Correlation Uncertainty, Heterogeneous Beliefs and Asset Prices

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Abstract

We construct an equilibrium model in the presence of correlation uncertainty and heterogeneous ambiguity-averse investors. Our model explains a number of well documented empirical puzzles on correlation, including *asymmetric correlation*, *underdiversification* and *limited participation*, *diversification premium*, *comovement* and *correlated trading patterns* in the market. When both agents have high correlation uncertainty or when uncertainty dispersion is small, each agent chooses the highest possible correlation endogenously, resulting in a full market participation. When the uncertainty dispersion among agents is large, the naive agent chooses not to participate due to a portfolio inertia feature, thus a limited participation equilibrium prevails. In this uniform framework, we demonstrate that asymmetric correlation occurs robustly and limited participation arises endogenously; the sophisticated (naive) agent always holds a well-diversified (under-diversified) portfolio compared with the market portfolio. In addition, the sophisticated agent achieves a better performance following a correlated-trading strategy. Moreover, we characterize the dispersion of assets' endogenous Sharpe ratios, and demonstrate that this dispersion decreases therefore assets move together more closely with the increase of correlation uncertainty.

A principle purpose of research on financial market is to study its correlated structure among all financial assets. Correlated structure is pervasive in financial market and plays a central role in finance since Markowitz (1952)'s seminal work on the portfolio choice, arbitrage pricing theory (Ross, 1976) and derivative pricing (Duffie et al, 2012), among many others. A voluminous number of studies also document significant empirical correlated structure-relevant phenomenons including asymmetric correlation (Ang and Chen, 2002; Longin and Solnik, 2001; Goetzmann, Li and Rouwenhorst, 2005), time-varying correlation structure (Bollerslev, Engle and Woolridge, 1988; Moskowitz, 2003, Ehling and Hererdahl-Larsen, 2013), underdiversification (Mankiw and Zeldes, 1991; Vissing-Jorgensen, 2002); limited participation (Goetzmann, Li and Rouwenhorst, 2005; Mankiw and Zeldes, 1991; Gomes and Livdan 2004), diversification discount and diversification premium (Campa and Kedia, 2002; Hoechel et. al., 2012), comovement (Barberis, Shleifer and Wurgler, 2005; Vedlhamp, 2006) and complicated correlation patterns among asset classes[1]. This paper develops a uniform framework underpinning these correlation-related puzzles and offer novel predictions for optimal portfolio choice and asset pricing related to the correlated structure in a financial market.

It is well known that the correlation estimation is more difficult than the estimation of the expected mean or volatility from both statistical perspective and econometric perspective (Chan, Karceski and Lakonishok, 1999; Ledoit, Santa-Clara and Wolf, 2003). The challenge of the correlation or covariance process stems from several reasons, for example, lack of enough market data source, limitation in estimation methodology, correlation process being unstable or very complicated,[2] not to mention the increasingly interconnected market pattern, making the whole correlation analysis much more demanding than the estimation of the marginal distribution.[3] Given its estimation challenge, the fragile sensitivity to many economic factors and investors' little knowledge about the future realization of asset correlation or its occurring probabilities under many circumstances, it is natural to study the

[1]See Forbes and Rigobon (2002) about the unconditional correlation coefficients during and after several financial crisis time periods. Wall Street Journal, August 21, 2011 also reports that "*S & P-500-index stocks have a correlation of 80%, even higher than 73% peak reached during the crisis in late 2008* ".

[2]See the Wald process in Bursaschi, Porchia and Trojani (2010); a stochastic matrix process in Tola et al (2006); and a dynamic multivariate GARCH model in Engle (2002).

[3]Alternatively, the correlation structure can be estimated from the option market. See Buss and Vilkov (2012); Kitiwiwattanachai and Pearson (2015). However, the complexity of the correlation process still preserves in spite of this implied estimation methodology.

correlation uncertainty in the financial market and investigate its implications to the market endogenous correlation in equilibrium.

This paper develops a correlation uncertainty framework to gauge a number of correlated phenomenons simultaneously by employing a multiple-priors[4] setting in Gilboa and Schmeidler (1989) on uncertainty. Specifically, agents' concern on the correlated structure is interpreted by ambiguity-aversion on the correlated structure, or virtually the correlation coefficient estimation. Each agent is both risk and ambiguity averse, and agents are heterogenous in terms of ambiguity aversion, reflecting their various levels of sophistication to deal with statistical data and estimation methodology.

To highlight the role of correlation ambiguity, we assume that agents have perfect knowledge of the marginal distributions for all assets; thus, they merely have concerns about the correlation structure. In this manner, our discussion on the correlation uncertainty significantly departs from previous literature in model uncertainty, where either expected return or the variance estimation is a concern. For instance, Garlappi, Uppal and Wang (2001), Boyle et al (2012), Easley and O'Hara (2009) investigate the expected return parameter uncertainty. Easley and O'Hara (2012), Epstein and Ji (2014) discuss volatility parameter uncertainty under recent high volatile market movement. In an information ambiguity setting, Illeditsch (2011) addresses the conditional distribution ambiguity of the signals. In those previous studies the correlation structure is always given exogenously. This paper develops an equilibrium model of heterogeneous *correlation uncertainty* and investigate its implications to the financial market.

We characterize the unique equilibrium at the presence of correlation uncertainty and demonstrate that the heterogeneous beliefs in correlation uncertainty, among others, play a fundamental role in the equilibrium. Specifically, when the dispersion of correlation uncertainty among agents is small, each agent chooses the *highest* available correlation in a full participation equilibrium. When the uncertainty dispersion is large, while the sophisticated agent still chooses her highest possible correlation coefficient, the naive agent's choice of correlation coefficient is no longer relevant in the equilibrium. The naive agent does not

[4]The multiple-priors framework has been applied in asset pricing theories by many authors. See, for instance, Easley and O'Hara (2009, 2012); Garlappi, Uppal and Wang (2007); Cao, Wang and Zhang (2005); Uppal and Wang (2003). For other approaches to address the ambiguity and its implication to asset pricing, see Bossarts et al (2010); Routledge and Zin (2014) and Illeditsch (2011).

participate in the market thus a partial participation equilibrium emerges. We show that this limited participation equilibrium follows from a remarkable *portfolio inertia* due to the correlation uncertainty.

In the portfolio inertia situation, the naive agent is irrelevant to the choice of the endogenous correlation coefficient, even though his optimal portfolio is unique. Cao, Wang and Zhang (2005) demonstrate a limited participation equilibrium model under expected return uncertainty without portfolio inertia. Our paper substantially differ from previous model uncertainty studies on expected mean and variance[5]. We show that the portfolio inertia can be obtained in the context of both portfolio choice and equilibrium under model uncertainty. Moreover, naive agents can be squeezed out from market by massive sophisticated investors, who mainly determine the equilibrium and stabilize the financial market. Hence, by introducing more institutional investors into the market and through enhancing investors' education/training and information transparency can reduce uncertainty on assets' correlated structure.[6]

Our equilibrium analysis has *several* key implications in understanding the empirical correlation-related puzzles. *First*, we demonstrate that correlation asymmetry occurs endogenously. While the benchmark/reference correlation coefficient is somewhat stable, an agent's correlation ambiguity is often larger in the weak market than in the strong market, yielding a higher correlation in a downside market movement. This correlation asymmetry phenomenon is shown to be persistent in spite of the heterogeneity in correlation estimation.

Second, we show that limited participation indeed occurs endogenously due to the agents' ambiguity dispersion on the correlated structure. By comparing with the market portfolio, we also show that the sophisticated agent always holds a more diversified (well-diversification) portfolio while the naive agent holds an underdiversification portfolio. Specifically, we propose a *dispersion measure* to quantify the dispersion among economical variables, inspired by the portfolio selection literature[7]. Previous studies such as Cao, Wang and Zhang (2005), Wang and Uppal (2003), Easley and O'Hara (2009) and Vissing-Jorgensen (2003), show that

[5]See Easley and O'Hara (2009, 2012); Garlappi, Uppal and Wang (2007); Cao, Wang and Zhang (2005); and Epstein and Ji (2014).

[6]Easley and O'Hara (2009) and Easley, O'Hara and Yang (2015) present similar insights from other perspectives.

[7]See Solnik and Roulet (2000); Hennessy and Lapan (2003); Ibragimov, Jaffee and Walden (2011).

underdiversification and limited participation occur when there is ambiguity on the distribution estimation. Nevertheless, we examine this issue for a large number of assets in a precise manner. We further demonstrate that when the risk dispersion is high enough, a limited participation equilibrium prevails regardless of the uncertainty degree.

Third, we analyze agents' trading activities when the level of correlation uncertainty varies. We show that each agent holds "long" position on *high risk* assets, but the naive agent might "sell" some endogenous *low risk* assets. We confirm that the naive agent's portfolio is less risky because his higher correlation uncertainty yields a higher implicit risk aversion.[8] We also show that the sophisticated agent achieves a better performance optimal portfolio.

Fourth, we identify circumstances in which the diversification premium is generated in a heterogeneous setting of correlation uncertainty. It is a debated topic whether diversification premium or diversification discount occurs.[9] Cao and Wang and Zhang (2005) find diversification discount in a heterogeneous setting of the expected return uncertainty. By contrast, we show that diversification premium can be generated in a heterogeneous correlation uncertainty framework.

Last, we conduct a comprehensive analysis of the sophisticated agent as well as the correlation uncertainty on asset prices, risk premiums and in particular, the dispersion of endogenous Sharpe ratios. Despite the dispersion of correlation uncertainty, we find that the dispersion of Sharpe ratios *decreases* endogenously as correlation uncertainty increases. Not only prices substantially drop in a downside market with larger correlation uncertainty, but also all risky assets are enforced to comove more since they offer similar investment opportunities. This analysis sheds light to further understand asset comovements from a correlation ambiguity perspective, given that assets moves closely together in a downside market and moves apart in an upside market.

The rest of the paper is organized as follows. Section 1 presents a model setting of correlation uncertainty. In Section 2, we study the portfolio choice problem under correlation uncertainty and demonstrate the persistent portfolio inertia feature. We also characterize the

[8]See Garlappi, Uppal and Wang (2007); Gollier (2011); Wang and Uppal (2003) for the discussion that ambiguity leads to risk aversion implicitly.

[9]See Campa and Kedia (2002), Rajan, Servases and Zingales (2000), Villalonga (2004), Hoechel *et. al.* (2012).

equilibrium in a homogeneous environment as a baseline model. In Section 3 we present the market equilibrium under heterogeneous correlation uncertainty and its effect on asset prices and risk premium. We also present the conditions of diversification premium in equilibrium. Section 4 concerns on the optimal portfolios. We demonstrate that the sophisticated agent has a well diversified, higher risk and better performance optimal portfolio than the naive agent. Section 5 investigates the effect of the sophisticated agent as well as the dispersion of risks to the financial market. Section 6 extends the model to multiple correlated structures. Section 7 concludes. Proofs and technical arguments are collected in Appendices.

1 A Model of Correlation Uncertainty

In this section we present a model of correlation uncertainty with heterogeneous beliefs among agents.

We consider a two-period economy with N risky assets and one risk-free asset which serves as a numeraire. The payoffs of these N risky assets are $\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_N$, respectively at time $t = 1$. The risk-free asset is in zero net supply, and the per capita endowment of risky asset i is $\overline{x}_i, i = 1, \cdots, N$. Each risky asset can be viewed as an investment asset, an investment fund, an asset class, or a market portfolio in a single-factor economy in an international market.

To focus entirely on the correlated risk and its effect on asset pricing, we investigate the correlation structure instead of the joint distribution of $(\tilde{a}_1, \cdots, \tilde{a}_N)$.[10] For simplicity, we assume that $(\tilde{a}_1, \cdots, \tilde{a}_N)$ has a multivariate Gaussian distribution as in Cao, Wang and Zhang (2005), Easley and O'Hara (2009, 2012); and each agent is confident in the estimation of expected mean $\overline{a}_i$ and variance σ_i^2 for each risky asset $i = 1, \cdots, N$. However, they are seriously concerned about the correlated structure estimation.

[10] By a copula theory (see McNeil, Frey and Embrechts, 2015), the joint distribution of $\tilde{a} = (\tilde{a}_1, \cdots, \tilde{a}_N)$ is characterized by the marginal distribution of each $\tilde{a}_i$, and a copula function which determines the correlation structure. In other words, the correlation structure can be independent of the marginal distributions. We consider a single-factor Gaussian copula correlation structure, but our setting itself is general enough to include an arbitrary copula correlation structure.

We confine ourselves on a *positive correlated* financial market[11]. That is, these risky assets have nonnegative correlation coefficients. Moreover, the correlation coefficient matrix R among risky assets has a simple structure that these risky assets have a same correlation coefficient ρ, i.e., $corr(\tilde{a}_i, \tilde{a}_j) = \rho$ for each $i \neq j$.This assumption on the correlation structure will be relaxed in Section 6 and it is well known that such as matrix is a correlation matrix of a multivariate Gaussian variable as long as $\rho \neq -\frac{1}{N-1}$.[12] Every agent encounters some correlation uncertainty; therefore,we assume that each agent has a plausible region for the correlation coefficient instead of one particular number.

There is a group of agents in this economy. Each agent has the CARA-type risk preference to maximize the worst-case diversification benefit

$$\min_{\rho} \mathbb{E}^{\rho}\left[u(W)\right], u(W) = -e^{-\gamma W}, \tag{1}$$

where ρ runs through a plausible correlation coefficient region, $\mathbb{E}^{\rho}[\cdot]$ represents the expectation operator under corresponding correlation coefficient ρ, and γ is the agent's absolute risk aversion parameter and we assume that each agent has the same absolute risk aversion. Since each agent is able to accurately estimate the marginal distribution of each $\tilde{a}_i$, the correlation coefficient ρ is the only one parameter with uncertainty that affects the diversification benefit in (1).

