpe	**0.1674**	0.1153	0.1674	0.1125
min. Sharpe	-0.0489	-0.0064	-0.0489	-0.0129
ave. Sharpe	0.0859	0.0600	0.0954	0.0594

The table lists the results for 100 randomly selected stocks (with average correlation of 0.2220). The numbers of stocks held in the optimal portfolios are 20 and 16, and the numbers of stocks not-held are 80 and 84, respectively, for two levels of ambiguity. At each level of ambiguity, average Sharpe ratios are reported in the table, as well as the maximum and minimum of Sharpe ratios. Basically, held stocks have higher Sharpe ratios on average than not-held stocks.

Table III. Comparison of Non-zero Positions

OP	-0.2776	-0.2288	0.3792	0.2185	0.1289	0.0964	0.0584	-0.1040	**-0.0483**	0.0013
	0.1986	-0.0175	0.4063	**0.1604**	1.0580	0.8835	0.3129	0.5768	0.1715	0.0252
MV	-0.5354	-1.1119	0.8954	2.1356	2.7047	1.4276	0.3582	-1.3984	**-2.5868**	0.4069
	2.0538	-1.3190	3.7618	**6.1355**	0.7032	5.1949	3.3031	1.4833	-0.0082	0.8841

"OP" represents the optimal portfolio under correlation ambiguity. "MV" represents the corresponding positions in the mean-variance portfolio. As shown in the table, the extreme positions (in bold fonts) in the MV are significantly reduced in the optimal portfolio under correlation ambiguity. The risk aversion A is 1.