We assume that agents are heterogeneous in their estimations of the correlation coefficient matrix. Specifically, there are two types of agent, *sophisticated* ("she") and *naive* ("he"). For the sophisticated agent, her available correlation coefficient region is $[\alpha_1, \beta_1]$; while for the naive agent, his correlation coefficient region is $[\alpha_2, \beta_2]$.[13] We assume that $[\alpha_1, \beta_1] \subseteq [\alpha_2, \beta_2]$, which reflects the fact that the estimation on correlation coefficient for the sophisticated agent is more accurate than the naive agent. The percentage of sophisticated agents in the market is ν and naive agents' percentage is $1 - \nu$.

[11]Technically speaking, our results hold when all correlation coefficients are strictly larger than $-\frac{1}{N-1}$. A positive correlated structure is driven by common shocks, or some factors in a financial market. Both the diversification benefits and synchronization are more critical in a positive correlated economy than in a negative correlated environment. For a positive correlated environment in terms of affiliated theory we refer to Milgrom and Weber (1982).

[12]It is well known that fat-tailed asset distribution can be generated while a correlation structure is simply preserved through normal mixture approach (McNeil, Frey and Embrechts, 2015). Therefore, our result virtually hold for a mean-variance setting to encounter tailed marginal distributions of asset prices.

[13]It is well known that the plausible linear correlation coefficient between any two variables X and Y is an interval, $[\rho_{min}, \rho_{max}]$. See NcNeil, Frey and Embrechts (2015).

The plausible correlation coefficient region $[\alpha, \beta]$ represents the correlation *magnitude* as well as *uncertainty*. An agent has in mind a benchmark or reference in an economy that represents his best estimate on the correlation, and the benchmark/reference correlation is implied by $\frac{\alpha+\beta}{2}$; however, and the level of correlation uncertainty is measured by $\frac{\beta-\alpha}{2}$. Precisely, the benchmark correlation coefficient signifies the market trend over all, and the level of correlation uncertainty indicates how far away plausible correlation coefficients move upon and below the benchmark. In most situations, econometricians are able to find the benchmark correlation coefficient through the calibration to a stochastic matrix process, and treat it as a market reference with some estimation errors (See Engle, 2002; Buraschi and Porchia and Trojani, 2010). In this case, we use $\overline{\rho}$ to denote the benchmark and ϵ the level of uncertainty.

As a special case to notice, both types of agents can agree on the same benchmark correlation coefficient, but have different ambiguities with respect to the correlation coefficient estimation. If so, the available correlation coefficient for the sophisticated agent is $[\overline{\rho} - \epsilon_1, \overline{\rho} + \epsilon_1]$, and the naive agent's plausible correlation coefficient is $[\overline{\rho} - \epsilon_2, \overline{\rho} + \epsilon_2]$ and $\epsilon_1 < \epsilon_2$. An extreme situation is $\epsilon_1 = 0$, where the sophisticated agent has the perfect knowledge about the correlation structure.[14] We illustrate our results with these special cases later.

2 Equilibrium in a homogeneous environment

In this section we first solve the portfolio choice problem under correlation uncertainty, then we present the equilibrium in a homogeneous environment.

[14] Alternatively, by adopting Cao et al. (2005), we can assume that there is a continuum of agents, say $[\overline{\rho} - \epsilon, \overline{\rho} + \epsilon]$, each type of investor's correlation uncertainty is captured by different parameter ϵ while ϵ is uniformly distributed among agents on $[\overline{\epsilon} - \delta, \overline{\epsilon} + \delta]$ with density $1/(2\delta)$. The main insights of this setting are fairly similar to ours whereas the impact of sophisticated agents in our current setting has a clearer expression. Our setting is in a manner reminiscent of Easley and O'Hara (2009, 2012) on the heterogeneity of agents' ambiguity aversion.

2.1 Portfolio Choices

Let x_i be the number of shares on the risky asset i, $i = 1, \cdots, N$, and W_0 is the initial wealth of the agent, then the investor's final wealth at time 1 is

$$W = W_0 + \sum_{i=1}^{N} x_i(\tilde{a}_i - P_i)$$

where P_i is the initial price of the risky asset i. Assuming the plausible range of the asset correlation coefficient ρ is $[\alpha, \beta]$ and there is no any trading constraint, the optimal portfolio choice problem for the agent is

$$\max_{x \in \mathbb{R}^N} \min_{\rho \in [\alpha,\beta]} \mathbb{E}^{\rho}\left[-e^{-\gamma W}\right]. \tag{2}$$

Under the CARA preference and the multivariate Gaussian distribution assumption of the asset returns, the above problem is reduced to be $u(A)$ and

$$A \equiv \max_{x} \min_{\rho \in [\alpha,\beta]} CE(x, \rho) \tag{3}$$

where $CE(x, \rho) = (\overline{a} - p)x - \frac{\gamma}{2} x^T \sigma^T R(\rho) \sigma x$ is the mean-variance utility of the agent when the demand vector on the risk assets is $x = (x_1, \cdots, x_N)^T$, where σ is a diagonal $N \times N$ matrix with diagonal vector $(\sigma_1, \cdots, \sigma_N)$ and $R(\rho)$ is a correlation matrix with a common correlation coefficient ρ. We use $\cdot^T$ to denote the transpose operator of a matrix. The certainty-equivalent of the uncertainty-averse agent is

$$CE(x) = \min_{\rho \in [\alpha,\beta]} CE(x, \rho) \tag{4}$$

and it is straightforward to see that

$$CE(x) = \begin{cases} CE(\beta, x), & \text{if } \sum_{i \neq j} (\sigma_i \overline{x}_i)(\sigma_j \overline{x}_j) > 0, \\ CE(\alpha, x), & \text{if } \sum_{i \neq j} (\sigma_i \overline{x}_i)(\sigma_j \overline{x}_j) < 0, \\ \sum_{i=1}^{N} \left((\overline{a}_i - p_i)x_i - \frac{\gamma}{2}\sigma_i^2 x_i^2\right), & \text{if } \sum_{i \neq j} (\sigma_i \overline{x}_i)(\sigma_j \overline{x}_j) = 0, \end{cases} \tag{5}$$

The insight of Equation (5) is straightforward. When the demand vector x has approximately same signs such that $\sum_{i \neq j} (\sigma_i \overline{x}_i)(\sigma_j \overline{x}_j) > 0$, the agent will choose the highest possible correlation coefficient to compute the certainty-equivalent in the worst case scenario under correlation uncertainty. On the other hand, if the holding positions on the risky assets are opposite such that $\sum_{i\neq j}(\sigma_i\overline{x}_i)(\sigma_j\overline{x}_j) < 0$, the correlation coefficient to compute the certainty-equivalent in the worse case scenario must be the smallest possible correlation coefficient. Finally, if *limited participation* occurs in the sense that $\sum_{i\neq j}(\sigma_i\overline{x}_i)(\sigma_j\overline{x}_j) = 0$,[15] then choice of the correlation coefficient is irrelevant to compute the certainty-uncertainty as $CE(\rho, x) = \sum_{i=1}^N \left((\overline{a}_i - p_i)x_i - \frac{\gamma}{2}\sigma_i^2 x_i^2\right)$ for each $\rho \in [\alpha, \beta]$.

To elaborate the certainty-equivalent and solve the portfolio choice problem, we introduce a dispersion measure, $\Omega(w)$, of a vector $w = (w_1, \cdots, w_N)$ with $\sum_{i=1}^N w_i \neq 0$ by

$$\Omega(w) \equiv \sqrt{\frac{1}{N-1}\left(N\frac{\sum_{i=1}^N w_i^2}{(\sum_{i=1}^N w_i)^2} - 1\right)}. \tag{6}$$

If further $\sum_{i=1}^N w_i = 1$, then $\Omega(w)^2 = \frac{1}{N-1}\left(\sum_{i=1}^N w_i^2 - 1\right)$ is up to a linear transformation the Herfindahl index $\sum_{i=1}^N w_i^2$. In Appendix B we present a formal justification of $\Omega(\cdot)$ being a dispersion measures of individual assets' risk.[16] This dispersion measure plays a crucial role in our equilibrium analysis for the endogenous correlation structure and its asset pricing implications in Section 2- Section 6.

By using the dispersion measure, we reformulate the certainty-equivalent utility of the uncertainty-averse agent as

$$CE(x) = \begin{cases} CE(\beta, x), & \text{if } \Omega(\sigma x) < 1, \\ CE(\alpha, x), & \text{if } \Omega(\sigma x) > 1, \\ \sum_{i=1}^N \left((\overline{a}_i - p_i)x_i - \frac{\gamma}{2}\sigma_i^2 x_i^2\right), & \text{if } \Omega(\sigma x) = 1, \end{cases} \tag{7}$$

Therefore, the optimal portfolio choice problem for the uncertainty-averse agent becomes

$$A = \max\left\{max_{\Omega(\sigma x)<1} CE(\beta, x), max_{\Omega(\sigma x)>1} CE(\alpha, x), max_{\Omega(\sigma x)=1} CE(\rho, x)\right\}. \tag{8}$$

The solution of the optimal portfolio choice problem (2) is given by the next result.

[15] It is evident for $N = 2$. When $x_1x_2 = 0$, then either $x_1 = 0$ or $x_2 = 0$, thus at most one component of x is non-zero. For an arbitrage N, if each $x_i \geq 0$, it is easy to show that $\sum_{i\neq j} x_i x_j = 0$ ensures that at most one component of x is non-zero; thus only invested in one risky asset. Similarly, when $N = 2$ and $x_1x_2 < 0$, it is a pair trading or a market-neutral strategy, and $N = 2, x_1x_2 > 0$, the portfolio yields synchronization strategy.

[16] The dispersion measure has been applied in the portfolio selection context. See Ibragimov, Jaffee and Walden (2011); Hennessy and Lapan (2003).

Proposition 1 *Let $\Omega(s)$ be the dispersion measure of Sharpe ratios (s_i) of all risky assets and assume that $S = \sum_{i=1}^{N} s_i \neq 0$, $s_i = (\overline{a}_i - p_i)/\sigma_i$.*

1. *If $\alpha > \tau(\Omega(s))$, then the agent chooses the optimal correlation coefficient $\rho^* = \alpha$, and the optimal demand is $x^* = x_\alpha$ in Problem (2).*
2. *If $\beta < \tau(\Omega(s))$, then the agent chooses the optimal correlation coefficient $\rho^* = \beta$, and the optimal demand is $x^* = x_\beta$ in Problem (2).*
3. *If $\alpha \leq \tau(\Omega(s)) \leq \beta$, then the agent is irrelevant to choose any correlation coefficient $\rho^* \in [\alpha, \beta]$, and the optimal demand is $x_{\tau(\Omega(s))}$ in Problem (2).*

For each $\rho \in [\alpha, \beta]$, $x_\rho \equiv \frac{1}{\gamma}\sigma^{-1}R(\rho)^{-1}s$ is the optimal portfolio at the absence of uncertainty when the correlation coefficient is ρ. τ is a linear fractional transformation: $\tau(t) \equiv \frac{1-t}{1+(N-1)t}$ for any real number $t \neq -\frac{1}{N-1}$.

The intuition of Proposition 1, (1), is as follows. When all available correlation coefficients are large enough such that $\alpha > \tau(\Omega(s))$, the dispersion of Sharpe ratios $\Omega(\sigma x_\alpha)$ is strictly larger than 1 (by Appendix A, Lemma 2). Therefore, (α, x_α) solves the optimal portfolio choice problem $CE(\alpha, x)$ under the demand constraint $\Omega(\sigma x) > 1$. By analyzing the dual-problem of Problem (2) (see its proof in Appendix A), we show that α is the best possible correlation coefficient for the uncertainty-averse agent in this situation; thus, (α, x_α) is the unique solution of the problem (2).

By the same reason, if all available correlation coefficients are small enough such that $\beta < \tau(\Omega(s))$, by Appendix A, Lemma 2 again, $\Omega(\sigma x_\beta) < 1$. Therefore, (β, x_β) maximizes $CE(\beta, x)$ for x satisfying the constraint that $\Omega(\sigma x) < 1$. Again, a dual-type analysis ensures that the highest correlation coefficient, β, is the best possible correlation coefficient. Hence, (β, x_β) is the unique solution for the uncertainty-averse agent's portfolio choice problem (2).

Proposition 1 is remarkable when the agent's available correlation coefficient region is large such that $\alpha \leq \tau(\Omega(s)) \leq \beta$ from several perspectives. First, the agent holds a limited participation portfolio to resolve the correlation uncertainty concern since the dispersion of

$x_{\tau(\Omega(s))}$ is one. Second, the optimal demand is unique and any other demand vector leads to a smaller maxmin expected utility in Problem (2). Third, while the optimal demand vector is unique, the choice of the correlation coefficient for the agent is irrelevant, that is, the *portfolio inertia* occurred if the correlation uncertainty is large. It is well documented that a high ambiguity might ensure portfolio inertia since Dow-Werlang (1992), Epstein-Schneider (2008) and Illeditsch (2011). However, this feature does not emerged in the Chiboa-Schmeidler maxmin expected utility setting where either mean or volatility is unknown and the worst case scenario corresponds to the extreme parameters.[17] If the agent has a full correlation ambiguity, say $[\alpha, \beta] = [-1, 1]$, Liu and Zeng (2015) show a strong version of limited participation under correlation uncertainty if one risky asset's Sharpe ratio dominates all other risky assets' Sharpe ratios.

According to Proposition 1, the dispersion of Sharpe ratios $\Omega(s)$ is fundamental in the characterization of the optimal portfolio choice problem and it deserves some comments. First, $\Omega(s)$ captures the *market investment capacity*. The higher $\Omega(s)$, the more diversified all risky assets' investment profile; thus, the higher expected utility for all agents. To see it, consider an economy in the presence of correlation uncertainty and the correlation coefficient is ρ. Then the maximum expected utility for a CRRA-type agent is $u\left(\frac{1}{2\gamma}G(\rho)\right)$, where

$$G(\rho) \equiv s' R(\rho)^{-1} s = \frac{S^2}{1-\rho}\left(\frac{(N-1)\Omega(s)^2+1}{N} - \frac{\rho}{1+(N-1)\rho}\right). \tag{9}$$

It is evident to see that the maximum expected utility is monotonically increasing with respect to $\Omega(s)$.

Second, the level of $\Omega(s)$ determines the optimal strategy under correlation uncertainty. We take an example of $N = 2$ to explain the unusual portfolio inertia feature under correlation uncertainty. In an economy with two risky assets, we have

$$\Omega(s) = |\frac{s_1 - s_2}{s_1 + s_2}|, \quad \tau(\Omega(s)) = \begin{cases} \min\left\{\frac{s_1}{s_2}, \frac{s_2}{s_1}\right\}, & \text{if } s_1 s_2 > 0, \\ \max\left\{\frac{s_1}{s_2}, \frac{s_2}{s_1}\right\}, & \text{if } s_1 s_2 < 0, \\ 0, & \text{if } s_1 s_2 = 0. \end{cases}$$

[17] See Garlappi, Uppal, Wang (2007), Easley and O'Hara (2009) and Epstein and Ji (2014).

$\tau(\Omega(s))$ measure the "similarity" of the two Sharpe ratios, or the investment opportunity offered from each risky asset. In an extreme situation that one asset (say, the first risky asset) has a very small Sharpe ratio, $\tau(\Omega(s))$ is close to zero. Since the expected return of holding the first risky asset is almost the same as holding the risk-free rate, the only reason to hold it in an optimal portfolio is for the diversification purpose. In order to take advantage of the diversification benefit, the optimal strategy in a mean-variance analysis for these two positively correlated assets must be one long and one short, the smallest possible correlation is thus chosen. Now assume another extreme case that $s_1 = s_2$ and $\tau(\Omega(s)) = 1$. These two risky assets have the same Sharpe ratios, so the unknown correlation coefficient becomes a major concern for diversification. Therefore, to hedge the correlation uncertainty in the worst-case scenario, the agent chooses naturally the highest possible correlation coefficient. In a general case with arbitrarily $s_1 > 0, s_2 > 0$, the diversification benefit can be written as (with a specific correlation coefficient ρ)

$$u\left(\frac{1}{2\gamma}\left(\frac{s_1^2 + s_2^2 - 2\rho s_1 s_2}{1-\rho^2}\right)\right).$$

Hence, the optimal correlation coefficient under correlation uncertainty depends on the level of $\tau(\Omega(s))$ with respect to the plausible region $[\alpha, \beta]$ of the correlation coefficient. In particular, when $\tau(\Omega(s)) \in [\alpha, \beta]$, the agent can choose any correlation coefficient while the optimal demand is chosen by pretending to choose $\tau(\Omega(s))$ as the "right" correlation coefficient. The intuition of Proposition 1 for $N \geq 3$ is similar.

2.2 Equilibrium in a homogeneous environment

The equilibrium in a homogeneous environment is characterized as follows.

Proposition 2 *Assume the plausible correlation coefficient is $[\alpha, \beta]$ in a homogeneous environment. There exists a unique uncertainty equilibrium in which the agent's endogenous correlation coefficient is the* highest *plausible correlation coefficient β.*

The price of the risky asset i is

$$p_i = \overline{a}_i - \gamma\sigma_i(1-\beta)\sigma_i\overline{x}_i - \gamma\sigma_i\beta(\sum_{n=1}^{N}\sigma_n\overline{x}_n). \quad (10)$$

The intuition of Proposition 2 is simple. The optimal portfolio must be the market portfolio $\sum_{i=1}^{N} \overline{x}_i \tilde{a}_i$ in equilibrium, thus, $\Omega(\sigma x^*) = \Omega(\sigma \overline{x}) = 1$. By Proposition 1, the representative agent chooses the highest possible correlation coefficient.

There are several important asset pricing implications of Proposition 2. To highlight the effect of correlation uncertainty, we use $\alpha = \overline{\rho} - \epsilon, \beta = \overline{\rho} + \epsilon$. Therefore, the risk premium $\overline{a}_i - p_i$ can be written as a sum of two components:

$$
\begin{aligned}
\overline{a}_i - p_i &= \gamma(1-\overline{\rho})\sigma_i^2 \overline{x}_i + \gamma \sigma_i \overline{\rho} \sum_{n=1}^{N} \sigma_n \overline{x}_n \\
&\quad + \epsilon\gamma \left(\sigma_i \sum_{j \neq i} \sigma_j \overline{x}_j \right) \qquad (11)
\end{aligned}
$$

where the first component represents the premium at the absence of correlation uncertainty, and the second one is the correlation-uncertainty premium.

First of all, Proposition 2 is useful to explain the equity premium puzzle in the presence of the correlation uncertainty due to the positive uncertainty premium. As an illustrative example, we report the percentage of the correlation-uncertainty premium to the reference Sharpe ratio, $(1-\overline{\rho})\sigma_i \overline{x}_i + \overline{\rho} \sum_{n=1}^{N} \sigma_n \overline{x}_n$, in a homogenous situation, in Table 1. Given parameters $\sigma_1 = 9\%, \overline{x}_1 = 1, \sigma_2 = 10\%, \overline{x}_2 = 5, \sigma_3 = 12\%, \overline{x}_3 = 10.5$ and $\gamma = 1$. Then $\Omega(\sigma \overline{x}) = 0.5558$. We assume that $\overline{\rho} = 40\%$. Without correlation uncertainty, $(s_1, s_2, s_3) = (0.79, 1.49, 2.70)$. Let the level of uncertainty, ϵ, move between 0 to 0.2, we see that the correlation-uncertainty premium increases in a reasonable amount. For instance, when $\epsilon = 0.08$, the percentage of the correlation-uncertainty premium adds about 17 percent, 16 percent and 12 percent to each asset respectively. With a high level of uncertainty, say, $\epsilon = 0.2$, the correlation-uncertainty premium can be very significant, adding 40 percent to the reference Sharpe ratio.

When we further compare the influence of correlation ambiguity on the excess equity premium with the mean and volatility ambiguity, a richer impact can be observed. For simplicity, we consider two independent risky assets and one risk-free asset with zero return. To be consistent with our setting, the joint distribution of asset returns is assumed to be a bivariate Gaussian distribution and the representative agent has a CARA-type preference. In economy A, the representative agent has no ambiguity on the variance of each asset, but the expected return $a_i \in [\overline{a}_i - \epsilon_i, \overline{a}_i + \epsilon_i]$ for each risky asset $i = 1, 2$. In economy B, the representative agent has no ambiguity concern on the expected return of each asset, but the plausible volatility $\sigma_i \in [\overline{\sigma}_i - \epsilon_i, \overline{\sigma}_i + \epsilon_i]$ for each asset i. These two economies have been applied in Garlappi, Uppal and Wang (2007), Cao, Wang and Zhang (2005) and Easley and O'Hara (2009) to study the asset price implication of ambiguity. It can be shown that [18] the Sharpe ratios in economy A and economy B are

$$s_i^A = s_i + \frac{\epsilon_i}{\sigma_i}; s_i^B = s_i \frac{\overline{\sigma}_i + \epsilon_i}{\overline{\sigma}_i} \tag{12}$$

where s_i is the Sharpe ratios in the absence of ambiguity (with expected mean $\overline{a}_i$ and volatility $\overline{\sigma}_i$ for each risky asset). In Equation (12), the uncertainty premium of each risky asset depends only on the ambiguity of the marginal distribution estimation. By contrast, the correlation-uncertainty premium of each risky asset i depends on the entire market structure, in particular, $\sigma_j, \overline{x}_j, j \neq i$.

Secondly, Proposition 2 can be used to explain the asymmetric correlation phenomenon. A plausible correlation coefficient region, $[\overline{\rho} - \epsilon, \overline{\rho} + \epsilon]$, relies on the benchmark correlation coefficient $\overline{\rho}$ and the level of the correlation uncertainty ϵ. It is clear that in the absence of correlation uncertainty, the risk premium increases as the market correlation coefficient $\overline{\rho}$ goes high. In addition, the level of uncertainty displays a counter-cyclical feature (see Krishnan, Petkova and Ritchken, 2009) hence $\overline{\rho}$ and ϵ have compound effects on the correlation asymmetric phenomenon as observed in Ang and Chen (2002) and Longin and Solnik (2002), where the market endogenous correlation coefficient is often larger in a weak market than in a strong economy.

Thirdly, according to Proposition 2,

$$\frac{\partial p_i}{\partial \epsilon} = -\gamma \sigma_i \sum_{j \neq i} \sigma_j \overline{x}_j < 0; \frac{\partial s_i}{\partial \epsilon} = \gamma \sum_{j \neq i} \sigma_j \overline{x}_j > 0, \tag{13}$$

Then, each risky asset's price *drops* with respect to ϵ; and the variance of the market portfolio, $\sum \overline{x}_i \hat{a}_i$, increases because of increasing correlation coefficient.Therefore, Proposition 2 gives

[18] The type of portfolio choice problem has been studied in Garlappi, Uppal and Wang (2007) and Easley and O'Hara (2009). We can employ the same method in proving Proposition 1 in two economies and also derive the unique equilibrium in a homogeneous environment. The details are available upon request from the authors.

an intuitive illustration about the 2007-2009 financial crisis in which a representative agent has high ambiguity on the correlation structure, so the agent chooses the highest possible correlation coefficient in equilibrium. As a consequence, the risky assets are more correlated in equilibrium. A high level of correlation uncertainty leads to significant increase in total market risk, excess covariance risk, or comovement in the equity market.[19]

Finally, the dispersion of Sharpe ratio in equilibrium is

$$\Omega(s) = \Omega(\sigma\overline{x})\tau(\beta)$$

and the market capacity $\Omega(s)$ depends negatively on ϵ. The intuition is also straightforward. The higher the uncertainty ϵ, the smaller the market capacity since all agent's expected utility under correlation uncertainty is reduced. Moreover, a smaller market capacity implies that all risky assets offer similar investment opportunities (Sharpe ratios).

3 Equilibrium in heterogeneous environment

This section characterizes the equilibrium under heterogeneous correlation uncertainty. We first demonstrate a unique equilibrium by characterizing the endogenous correlation coefficient for each agent, then we present the correlation uncertainty effect on the asset price as well as on the risk premiums. As an application, we show that diversification premium puzzle can be explained in the presented correlation uncertainty setting.

3.1 A characterization of the equilibrium

In characterizing the equilibrium, it is vital to determine the optimal correlation coefficient for each agent. The agent can choose either the highest, the lowest one, or even any possible correlation coefficient in a portfolio choice setting; the market clearing condition enforces the representative agent, in a homogeneous environment, to choose the correlated structure with the highest possible correlation coefficient under the diversification concern. In a heterogeneous environment, however, the diverse choices among agents and the dispersion of correlation uncertainty lead to strikingly different features in equilibrium.

[19]The excess covariance risk is closely linked to the comovement and it can not be explained entirely by economic fundamentals. See Barberis, Shleifer and Wurgler (2005), Pindyck and Rotemberg (1993) and Vedlhamp (2006). Our result provides an alternative explanation of the excess covariance due to correlation.

We define several auxiliary functions to capture the parameters $\{\nu, \beta_1, \beta_2\}$ in a heterogeneous economy. Put

$$K(\beta_1) \equiv \frac{\frac{1}{1-\beta_1} - \frac{\Omega(\sigma\overline{x})}{1+(N-1)\beta_1} + \frac{(1-\nu)(1-\Omega(\sigma\overline{x}))}{\nu}}{\frac{1}{1-\beta_1} + \frac{(N-1)\Omega(\sigma\overline{x})}{1+(N-1)\beta_1}}. \tag{14}$$

and for each real number $x, y \neq 1, -\frac{1}{N-1}$,

$$m(x,y) \equiv \frac{\nu}{1-x} + \frac{1-\nu}{1-y}, n(x,y) \equiv \frac{\nu x}{(1-x)(1+(N-1)x)} + \frac{(1-\nu)y}{(1-y)(1+(N-1)y)}. \tag{15}$$

Proposition 3 *The market equilibrium is presented in two separable cases.*

1. (Full Participation Equilibrium) *If β_1 is large enough such that*

$$\beta_1 \geq \frac{1}{N-1}\left\{\frac{\nu}{1-\nu}\frac{\Omega(\sigma\overline{x})}{1-\Omega(\sigma\overline{x})}N - 1\right\}, \tag{16}$$

or if β_2 is strictly smaller than $K(\beta_1)$, there exists a unique equilibrium in which each agent chooses the corresponding highest *correlation coefficient, respectively.*

2. (Limited Participation Equilibrium) *If Equation (16) fails and β_2 is larger than $K(\beta_1)$, there exists a unique equilibrium in which the sophisticated agent chooses her highest possible correlation coefficient β_1 while the choice of the naive agent is irrelevant. However, the naive agent's optimal demand $x^{(n)}$, is uniquely determined by the endogenous Sharpe ratios in the equilibrium.*

3. *In either equilibrium, the market price for asset i is, for each $i = 1, \cdots, N$,*

$$p_i = \overline{a}_i - \frac{\gamma\sigma_i}{m(\beta_1, \rho_2)}\left(\sigma_i\overline{x}_i + \frac{n(\beta_1, \rho_2)}{m(\beta_1, \rho_2) - Nn(\beta_1, \rho_2)}\sum_{j=1}^{N}\sigma_j\overline{x}_j\right). \tag{17}$$

where $\rho_2 = \beta_2$ in the full participation equilibrium and $\rho_2 = K(\beta_1)$ in the limited participation equilibrium. Furthermore, each risky asset is priced at discount in equilibrium. That is, $p_i < \overline{a}_i$ and $s_i > 0$ for each $i = 1, \cdots, N$.

The full participation equilibrium prevails when all agents participate in the market. If agents are relatively homogeneous, i.e. both β_1 and β_2 are large under condition (16) or both β_1 and β_2 are small in the sense that $\beta_2 < K(\beta_1)$, both agents participate in the market by choosing the corresponding highest possible correlation coefficient under the diversification concern. This full participation condition indicates that whether agents participate in the market really depends on both the heterogeneity of agents as well as the dispersion of correlation uncertainty.

There are two useful situations for which Equation (16) holds. The first situation relates to a high reference correlation coefficient, say $\overline{\rho} \geq \frac{1}{N-1}\left\{\frac{\nu}{1-\nu}\frac{\Omega(\sigma\overline{x})}{1-\Omega(\sigma\overline{x})}N - 1\right\}$. For instance, $\overline{\rho} = 0.8, \Omega(\sigma\overline{x}) = 0.5, N = 10, \nu \leq 40\%$. In this case, each agent chooses the highest possible correlation coefficient regardless of the uncertainty degree. In the second situation, the benchmark correlation coefficient is small, say $\overline{\rho} = 0.3$, but each agent has high correlation uncertainty in the sense that $\overline{\rho} + \epsilon_1 \geq \frac{1}{N-1}\left\{\frac{\nu}{1-\nu}\frac{\Omega(\sigma\overline{x})}{1-\Omega(\sigma\overline{x})}N - 1\right\}$. Each agent chooses the highest possible correlation coefficient in equilibrium in order to hedge the worst-case correlation uncertainty, regardless of the uncertainty dispersion among agents. In general, the heterogeneity among agents and the dispersion of risk jointly affect the correlation bound in Equation (16). This correlation bound is positively related to ν and $\Omega(\sigma\overline{x})$. We discuss in detail how ν and $\Omega(\sigma\overline{x})$ affects the asset pricing implications in Section 5.

In the limited participation equilibrium of Proposition 3, the naive agent will not participate in the market if there is a large amount of heterogeneity of correlation estimation among agents. Cao, Wang and Zhang (2005) demonstrate an endogenous limited participation equilibrium when investors are heterogeneous in terms of expected return uncertainty. Proposition 3(2), demonstrates precisely an endogenous limited participation equilibrium under correlation uncertainty. Moreover, both the prices and the risk premiums depend only on the sophisticated agent when the naive agent's correlation uncertainty meets a threshold. Even though the naive agent is irrelevant in the choice of the correlation coefficient, his optimal demand, $x^{(n)}$, is uniquely determined by the endogenous Sharpe ratios in the equilibrium such that

$$x_i^{(n)} = \frac{1}{\gamma\sigma_i}\frac{S(1+(N-1)\Omega)}{N\Omega}\left(\frac{s_i}{S} - \frac{1-\Omega}{N}\right) \tag{18}$$

where $\Omega = \tau(K(\beta_1))$ is the dispersion of Sharpe ratios.

Both the heterogeneity of agents and the dispersion of risk are involved in the limited participation equilibrium as well. When the sum of sophisticated agents and the risk dispersion, $\nu + \Omega(\sigma\overline{x})$, is greater than 1, condition (16) fails. Further if the naive agent is very uncertain about the correlation coefficient such that $\epsilon_2 > K(\overline{\rho} + \epsilon_1) - \overline{\rho}$, he decides not to participate in the market, or equivalently, any choice of the correlation coefficient in his plausible range is irrelevant in equilibrium. This indicates that when there are many sophisticated agents in the market, naive agents are squeezed out of the market. As a result, sophisticated agents dominate the trading in the market. It is also the case that when the naive agent has a big uncertainty issue, no-diversification becomes his optimal strategy. As explained in the portfolio choice context, if his plausible correlation coefficient range is large enough to include $\tau(\Omega(s))$ and since $\tau(\Omega(s)) = K(\beta)$ in equilibrium, his optimal demand is given as $x_{K(\beta)}$, which is determined by the sophisticated agent's endogenous correlation coefficient β_1.

Figure 1 illustrates the property of the function $K(\beta_1)$ with respect to β_1 and how this correlation boundary determines different equilibrium cases accordingly. As shown, the dark curve line representing $K(\beta_1)$ is strictly above the straight line of β_1. The limited participation equilibrium occurs only in the yellow region whereas a full participation equilibrium happens elsewhere. As the correlation uncertainty of sophisticated investors increases, the endogenous correlation of naive investors rises consequently.

Figure 2 illustrates the Sharpe ratios of all assets in a full participation equilibrium. The parameters are $N = 3$, $\bar{a}_1 = 50$, $\sigma_1 = 9\%$, $\bar{x}_1 = 5$; $\bar{a}_2 = 60$, $\sigma_2 = 10\%$, $\bar{x}_2 = 3.7$; $\bar{a}_3 = 15$, $\sigma_3 = 12\%$, $\bar{x}_3 = 10.5$. Hence $\Omega(\sigma\overline{x}) = 0.41$. As seen in Figure 3, the plot displays similar magnitude of the correlation-uncertainty premium as in a homogeneous environment (Table 1). The correlation-uncertainty premium can also be identified in Table 2 for a limited participation equilibrium.

3.2 Diversification Premium

In this section we demonstrate that the correlation uncertainty could lead to diversification premium under certain circumstances due to the heterogeneity among the agents and the correlation uncertainty.

We assume each risky asset presents a common stock of a company with the same number of shares $\overline{x}$ and consider a situation where these companies merge into one conglomerate firm, and the cash flow of the conglomerate firm is $\tilde{a} = \tilde{a}_1 + \cdots + \tilde{a}_N$. Under the same assumption as in Section 2, $(\tilde{a}_1, \cdots, \tilde{a}_N)$ has a multivariate Gaussian distribution and each agent has a precise estimation on the marginal distribution of $(\tilde{a}_1, \cdots, \tilde{a}_N)$; the piecewise-correlation coefficient $\rho = corr(\tilde{a}_i, \tilde{a}_j)$ belongs to an interval $[\alpha, \beta]$. Therefore, $\tilde{a}$ has a normal distribution with mean $\overline{a} = \sum_{i=1}^N \overline{a}_i$, but the variance is unknown.[20] Let P_M be the equilibrium of the conglomerate firm's stock price (if exists).

Proposition 4 *There exists a unique conglomerate equilibrium in both the homogeneous and the heterogeneous environment.*

1. *In a homogeneous environment, $P_M = \sum_{i=1}^N P_i$.*
2. *In a heterogeneous environment, assume the benchmark correlation coefficient is large enough such that Equation (16) holds. Let $m = m(\beta_1, \beta_2), n = n(\beta_1, \beta_2)$.*
   - *(Diversification premium) $P_M > \sum_{i=1}^N P_i$ if, and only if*

$$(\sum_{i=1}^N \sigma_i^2)(m-2)(m-Nn) < (\sum_{i=1}^N \sigma_i)^2[(N+1)n-m)]. \tag{19}$$

   - *If further each σ_i is the same, then diversification premium always exists as long as N is large. If $N = 2$ and each σ_i is the same, diversification premium exists if $m^2 < 2n(m+1)$.*

[20]See Easley and O'Hara, 2009; Epstein and Ji, 2014 for the unknown variance setting. Precisely, $Var(\tilde{a}) = \sum_{i=1}^N \sigma_i^2 + 2\sum_{i\neq j}\sigma_i\sigma_j\rho$ runs through an interval $[\sum_{i=1}^N \sigma_i^2 + 2\sum_{i\neq j}\sigma_i\sigma_j\alpha, \sum_{i=1}^N \sigma_i^2 + 2\sum_{i\neq j}\sigma_i\sigma_j\beta]$.

Proposition 4 shows that the heterogeneous beliefs on correlation uncertainty plays a crucial role in the diversification premium puzzle. While the diversification discount can be explained in a heterogeneous mean uncertainty setting in Cao, Wang and Zhang (2005), and well documented in Janan, Servaes and Zingales (2000), recent literature also identify diversification premium in many situations. For example, Villalonga (2004), and Hoechle, et. al (2012), etc. Proposition 4 demonstrates that there is diversification premium when two identical firms are merged and $m^2 < 2n(m+1)$ is satisfied. For instance, when $\Omega(\sigma\overline{x}) = 0.5$ and the benchmark correlation coefficient $\overline{\rho} = 0.7$, $\nu = 0.8$ and the sophisticated agent has a perfect knowledge about the correlation coefficient. Then $m < 2$ as long as $\epsilon \leq 0.06$ for the naive agent. Hence, $m^2 < 2n(m+1)$ holds naturally and there is a diversification premium. We also observe the diversification premium feature when the number of identical firms is large.

3.3 Sharpe ratio and the market capacity

In this section we describe properties of the endogenous risk premium, Sharpe ratios and the market capacity in the setting. These properties are presented by the following proposition.

Proposition 5 *For each asset $i \neq j$,*

1. *$s_i \geq s_j$ if, and only if $\sigma_i\overline{x}_i \geq \sigma_j\overline{x}_j$; It is also equivalent to $\sigma_i x_i^{(s)} \geq \sigma_j x_j^{(s)}$ as well as $\sigma_i x_i^{(n)} \geq \sigma_j x_j^{(n)}$, where $x^{(j)}$ is the optimal demand vector for the agent $j \in \{s, n\}$.*

2. *s_i is larger than the average Sharpe ratio, $\frac{S}{N}$, if and only if $\sigma_i\overline{x}_i \geq \frac{L}{N}$;*

3. *If $\frac{\sigma\overline{x}_i}{L} \leq \frac{1}{N}$ for asset i, then the higher the level of uncertainty for the agents the smaller the Sharpe ratio for the asset i; the effect of correlation uncertainty on the Sharpe ratio of asset with higher risk, $\frac{\sigma\overline{x}_i}{L} > \frac{1}{N}$ is ambiguous.*

4. *$\Omega(s)$ depends negatively on the level of correlation uncertainty, but it increases with respect to the number of sophisticated agents.*

Proposition 5 first demonstrates the symmetric property between the Sharpe ratio with the associated risk among all assets. Proposition 5(1) states that the a higher Sharpe ratio always corresponds to a higher risk among all risky assets, and each agent invests more on the assets with higher Sharpe ratios. Specially, if we re-order the risky assets $i = 1, \cdots, N$ in the decreasing order by $\sigma_i \overline{x}_i, i = 1, \cdots, N$, both the sequence $\sigma_i x_i^{(s)}$ and $\sigma_i x_i^{(n)}$ depict a similar decreasing feature. We discuss in details about these two optimal portfolios in the next section.

To understand Proposition 5(2) and (3), we decompose the Sharpe ratio of risky asset i into two components:

$$s_i = \frac{S}{N} + \frac{\gamma}{m}\left(\sigma_i \overline{x}_i - \frac{L}{N}\right) \tag{20}$$

where the first component is the *average Sharpe ratio*, and the second one represents how much it differs from the average Sharpe ratio. We call the second component as a *specific Sharpe ratio*. According to this decomposition, the specific Sharpe ratio of asset i is proportional to the difference between the individual risk and the average risk, $\sigma_i \overline{x}_i - \frac{L}{N}$, and the endogenous Sharpe ratio is determined jointly by the average Sharpe ratio and the specific Sharpe ratio.

Proposition 5(2) follows from Equation (20) immediately that a risky asset's Sharpe ratio is above the average Sharpe ratio only when its risk is above the average level. Proposition 5(3) displays the crucial difference between asset with high risk and low risk. We first note that both the average Sharpe ratio and $m(\beta_1, \rho_2)$ depend negatively on the level of uncertainty,[21] For an asset with low risk, $\sigma_i \overline{x}_i \leq \frac{L}{N}$, the specific Sharpe ratio also inversely depends on the level of uncertainty. Therefore, when the correlation uncertainty increases, the Sharpe ratio decreases.

However, for high risk assets, the effect of correlation uncertainty is ambiguous since it depends on two opposing effects of the average Sharpe ratio and the specific Sharpe ratio. With increasing uncertainty, the average Sharpe ratio decreases, and the Sharpe ratio

[21] It can be seen easily by the expression of $m(x, y), n(x, y)$ and the fact that $S = \frac{\gamma L}{m(x,y) - N n(x,y)}$ and $\beta_1 < \rho_2^*$ in Proposition 3.

decreases only when the specific Sharpe ratio is very large such that

$$-\frac{\partial}{\partial\beta}\left(\frac{1}{m}\right)\left(\sigma_i\overline{x}_i - \frac{L}{N}\right) > \frac{\partial}{\partial\beta}\left(\frac{1}{m-Nn}\right).$$

Otherwise, if $\sigma_i\overline{x}_i - \frac{L}{N}$ is fairly small, the positive effect of the specific Sharpe ratio dominates the negative effect of the average Sharpe ratio, thus reaching a positive impact in total.

The intuition of Proposition 5(4) also follows in essence from the above decomposition of the Sharpe ratio. Reformulate Equation (20) as a multiplication version of the decomposition of the Sharpe ratio in terms of the average Sharpe Ratio.

$$\frac{s_i}{S} - \frac{1}{N} = \frac{m-Nn}{m}\left(\frac{\sigma_i\overline{x}_i}{L} - \frac{1}{N}\right). \tag{21}$$

Since $\frac{m-Nn}{m}$ is negatively related to the correlation uncertainty[22], the sensitivity of the relative Sharpe ratio $\frac{s_i}{S}$ relies on $\frac{\sigma_i\overline{x}_i}{L} - \frac{1}{N}$: it decreases for assets with $\frac{\sigma_i\overline{x}_i}{L} > \frac{1}{N}$, and increases for assets with $\frac{\sigma_i\overline{x}_i}{L} < \frac{1}{N}$. By Proposition 5, (2), $\frac{\sigma_i\overline{x}_i}{L} > \frac{1}{N}$ is associated with $\frac{s_i}{S} > \frac{1}{N}$. Therefore, with increasing correlation uncertainty, $\frac{s_i}{S}$ decreases if $\frac{s_i}{S} > \frac{1}{N}$ and increases if $\frac{s_i}{S} < \frac{1}{N}$. Put it together, all $\frac{s_i}{S}$ are closer with higher correlation uncertainty, so the dispersion of Sharpe ratio decreases, as presented in Proposition 5, (4).

4 Optimal portfolios

In this section we discuss and compare the optimal portfolio, $x^{(s)}$ and $x^{(n)}$. Our comparison between these two optimal portfolios is summarized by the next proposition.

Proposition 6 *1. (Underdiversification and well-diversification). Comparing with the market portfolio $\sum_{i=1}^{N}\overline{x}_i\tilde{a}_i$, the naive agent has an underdiversified portfolio while the sophisticated agent has a more diversified (well-diversification) portfolio.*

[22] By the proof of Proposition 3, $\frac{m-Nn}{m} = k(\beta_1, \rho_2)$ in Appendix, Equation (A-17), and it is easy to check its property as desired.

2. *(Portfolio Risk) The sophisticated agent holds a riskier portfolio than the naive agent. Specifically, the variance of $\sum_i \tilde{a}_i x_i^{(s)}$ is strictly larger than the variance of $\sum_i \tilde{a}_i x_i^{(n)}$.*

3. *(Portfolio Position) The sophisticated agent holds long position on all risky assets; the naive agent hold long positions only on the high risk assets and short positions on the low risk positions.*

4. *(Portfolio Performance) The sophisticated agent has a better portfolio performance than the naive agent in the sense that the Sharpe ratio of her optimal portfolio is strictly larger than the Sharpe ratio of the naive agent's optimal portfolio.*

5. *(Maxmin Expected Utility) The sophisticated agent has a higher maxmin expected utility than the naive agent.*

6. *(Comovement) From both the sophisticated agent and the naive agent's perspective, the optimal portfolios have a comovement feature. Specifically, the covariance of these two optimal portfolios are greater than $(\frac{S}{N})^2$, which is always positive.*

Proposition 6, (1), demonstrates that, in a precise manner, the sophisticated agent always chooses a more diversified optimal portfolio than the naive agent, and it sheds some lights on the underdiversification puzzle. Compared with the market portfolio, the naive agent's portfolio is always under-diversified according to the dispersion measure while the sophisticated agent has a more diversified portfolio.

It is well documented that underdiversification can happen due to model misspecification (Uppal and Wang, 2003; Easley and O'Hara, 2009), heterogeneous beliefs (Milton and Vorkink, 2008). For instance, Easley and O'Hara (2009) demonstrates limited participation happens at the presence of the marginal distribution ambiguity while assets are assumed to be independent. Uppal and Wang (2003) considers the ambiguity for both the joint distribution and the marginal distributions from a portfolio choice setting. Uppal and Wang (2003) shows that numerically, when the overall ambiguity about the joint distribution is high, a small difference in ambiguity for the marginal return distribution will result in an under-diversified portfolio. By contrast, our result shows that underdiversification can be generated endogenously from the dispersion of correlation uncertainty, even without ambiguity on any marginal distribution. In a particular case, the naive agent could even hold a limited participation portfolio when his ambiguity on the correlated structure is large enough.[23] Furthermore, we demonstrate a well-diversified portfolio is associated with a sophisticated agent given her sophistication on the correlated structure estimation.

We draw numerically the dispersion of the sophisticated agent's and the naive agent's optimal portfolio in Figure 8. Figure 8 is concerned with a full participation equilibrium in a heterogeneous setting. Clearly, the dispersion of the sophisticated agent's optimal portfolio, as drawn in the upper plot, is smaller than the corresponding dispersion of the naive agent's optimal portfolio in the lower plot.

To show further the application of the dispersion measure to the limited participation, we consider a limited participation equilibrium and compute the optimal portfolio's dispersion for each agent, in Table 3. As shown, the naive agent's dispersion is fairly close to one, which reflects to his extremely under-diversified portfolio. By contrast, the dispersion of the sophisticated agent's optimal portfolio is between 0.625 and 0.629, indicating a more diversified holding.

Proposition 6, (2) concerns with the portfolio risk, which states that the sophisticated agent is willing to take a riskier portfolio than the naive agent due to the dispersion of correlation uncertainty among agents. The intuition is simple. Since ambiguity-aversion leads to risk-aversion, the naive agent behaves more risk-averse; hence, he has a smaller risk in his optimal portfolio. Proposition 6, (2), demonstrates that ambiguity-aversion leads to risk-aversion in the correlated structure,[24] and a higher level of correlation uncertainty yields a higher level of risk-aversion; thus the corresponding optimal portfolio is less risky. The comparison of portfolio risk can be also derived by Proposition 6, (3) which gives the position on each risky asset. Actually, by the explicit expression of $x^{(s)}, x^{(n)}$ in Proposition 3 and Proposition 1, the sophisticated agent holds all long positions on all risky assets but the naive agents might hold short positions on some risky asset. Therefore, the sophisticated agent's portfolio risk is higher.

Proposition 6, (4)-(5) further compare the performance between these two optimal portfolios. As expected, the sophisticated agent has a higher Sharpe ratio portfolio than the

[23] See Mankiw and Zeldes (1991) and Vissing-Jorgensen (2003).

[24] It is well known that uncertainty aversion yields risk aversion in the ambiguity literature. See Easley and O'Hara (2008, 2012); Cao, Wang and Zhang (2005); Gollier (2011); Garlappi, Uppal and Wang (2012).

naive agent. By the same token, the sophisticated agent also has a higher maxmin expected utility.

At last, Proposition 6, (6), investigates the comovement between the agents from their own perspectives and we observe a robust comovement pattern among the agents (See Simsek, 2013 and Towensend, 1994 for a comovement among householders' optimal equity portfolios).

5 Sophisticated agents and risk dispersion

In this section we examine the implication of sophisticated agents and risk dispersion. ν represents the percentage of sophisticated agent or institutional investor. $\sigma_i \overline{x}_i$ represents the total risk of the firm to the market. It is different from the volatility σ_i or the market capitalization $p_i \overline{x}_i$ that are often used as proxies of stock characteristics. We use $\Omega(\sigma\overline{x})$ to represent the risk dispersion in the economy. We derive several testable hypothesis in this regard. To emphasize the impact of $\nu, \Omega(\sigma\overline{x})$ we highlight on these two variables as extra components in this section.

5.1 Sophisticated agents

We first discuss the effect to the equilibrium. For this purpose, we let

$$L(\beta_1) = \lim_{\nu \to 1} K(\beta_1, \nu) = \frac{1 - \Omega(\sigma\overline{x}) + \beta_1(\Omega(\sigma\overline{x}) + N - 1)}{1 + (N-1)\Omega(\sigma\overline{x}) + (N-1)\beta_1(1 - \Omega(\sigma\overline{x}))}$$

If both agents have fairly close correlation uncertainty that β_2 is smaller than $L(\beta_1)$, then $\beta_2 \leq K(\beta_1, \nu)$ regardless of the amount of sophisticated agents and therefore, by Proposition 3, there is a full participation equilibrium. The number of sophisticated agent has significant effect on equilibrium if the level of correlation uncertainty differs from each, say, $\beta_2 > L(\beta_1)$. For a small number of sophisticated agents, $\beta_2 \leq K(\beta_1, \nu)$. However, if there are so many sophisticated agents that $\beta_2 > K(\beta_1, \nu)$, a limited participation equilibrium prevails. In an extreme case, $\nu \to 1$, the limited participation equilibrium becomes the homogeneous equilibrium as in Proposition 2.

Proposition 7 *1. (Equilibrium) If $\beta_2 \leq L(\beta_1)$, there exists no limited participation. However, if $\beta_2 > L(\beta_1)$, there is only limited participation equilibrium as long as there are many sophisticated agents.*

2. (Price) If $\frac{\sigma_i \overline{x}_i}{L} \leq \frac{1}{N}$ for asset i, then $\frac{\partial}{\partial \nu}(s_i) < 0, \frac{\partial}{\partial \nu}(p_i) > 0$. The effect of ν on the price is ambiguous for asset i with $\frac{\sigma_i \overline{x}_i}{L} > \frac{1}{N}$.

3. (Volume) If $\frac{\sigma_i \overline{x}_i}{L} \leq \frac{1}{N}$ for asset i, then $x_i^{(s)} > x_i^{(n)}$, and $\frac{\partial}{\partial \nu} x_i^{(s)} < 0, \frac{\partial}{\partial \nu} x_i^{(n)} < 0$. On the other hand, if $\frac{\sigma_i \overline{x}_i}{L} \geq \frac{1}{N}$, then $\frac{\partial}{\partial \nu}\left(\nu x_i^{(s)}\right) > 0$.

Proposition 7, (1) follows directly from the above discussion. Proposition 7, (2) follows from the decomposition of the Sharpe ratio, Equation (A-31) and its intuition is similar to Proposition 5, (2). Proposition 7 further justifies the impact of institutional investors on the price from the demand perspective. The sophisticated agent has higher demand on the low risk assets than the naive agent, leading to a price increase of the low risk asset. It is also interesting to see that with increasing sophisticated agent, the total volume increases on the high risk asset. However, regarding the individual demand for each agent, the impact of ν on the high risk asset is not as clear as it is on the low risk asset. Furthermore, the demand of the sophisticated agent on the low risk asset depends on two opposite effects of the average Sharpe ratio and the specific Sharpe ratio; thus the effect of ν is ambiguous.

5.2 The risk dispersion $\Omega(\sigma\overline{x})$

The market risk distribution $\Omega(\sigma\overline{x})$ performs another vital role in the heterogeneous equilibrium and displays important asset pricing implications from several aspects.

First, the level of $\Omega(\sigma\overline{x})$ affects the market equilibrium. When each firm (risky assets) contributes the similar risk to the market, it generates a full participation equilibrium. In one extreme case, if each risky asset contributes the same risk, $\Omega(\sigma\overline{x}) = 0$, then $\Omega(s) = 0$. On the other hand, when the risky assets contribute very different contribution, or alternatively, they offers a skewed risk distribution and $\Omega(\sigma\overline{x})$ is close to one, a limited participation equilibrium is generated. To understand it, we observe that

$$\lim_{\Omega(\sigma\overline{x})\to 1} K(\beta_1, \Omega(\sigma\overline{x})) = \beta_1. \tag{22}$$

Therefore, any naive agent with $\beta_2 > \beta_1$ must hold a limited portfolio for a skewed enough risk distribution, that is, $\Omega(\sigma\overline{x})$ is close enough to one.

Our analysis regarding the risk dispersion demonstrates another channel for the limited participation phenomena, which can be generated due to a large risk dispersion. For instance, if the equity market and the bond market contributes a far different risk, naive agents or households (retail investors) want to optimally hold a limited participation portfolio. As another example, if one country (say, U.S. equity market) dominates other equity markets in an international setting, households want to focus on the investment on the U.S equity market and there is no much incentive to participate in the international market.

Second, the Sharpe ratios and the dispersion of Sharpe ratios (the market capacity) depend on the risk distribution. Equation (A-15) presents this fundamental relationship. In both the full participation and the limited participation equilibrium (by Equation A-32) , we have

$$\frac{\partial\Omega(s)}{\partial\Omega(\sigma\overline{x}_i)} > 0. \tag{23}$$

Due to this positive relation, the higher the risk dispersion the higher the market capacity.

6 Extension

So far, we assume that any two risky assets have the same, or very close, correlation coefficient. In this section, we justify this assumption first and then present its extensions.

We consider a two-period economy with N risky assets and one zero supply risk-free asset, and these N risky assets follow a one factor model such as $\tilde{a}_i = \beta_i \tilde{m} + \epsilon$ where m represents one fundamental factor and each ϵ_i is the specific risk for the asset i. The correlation coefficient between the asset i and asset j is

$$corr(\tilde{a}, \tilde{a}_j) = \frac{(\beta_i\sigma_m)(\beta_j\sigma_m)}{\sqrt{(\beta_i\sigma_m^2 + \sigma_i^2)(\beta_j\sigma_m^2 + \sigma_j^2).}}$$

Thus, this correlated structure is one special case of a correlated structure in which $corr(\tilde{a}_i, \tilde{a}_j) = b_i b_j, \forall i \neq j$ for a sequence of positive numbers $b_1, \cdots, b_N$. If in particular, $b_1 = \cdots = b_N$, the correlation matrix becomes $R(\rho)$ in Section 2. The general situation can be studied in a portfolio choice setting as in proposition 1.

As another example and without relying the factor-model, we decompose these risky assets are decomposed into K sectors based on their firm characterizes, business and so on. Therefore, the risky assets in each factor have relatively close correlation coefficient. Specifically, we group assets into several classes, $A_1, A_2, \cdots, A_k$, such that the piece-wise correlation coefficients among assets in each class A_i is very close to each other and the common correlation coefficient is denoted by ρ_i. We assume that $\rho_1 >> \rho_2 >> \cdots >> \rho_k$. [25] In this way, ρ_1 represents the largest possible correlation coefficient among all assets and any pair of assets in class A_1 has a correlation coefficient (or very close to) ρ_1. The correlation coefficient ρ_2 denotes the second largest possible correlation coefficient, to some contexts, among all assets in the market. For this reason we assume that ρ_1 dominates ρ_2, written as $\rho_1 >> \rho_2$.

Under the above hierarchical correlation structure, we are able to extend the model setting in Section 2 to each asset class $A_1, A_2, \cdots, A_k$ separably, since the piece-wise correlation coefficient in each asset class A_i is fairly close to each other. In fact, this result can be applied to any asset class in which the piece-wise correlation coefficient is close to each other, when the correlation uncertainty within this asset class is a concern. For instance, because of the too big to fail issue and the substantial systemic risk, it is essential to understand the correlation structure among these big financial institutions as a group and how it changes with respect to macro-economic shocks[26]. Therefore, to investigate the correlation uncertainty among those big financial institutions is important and our previous results can be applied virtually in this setting.

To demonstrate our extended setting, we consider two different correlated structures among asset classes in the hierarchical structure. In the first one, we assume that each asset

[25] We use $a >> b$ to represent that the number a is *way larger* than b. This decomposition of the correlation structure is similar to the principal component analysis in which the eigenvalues of the covariance matrix has a decreasing order in terms of "$>>$".

[26] For instance, for big financial institutions such as Bank of America, Citi, AIG, Wells Fargo, JP Morgan, our computation shows that any pair of two big financial institutions in this group stays between 70% to 80% from 2001 to 2014, including pre-crises and post-crisis time period.

class is independent to each other, or have extremely small correlation coefficients at the least. For simplicity we use two asset classes to illustrate, the correlation matrix can be written as

$$R_1 = \begin{bmatrix} R(\rho_1) & 0 \\ 0 & R(\rho_2) \end{bmatrix}.$$

with $\rho_1 >> \rho_2$ and each $R(\rho_i)$ is a correlation matrix with element ρ_i off the diagonal and the component being one along the diagonal.

For the second correlated structure, we consider two asset classes while the first asset class contains asset $i \in I$ and the second asset class contains all other assets $j \in J$. Given the hierarchical structure, we assume that for each pair $i_1 \neq i_2 \in I$, $corr(\tilde{a}_{i_1}, \tilde{a}_{i_2}) = \rho_1$, and for each pair $j_1 \neq j_2$ in J, $corr(\tilde{a}_{j_1}, \tilde{a}_{j_2}) = \rho_2$. We assume that $\rho_1 >> \rho_2$. Since the correlation between asset class I and asset class J is not substantial, we assume that $corr(\tilde{a}_i, \tilde{a}_j) = \rho_1\rho_2$ for $i \in I, j \in J$. Precisely, the correlation matrix is written as[27]

$$R^{II} = \begin{bmatrix} R(\rho_1) & \rho_1\rho_2 B \\ \rho_1\rho_2 B' & R(\rho_2) \end{bmatrix}.$$

with $\rho_1 >> \rho_2$ and $R(\rho_1)$ is a $M \times M$ matrix, $R(\rho_2)$ is a $J \times J$ matrix and B is a $M \times J$ matrix with components one. Since we are interested in the positive correlated environment, we assume that $\rho_i > 0$ for each $i = 1, \cdots, K$ in each correlated structure.

Proposition 8 *In a homogeneous environment the representative agent has a correlation uncertainty* $[\overline{\rho}_k - \epsilon_k, \overline{\rho}_k + \epsilon_k]$ *on the correlation coefficient* ρ_k *in the asset class* A_k *for* $k = 1, 2$. *We assume that the correlation uncertainty is also consistent with the correlated structure in the sense that* $\overline{\rho}_1 - \epsilon_1 > \overline{\rho}_2 + \epsilon_2$ *for the first hierarchical structure, and* $\overline{\rho}_1 - \epsilon_1 >> \overline{\rho}_2 + \epsilon_2$. *The agent chooses the highest possible correlation coefficients in equilibrium in the above two correlation structures with correlation uncertainty.*

According to Proposition 8, the representative agent chooses the highest plausible correlation coefficients in a relatively straightforward hierarchical correlated structure. Similar

[27] We can check that R_2 is a positive definite matrix largely, thus, R_2 is a correlation matrix for a multivariate Gaussian variable.

to our previous discussion in Section 1, $\overline{\rho}_k + \epsilon_k$ relies on both the benchmark and the level of correlation uncertainty for each asset class A_k. In an extremely weak market situation where the level of correlation uncertainty is high, the endogenous correlation coefficients between assets will be high, and the endogenous risk premiums will increase and prices will drop enormously. Therefore, our previous arguments can be applied to the general correlation structure.

The endogenous correlation pattern in hierarchical correlated structure is richer than the one exhibited in Section 6, attributable to the heterogeneous uncertainty levels among agents. The endogenous market correlation is time-varying, depending on the benchmark of the correlation coefficients in each asset class and the level of uncertainty. Besides, the endogenous correlation inside each asset class or between asset classes increase with the number of sophisticated agents.

7 Conclusion

To investigate the on-going complicated correlation structure among asset classes and the nature of well-documented empirical correlated-related facts, this paper develops an equilibrium model at the presence of correlation uncertainty where two types of agents have heterogeneous beliefs in their correlation estimation. We find that those correlation-related phenomena can be essentially connected through the disagreement among agents in the correlation structure, the dispersion of the agents and assets' risk, when the marginal distribution of each risky asset is a perfect knowledge.

Specifically, when the disagreement on correlation estimation is large, the naive agent does not participate in the market, thus a limited participation equilibrium. This is also true when more sophisticated agents emerge in the market or when the dispersion of assets' risk is high. The naive agent is irrelevant to choose his correlation coefficient even though his optimal portfolio is uniquely determined in equilibrium. Our portfolio analysis demonstrates that the sophisticated agent always holds a diversified portfolio in contrary to the naive agent who is underdiversified. Our equilibrium model is helpful to explain asymmetric correlation, correlation trading, comovement and diversification premium (discount). This paper contributes further to the literature on the asset pricing implication of Knightian uncertainty.

Appendix A: Proof

The following Sherman-Morrison formula in linear algebra is useful in the subsequent derivations.

Lemma 1 *Suppose A is an invertible $s \times s$ matrix and u, v are $s \times 1$ vectors. Suppose further that $1 + v^T A^{-1} u \neq 0$. Then the matrix $A + uv^T$ is invertible and*

$$(A + uv^T)^{-1} = A^{-1} - \frac{A^{-1} u v^T A^{-1}}{1 + v^T A^{-1} u}. \tag{A-1}$$

The next lemma computes the dispersion of optimal demand in terms of the dispersion of the Sharpe ratios.

Lemma 2 *Let $\rho \neq 1, \rho \neq -\frac{1}{N-1}$. Then*

$$\Omega(\sigma x_\rho) = \Omega(s) \frac{1 + (N-1)\rho}{1 - \rho}; \Omega(s) = \Omega(\sigma x_\rho) \frac{1 - \rho}{1 + (N-1)\rho} \tag{A-2}$$

Proof. Note that $\sigma x_\rho = \frac{1}{\gamma} R(\rho)^{-1} s$. By Lemma 1,

$$R(\rho)^{-1} = \frac{1}{1-\rho} I_N - \frac{1}{1 + (N-1)\rho} \frac{\rho}{1-\rho} ee^T. \tag{A-3}$$

Then $\Omega(\sigma x_\rho) = \Omega(t)$, where $t_i = s_i - \frac{\rho}{1+(N-1)\rho} S$. We obtain $\sum_{i=1}^N t_i = \frac{1-\rho}{1+(N-1)\rho} S$, and

$$\begin{aligned} \sum_{i=1}^N t_i^2 &= \sum_{i=1}^N \left(s_i^2 - 2 s_i S \frac{\rho}{1 + (N-1)S} + \left(\frac{\rho}{1 + (N-1)\rho} \right)^2 S^2 \right) \\ &= \sum_{i=1}^N s_i^2 - \frac{(N-2)\rho^2 + 2\rho}{(1 + (N-1)\rho)^2} S^2. \end{aligned}$$

By straightforward computation, we obtain

$$\Omega(t) = \Omega(s) \frac{1 + (N-1)\rho}{1 - \rho}. \tag{A-4}$$

$\square$

Proof of Proposition 1.

By Sion's theorem (1958),

$$
\begin{aligned}
A &= \max_{x\in\mathbb{R}^N} \min_{\rho\in[\alpha,\beta]} \left\{ (\overline{a}_i - p_i)x_i - \frac{1}{2}\sum_{i,j=1}^{N} x_i x_j \sigma_i \sigma_j R_{ij} \right\} \\
&= B \equiv \min_{\rho\in[\alpha,\beta]} \max_{x\in\mathbb{R}^N} \left\{ (\overline{a}_i - p_i)x_i - \frac{1}{2}\sum_{i,j=1}^{N} x_i x_j \sigma_i \sigma_j R_{ij} \right\}
\end{aligned}
$$

and both A and B are equal to $\min_{\rho\in[\alpha,\beta]} \frac{1}{2\gamma}G(\rho)$, where

$$
G(\rho) \equiv s^T R^{-1} s = \frac{N\sum_{n=1}^N s_n^2 - (\sum_{n=1}^N s_n)^2}{N(1-\rho)} + \frac{(\sum_{n=1}^N s_n)^2}{N(1+(N-1)\rho)}. \tag{A-5}
$$

$G(\rho)$ can be rewritten in term of the dispersion measure as follows.

$$
G(\rho) = \frac{S^2}{N}\left(\frac{(N-1)\Omega(s)^2}{1-\rho} + \frac{1}{1+(N-1)\rho} \right).
$$

Lemma 3 *Notations as above. The solution of* $\min_{\rho\in[\alpha,\beta]} G(\rho)$ *is given by*

$$
\rho^* = \begin{cases} \alpha, & \text{if } \alpha > \tau(\Omega(s)), \\ \beta, & \text{if } \beta < \tau(\Omega(s)), \\ \tau(\Omega(s)) & \text{if } \tau(\Omega(s)) \in [\alpha,\beta]. \end{cases} \tag{A-6}
$$

Proof of Lemma 3. For simplicity we assume that $\Omega(s) \neq \frac{1}{N-1}$ and let $\hat{\tau}(\Omega(s)) \equiv \frac{1+\Omega(s)}{1-(N-1)\Omega(s)}$. Then,

$$
G'(\rho) = \frac{(N-1)S^2}{N(1-\rho)^2[1+(N-1)\rho]^2}[\Omega(s)^2(N-1)^2 - 1][\rho - \tau(\Omega(s))][\rho - \hat{\tau}(\Omega(s))]. \tag{A-7}
$$

Thus, $\rho^* = argmin_{\rho\in[\alpha,\beta]}G(\rho)$ is determined in the following three cases, respectively.

- If $\Omega(s) \geq \frac{2}{N-2}$, then $|\tau(\Omega(s))| \leq 1, |\hat{\tau}(\Omega(s))| \leq 1$ and $\hat{\tau}(\Omega(s)) < \tau(\Omega(s))$. Moreover $\hat{\tau}(\Omega(s)) \leq 0 \leq \alpha$.

- If $\frac{2}{N-2} > \Omega(s) > \frac{1}{N-1}$, then $\hat{\tau}(\Omega(s)) < -1$, and $0 < \tau(\Omega(s)) < 1$.

- If $\Omega(s) < \frac{1}{N-1}$, then $|\tau(\Omega(s))| \leq 1, \hat{\tau}(\Omega(s)) > 1$; but $\Omega(s)^2(N-1) - 1 < 0$ so $G'(\rho) = (-) \times (\rho - \tau(\Omega(s))) \times (\rho - \hat{\tau}(\Omega(s)))$.

Then the proof of Lemma 3 is finished. □

Proof of Proposition 1, Continue.

By the above argument, we have shown that

$$A = B = \frac{1}{2\gamma} G(\rho^*) = CE(\rho^*, x_{\rho^*}). \tag{A-8}$$

(1). If $\alpha > \tau(\Omega(s))$, then by Lemma 2, $\Omega(\sigma x_\alpha) > 1$ and $\max_{\Omega(\sigma x)>1} CE(\alpha, x) = CE(\alpha, x_\alpha) = A$. By equation (A-8) and the property of $G(\rho)$ stated in Lemma 3, we know that $\max_{\Omega(\sigma x)<1} CE(\beta, x) \leq max_x CE(\beta, x) < CE(\alpha, x_\alpha)$. Similarly, $\max_{\Omega(\sigma x)=1} CE(\rho, x) < CE(\alpha, x_\alpha)$ for each $\rho \in [\alpha, \beta]$. Then the unique solution of the problem (2) is $\rho^* = \alpha$, and $x^* = x_\alpha$.

(2). If $\beta < \tau(\Omega)$, by Lemma 2, $\max_{\Omega(\sigma x)<1} CE(\beta, x) = CE(\beta, x_\beta) = A$. Moreover, by equation (A-8) and Lemma 3, we have $\max_{\Omega(\sigma x)>1} CE(\alpha, x) \leq CE(\alpha, x_\alpha) < CE(\beta, x_\beta)$ and $\max_{\Omega(\sigma x)=1} CE(\rho, x) < CE(\beta, x_\beta)$ for each $\rho \in [\alpha, \beta]$. Therefore, $\rho^* = \beta, x^* = x_\beta$ is the unique solution of the portfolio choice problem (2).

(3). Assume that $\alpha \leq \tau(\Omega) \leq \beta$. By equation (A-8), $A = B = CE\left(\tau(\Omega(s)), x_{\tau(\Omega(s))}\right)$. Moreover, by Lemma 2, $\Omega\left(x_{\tau(\Omega(s))}\right) = 1$. By straightforward calculation, we have

$$A = B = CE\left(\tau(\Omega(s)), x_{\tau(\Omega(s))}\right) = \frac{(\sum_i s_i)^2}{2\gamma} \left(\frac{1 + (N-1)\Omega(s)}{N}\right)^2.$$

For any x with $\Omega(\sigma x) < 1$, by Lemma 3 and the last equation, we have

$$CE(\beta, x) \leq CE(\beta, x_\beta) < A = CE\left(\tau(\Omega(s)), x_{\tau(\Omega(s))}\right).$$

By the same reason, for any x with $\Omega(\sigma x) > 1$, we see that $CE(\alpha, x) < CE\left(\tau(\Omega(s)), x_{\tau(\Omega(s))}\right)$. Finally, for each x^* with $\Omega(\sigma x^*) = 1$ and $CE(\tau(\Omega(s)), x^*) = \max_{\Omega(\sigma x)=1} CE(\tau(\Omega(s)), x^*)$, we have $CE(\tau(\Omega(s)), x^*) = CE(\tau(\Omega(s)), x_{\tau(\Omega(s))})$, and because of the uniqueness x_ρ for maximizing $CE(\rho, x)$, $x^* = x_{\tau(\Omega(s))}$. Therefore, we have shown that the unique demand for the uncertainty-averse agent is $x_{\tau(\Omega(s))}$, but the agent is irrelevant to choose any correlation coefficient $\rho \in [\alpha, \beta]$ since $CE(\rho, x_{\tau(\Omega(s))}) = A$ for each $\rho \in [\alpha, \beta]$.

The proof of Proposition 1 is completed. □

Proof of Proposition 2.

The optimal demand x^* is presented by Proposition 1. By the market clearing condition, $x^* = \overline{x}$ in equilibrium. Then $\Omega(\sigma x^*) = \Omega(\sigma \overline{x})$. Since $\Omega(\sigma \overline{x}) < 1$. Therefore, the optimal demand in equilibrium satisfies $\Omega(\sigma x^*) < 1$. Then, by Proposition 1 again, the optimal correlation coefficient is $\rho^* = \beta$, the highest possible correlation coefficient. □

In what follows we do not distinguish agent $j = 1, 2$ or $j = s, n$ for sophisticated agent and naive agent, respectively.

Proof of Proposition 3.

The unique optimal demand of type $j = 1, 2$ agent is $x_j = \frac{1}{\gamma}\sigma^{-1} R_j^{-1} s$ and R_j corresponds to an endogenous correlation coefficient $\rho_j \in [\alpha_j, \beta_j]$. Note that x_j is unique regardless of the optimal correlation coefficient in Proposition 1, (3) and in this case let $\rho = \tau(\Omega(s))$. In equilibrium, we have $\nu x_1 + (1-\nu)x_2 = \overline{x}$. Then

$$\frac{1}{\gamma}(\nu R_1^{-1} + (1-\nu) R_2^{-1}) \cdot s = \sigma \overline{x}. \tag{A-9}$$

The coefficient matrix of the last equation, $X = \nu R_1^{-1} + (1-\nu) R_2^{-1}$, is written as

$$X \equiv \left(\frac{\nu}{1-\rho_1} + \frac{1-\nu}{1-\rho_2}\right) I_N - \left(\frac{\nu \rho_1}{(1-\rho_1)(1+(N-1)\rho_1)} + \frac{(1-\nu)\rho_2}{(1-\rho_2)(1+(N-1)\rho_2)}\right) ee^T \tag{A-10}$$

Let $m \equiv \frac{\nu}{1-\rho_1} + \frac{1-\nu}{1-\rho_2}, n \equiv \frac{\nu\rho_1}{(1-\rho_1)(1+(N-1)\rho_1)} + \frac{(1-\nu)\rho_2}{(1-\rho_2)(1+(N-1)\rho_2)}$ and notice that $m - Nn = \frac{\nu}{1+(N-1)\rho_1} + \frac{1-\nu}{1+(N-1)\rho_2} > 0$. Then by Lemma 1, X is invertible and its inverse matrix is (with

$\kappa \equiv \frac{m-Nn}{m}$)

$$X^{-1} = \frac{1}{m}\left(I_N + \frac{n}{\kappa m} ee^T\right). \tag{A-11}$$

Therefore, $s = \gamma X^{-1} \cdot (\sigma x)$. Precisely,

$$s_i = \frac{\gamma}{m}\left(\overline{y}_i + \frac{n}{\kappa m} L\right), L \equiv \sum_{i=1}^{N} \sigma_i \overline{x}_i. \tag{A-12}$$

Hence

$$\sum s_i = \frac{\gamma}{m}\left(1 + \frac{Nn}{\kappa m}\right) L = \frac{\gamma L}{m - nN} \tag{A-13}$$

and

$$\begin{aligned} \sum \left(\frac{s_i}{\gamma}\right)^2 &= \frac{C^2}{m^2} + \frac{L^2}{m^2}\left(\frac{Nn^2}{\kappa^2 m^2} + 2\frac{n}{\kappa m}\right) \\ &= \frac{C^2}{m^2} + \frac{L^2}{m^2}\left(\frac{2n}{m - Nn} + \frac{Nn^2}{(m - Nn)^2}\right) \\ &= \frac{C^2}{m^2} + \frac{L^2}{m^2}\frac{2mn - Nn^2}{(m - Nn)^2} \end{aligned}$$

where $C^2 = \sum_{i=1}^{N} (\sigma_i \overline{x}_i)^2$. Then

$$\begin{aligned} \frac{\sum s_i^2}{(\sum s_i)^2} &= \frac{C^2}{L^2}\frac{(m - Nn)^2}{m^2} + \frac{2mn - Nn^2}{m^2} \\ &= \frac{C^2}{L^2}\kappa^2 + \frac{1}{N}(1 - \kappa^2). \end{aligned}$$

Therefore,

$$\Omega(s)^2 = \frac{1}{N-1}\left(N\frac{\sum s_i^2}{(\sum s_i)^2} - 1\right) = \kappa^2 \Omega(\sigma \overline{x})^2, \tag{A-14}$$

and then a fundamental relationship between the dispersion of Sharpe ratios and the dispersion of risks as follows

$$\Omega(s) = \kappa \Omega(\sigma \overline{x}). \tag{A-15}$$

Assume first that $\Omega(\sigma\overline{x}) = 0$, then each $\sigma_i\overline{x}_i = c$ and Equation (A-15) ensures that $\Omega(s) = 0$ and $\tau(\Omega(s)) = 1$. Therefore, each agent chooses her highest correlation coefficient in equilibrium by Proposition 1. Moreover, all Sharpe ratios are the same and equal to

$$s_i = \frac{c}{m}\left(1 + \frac{nN}{m - nN}\right) = \frac{c}{m - nN}. \tag{A-16}$$

We next assume that $\Omega(\sigma\overline{x}) \in (0, 1)$ and characterize the equilibrium in general. By using Proposition 1, there are five different cases regarding the equilibrium.

Case 1: $\tau(\Omega(s)) \leq \alpha_2$. Each agent chooses the smallest possible correlation coefficient, respectively.

Case 2: $\alpha_2 < \tau(\Omega(s)) < \alpha_1$. The sophisticated agent chooses the smallest possible correlation coefficient; the naive agent is irrelevant of the correlation coefficient and his optimal holding is $x_{\tau(\Omega(s))}$.

Case 3: $\tau(\Omega(s)) \in [\alpha_1, \beta_1]$. Each agent is irrelevant of the correlation coefficient and the corresponding optimal holding is $x_{\tau(\Omega(s))}$.

Case 4: $\beta_1 < \tau(\Omega(s)) \leq \beta_2$. The sophisticated agent chooses the highest possible correlation coefficient; the naive agent is irrelevant of the correlation coefficient and his optimal holding is $x_{\tau(\Omega(s))}$.

Case 5: $\beta_2 < \tau(\Omega(s))$, each agent chooses her highest possible correlation coefficient.

We prove next that Case 1 - Case 3 is impossible in equilibrium and characterize the equilibrium condition in both Case 4 and Case 5.

To proceed we state a simple lemma below and investigate each case respectively. Recall

$$\kappa = \kappa(\rho_1, \rho_2) \equiv \frac{\frac{\nu}{1+(N-1)\rho_1} + \frac{1-\nu}{1+(N-1)\rho_2}}{\frac{\nu}{1-\rho_1} + \frac{1-\nu}{1-\rho_2}}. \tag{A-17}$$

Lemma 4 *Assume that* $\kappa = \frac{\nu a+(1-\nu)b}{\nu c+(1-\nu)d}$ *with* $a, b, c, d > 0$ *and* $\nu \in (0, 1)$. *Then*

$$\min\left\{\frac{a}{c}, \frac{b}{d}\right\} \leq \kappa \leq \max\left\{\frac{a}{c}, \frac{b}{d}\right\}. \tag{A-18}$$

The inequalities are strictly if $\frac{a}{c} \neq \frac{b}{d}$.

Case 1. We show that this situation is impossible in equilibrium. Since $\tau(\Omega(s)) \leq \alpha_2 < \alpha_1$ ensures that $\Omega(s) \geq \tau(\alpha_2) > \tau(\alpha_1)$. Then, by Equation (A-15),

$$\kappa = \kappa(\alpha_1, \alpha_2) = \frac{\Omega(s)}{\Omega(\sigma\overline{x})} > \Omega(s) > \tau(\alpha_1), \tau(\alpha_2), \tag{A-19}$$

which is impossible by Lemma 4.

Case 2. We prove that it is impossible in equilibrium in this situation that $\alpha_2 < \tau(\Omega(s)) \leq \alpha_1$.

In this case, $\tau(\alpha_2) > \Omega(s) \geq \tau(\alpha_1)$. But

$$\kappa = \kappa(\alpha_1, \tau(\Omega(s))) > \kappa\Omega(\sigma\overline{x}) = \Omega(s) \geq \frac{1-\alpha_1}{1+\alpha_1(N-1)}, \kappa > \Omega(s) = \frac{1-\tau(\Omega(s))}{1+(N-1)\tau(\Omega(s))}. \tag{A-20}$$

Hence, it is impossible by using Lemma 4 again.

Case 3. We prove that it is impossible that $\tau(\Omega(s)) \in [\alpha_1, \beta_1]$ in equilibrium.

Otherwise, the optimal holding of each agent is $x_{\tau(\Omega(s))}$ by Proposition 1. Then the market clearing condition yields that $x_{\tau(\Omega(s))} = \overline{x}$. However, by Lemma 2, $\Omega(\sigma x_{\tau(\Omega(s))}) = 1$ but $\Omega(\sigma\overline{x}) < 1$. Therefore, Case 3 is not possible in equilibrium.

Case 4. We characterize the equilibrium in which $\beta_1 < \tau(\Omega(s)) \leq \beta_2$. In equilibrium, $\rho_1 = \beta_1$, the optimal holding for the naive agent is $x_{\tau(\Omega(s))}$, and

$$\tau(\beta_2) \leq \Omega(s) < \tau(\beta_1). \tag{A-21}$$

By definition of κ, we have

$$\kappa = \kappa(\beta_1, \tau(\Omega(s))) = \frac{\Omega(s)}{\Omega(\sigma\overline{x})} = \frac{\frac{\nu}{1+(N-1)\beta_1} + \frac{1-\nu}{N}(1+(N-1)\Omega(s))}{\frac{\nu}{1-\beta_1} + \frac{1-\nu}{\Omega(s)N}(1+(N-1)\Omega(s))}. \tag{A-22}$$

By solving the last equation in $\Omega(s)$, we obtain

$$\Omega(s) = \frac{\frac{\nu}{1+(N-1)\beta_1}\Omega(\sigma\overline{x}) - \frac{1-\nu}{N}(1-\Omega(\sigma\overline{x}))}{\frac{\nu}{1-\beta_1} + \frac{(1-\nu)(N-1)}{N}(1-\Omega(\sigma\overline{x}))}. \tag{A-23}$$

In particular, $\Omega(s) \geq 0$ ensures that

$$\beta_1 \leq \frac{1}{N-1}\left\{\frac{\nu}{1-\nu}\frac{\Omega(\sigma\overline{x})}{1-\Omega(\sigma\overline{x})}N - 1\right\}. \tag{A-24}$$

Moreover, the left side of Equation (A-21) is translated as $\beta_2 \geq \tau(\Omega(s)) = K(\beta_1)$, and the right side of Equation (A-21) is $K(\beta_1) \geq \beta_1$ which holds always. Then there exists a unique equilibrium in Case 4, a limited participation equilibrium, under conditions presented in Proposition 3.

Case 5. We characterize the equilibrium in which $\tau(\Omega(s)) > \beta_2$.

In equilibrium, $\rho_1 = \beta_1, \rho_2 = \beta_2$, and $\Omega(s) < \tau(\beta_2) < \tau(\beta_1)$, which is equivalent to $\kappa(\beta_1, \beta_2)\Omega(s) < \frac{1-\beta_2}{1+(N-1)\beta_2}$. By straightforward computation, this condition equals to

$$\left\{\frac{\nu}{1-\beta_1} + \frac{\Omega(\sigma\overline{x})\nu(N-1)}{1+(N-1)\beta_1}\right\}\beta_2 < \frac{\nu}{1-\beta_1} - \frac{\Omega(\sigma\overline{x})\nu}{1+(N-1)\beta_1} + (1-\nu)(1-\Omega(\sigma\overline{x})) \tag{A-25}$$

or equivalently, $\beta_2 < K(\beta_1)$.

To the end, we note that when β_1 is large enough such that

$$\frac{\Omega(\sigma\overline{x})\nu N}{(1-\nu)(1-\Omega(\sigma\overline{x}))} \leq 1 + (N-1)\beta_1, \tag{A-26}$$

or equivalently,

$$\beta_1 \geq \frac{1}{N-1}\left\{\frac{\nu}{1-\nu}\frac{\Omega(\sigma\overline{x})}{1-\Omega(\sigma\overline{x})}N - 1\right\},$$

then $\beta_2 < K(\beta_1)$ holds by natural since $\beta_2 < 1$. Then we have characterized the equilibrium in Proposition 3.

Finally, by using this characterization, we see that each $s_i > 0$, thus, each risky asset is priced at discount in equilibrium. $\square$

Proof of Proposition 4.

We first solve the portfolio choice problem under correlation uncertainty in a conglomerate economy. which is virtually one economy with one risky asset in the presence of variance ambiguity. It is easy to see that the optimal variance must be the largest possible variance to hedge the ambiguity concern, which in turn leads to the highest possible correlation coefficient. Therefore, the optimal demand on the conglomerate firm is

$$x_M^* = \frac{\overline{a} - p_M}{\sum_{i=1}^N \sigma_i^2 + 2\sum_{i\neq j}\sigma_i\sigma_j\beta}. \tag{A-27}$$

(1) We first consider the homogeneous environment, $x_M^* = \overline{x}$. Then it follows from Equation (A-27) that

$$p_M = \overline{a} - \gamma\overline{x}\left\{\sum_{i=1}^N \sigma_i^2 + 2\sum_{i\neq j}\sigma_i\sigma_j\beta\right\}.$$

By using Proposition 2, we see that $P_M = \sum_{i=1}^N P_i$ holds,thus neither diversification discount nor diversification premium.

(2) We consider the heterogeneous environment, the optimal demand $x_{i,M}^*$ for each agent $i = 1, 2$ satisfies

$$\overline{a} - p_M = \gamma x_{i,M}^*\left\{\sum_{i=1}^N \sigma_i^2 + 2\sum_{i\neq j}\sigma_i\sigma_j\beta_j\right\}$$

since each agent chooses the corresponding highest correlation coefficient. Since $\nu x_{1,M}^* + (1-\nu)x_{2,M}^* = \overline{x}$, we obtain, for $m = m(\beta_1, \beta_2), n = n(\beta_1, \beta_2)$,

$$\overline{a} - p_M = \frac{\gamma x}{m}\left\{m\sum_{i=1}^N \sigma_i^2 + 2\sum_{i\neq j}\sigma_i\sigma_j\right\}. \tag{A-28}$$

On the other hand, by Proposition 3, each stock's market price, if traded separably, is

$$p_i = \overline{a}_i - \frac{\gamma\overline{x}}{m}\left\{\sigma_i^2 + \sigma_i\frac{n}{m - Nn}\sum_{j=1}^N \sigma_j\right\}.$$

Then,

$$\sum_{i=1}^N p_i = \sum_{i=1}^N \overline{a}_i - \frac{\gamma\overline{x}}{m}\left\{\sum_{i=1}^N \sigma_i^2 + \frac{n}{m - Nn}(\sum_{i=1}^N \sigma_i)^2\right\}.$$

Then,

$$\overline{a} - \sum_{i=1}^{N} p_i = \frac{\gamma x}{m} \left\{ \sum_{i=1}^{N} \sigma_i^2 + \frac{n}{m - Nn} (\sum_{i=1}^{N} \sigma_i)^2 \right\}. \quad \text{(A-29)}$$

Therefore, $p_M > \sum_{i=1}^{N} p_i$ if, and only if

$$(\sum_{i=1}^{N} \sigma_i)^2 [(N+1)n - m] > \sum_{i=1}^{N} \sigma_i^2 (m-2)(m-nN).$$

In particular, if each σ_i is the same, the last formula is equivalent to

$$N[(N+1)n - m] > (m-2)(m-Nn)$$

which obviously holds when N is large enough. The proof of Proposition 4 is completed. □

Proof of Proposition 5.

The first part of (1) follows from the characterization of the Sharpe ratio in Proposition 3. The statement for the sophisticated agent follows from the expression of $\sigma_i x_i^{(s)}$:

$$\gamma \sigma_i x_i^{(s)} = \frac{1}{1-\beta_1} \left\{ s_i - \frac{\beta_1}{1 + (N-1)\beta_1} S \right\}. \quad \text{(A-30)}$$

Then, $\sigma_i x_i^{(s)} \geq \sigma_j x_j^{(s)}$ if and only if $s_i \geq s_j$. The proof for the naive agent is the same.

As explained in the text, both (2) and (3) follows from the following decomposition of the Sharpe ratio

$$s_i = \frac{S}{N} + \frac{\gamma}{m} \left(\sigma_i \overline{x}_i - \frac{L}{N} \right). \quad \text{(A-31)}$$

For (4), according to Equation (A-15), $\Omega(s) = \kappa \Omega(\sigma \overline{x})$. In the limited participation equilibrium, we have

$$\Omega(s) = \frac{\frac{\Omega(\sigma\overline{x})\nu N}{1+(N-1)\beta_1} - (1-\nu)(1-\Omega(\sigma\overline{x}))}{\frac{\nu N}{1-\beta_1} + (1-\nu)(1-\Omega(\sigma\overline{x}))(N-1)}. \quad \text{(A-32)}$$

Clearly, $\frac{\partial \Omega(\sigma\overline{x})}{\partial \beta_1} < 0$ and $\frac{\partial \Omega(\sigma\overline{x})}{\partial \nu} > 0$. The proof for the full participation equilibrium follows from the fact that $\frac{\kappa(\rho_1,\rho_2)}{\partial \rho_1} < 0$, $\frac{\kappa(\rho_1,\rho_2)}{\partial \rho_2} < 0$, and $\frac{\kappa(\rho_1,\rho_2)}{\partial \nu} > 0$.

(3). We have

$$\frac{\partial}{\partial \beta}\left(\frac{s_i}{S}\right) = \frac{\partial}{\partial \beta}\left(\frac{n}{m}\right)\left(1 - N\frac{\sigma_i \overline{x}_i}{L}\right). \tag{A-33}$$

By formula (A-15), $\frac{\partial}{\partial\beta}\left(\frac{n}{m}\right) = -\frac{\Omega(s)}{N}\frac{\partial\Omega(s)}{\partial\beta}$. By Proposition 5, (4), $\frac{\partial\Omega(s)}{\partial\beta} < 0$ in equilibrium and equilibrium B where β is the corresponding endogenous correlation coefficient(s). Therefore, $\frac{\partial}{\partial\beta}\left(\frac{n}{m}\right) > 0$. By the same argument we can show that $\frac{\partial\Omega(s)}{\partial\nu} > 0$. The proof of Proposition 5 is finished. □

Proof of Proposition 6.

(1). We compare the agents' optimal portfolios with the market portfolio in term of the market capacity $\Omega(s)$. First, Equation (A-15) states that $\Omega(s) = \kappa\Omega(\sigma\overline{x})$. In the full participation equilibrium, Lemma 2 implies that $\Omega(s) = \Omega(\sigma x^{(j)})\frac{1-\rho_j}{1+(N-1)\rho_j}$. Then, by Lemma 4 we have

$$\Omega(\sigma x^{(1)}) < \Omega(\sigma\overline{x}) < \Omega(\sigma x^{(2)}). \tag{A-34}$$

The proof in the limited participation equilibrium is the same. Since $\rho_1 \leq \tau(\Omega(s))$, Lemma 2 and Lemma 4 together imply that $\Omega(\sigma x^{(1)}) < \Omega(\sigma\overline{x})$. Moreover $\Omega(\sigma\overline{x}) < 1 = \Omega(\sigma x^{(2)})$.

(2). Let $X^{(1)} = \sum_i \tilde{a}_i x_i^{(1)}$ be the optimal portfolio of the sophisticated agent and $X^{(2)}$ is the optimal portfolio of the naive agent. We have $Var(X^{(1)}) = \frac{1}{\gamma^2}s'R(\rho_1)^{-1}s$ where the correlation coefficient is ρ_1, and $Var(X^{(2)}) = s'R(\rho_2)^{-1}s$ with the correlation coefficient ρ_2. By using the same notation as above, we have

$$Var(X^{(1)}) - Var(X^{(2)}) = \frac{1}{\gamma^2}\left\{G(\rho_1) - G(\rho_2)\right\}. \tag{A-35}$$

Applying Proposition 3, since the portfolio choice is under the worst-case correlation uncertainty, $G(\rho_1) > G(\rho_2)$ holds. Therefore, $Var(X^{(1)}) > Var(X^{(2)})$.

(3). For the sophisticated agent, it suffices to show that

$$s_i > \frac{\beta_1}{1+(N-1)\beta_1}S. \tag{A-36}$$

In fact, we have,

$$s_i = \frac{\gamma}{m}\sigma_i\overline{x}_i + S\frac{n}{m} > S\frac{n}{m}$$

and Lemma 4 implies that $\frac{n}{m} > \frac{\beta_1}{1+(N-1)\beta_1}$. We can easily formula a lower bound for s_i/S or σ_i/L for which $x_i^{(n)} > 0$.

(4). $\mathbb{E}[X^{(1)}] = \sum_{i=1}^N x_i^{(1)}(\overline{a}_i - p_i) = (\sigma x^{(1)})' s = \frac{1}{\gamma}s' R(\rho_1)s$. By (2), the variance of $X^{(1)}$ is $\frac{1}{\gamma^2}s'R(\rho_1)^{-1}s$. Then the Sharpe ratio of the portfolio $X^{(1)}$ is

$$SR(X^{(1)}) = \sqrt{s'R(\rho_1)^{-1}s}.$$

Therefore, $SR(X^{(s)}) > SR(X^{(n)})$ follows from $G(\rho_1) > G(\rho_2)$.

(5). By the proof of Proposition 6, the sophisticated agent's maxmin expected utility is $CH(\rho_1, x_{\rho_1}) = \frac{1}{2}G(\rho_1)$. Again, the fact that $G(\rho_1) > G(\rho_2)$ ensures that the sophisticated agent has a higher maxmin expected utility than the naive agent.

(6). For agent 1, the covariance between two portfolios $X^{(1)}$ and $X^{(2)}$ can be calculated as $(\sigma x^{(1)})' R(\rho_1)(\sigma x^2)'$, which is the same as $s'R(\rho_2)^{-1}s$. Similarly, the covariance of agent 2 is $s'R(\rho_1)^{-1}s$. By the proof in Proposition 1, both covariances are bounded from below by $s'R(\tau(\Omega(s)))^{-1}s$. It suffices to show that $s'R(\tau(\Omega(s)))^{-1}s > 0$. By simple calculation, we have

$$s'R(\tau(\Omega(s)))^{-1}s = S^2\left(\frac{1+(N-1)\Omega(s)}{N}\right)^2 \tag{A-37}$$

which is always greater than $\left(\frac{S}{N}\right)^2$. When $N = 2$, it reduces to $m^2 < 2n(m+1)$. The proof of Proposition 6 is completed. □

Proof of Proposition 7.

The first part, Proposition 7, (1) follows from the discussion in the text.

(2). By Equation (A-31) and using the fact that $\frac{\partial}{\partial\nu}\left(\frac{S}{N}\right) < 0$ and $\frac{\partial}{\partial\nu}(m) < 0$.

(3). If $\frac{\sigma \overline{x}_i}{L} \leq \frac{1}{N}$, then by Proposition 5, (1), $s_i \leq S/N$. By the expression of $x_i^{(s)}$ and $x_i^{(n)}$, it suffices to show that

$$\frac{1}{1-\beta_1}\left(s_i - \frac{\beta_1}{1+(N-1)\beta_1}S\right) > \frac{1}{1-\rho_2}\left(s_i - \frac{\rho_2}{1+(N-1)\rho_2}S\right).$$

The last equation is equivalent to

$$\frac{s_i}{S}\frac{\rho_2-\beta_1}{(1-\beta_1)(1-\rho_2)} < \frac{\rho_2}{(1-\rho_2)(1+(N-1)\rho_2)} - \frac{\beta_1}{(1-\beta_1)(1+(N-1)\beta_1)}$$

which follows from the fact that (for any $\beta_1 < \rho_2$)

$$\frac{1}{N}\frac{\rho_2-\beta_1}{(1-\beta_1)(1-\rho_2)} < \frac{\rho_2}{(1-\rho_2)(1+(N-1)\rho_2)} - \frac{\beta_1}{(1-\beta_1)(1+(N-1)\beta_1)}.$$

At last, we write the following decomposition

$$\gamma\sigma_i x_i^{(s)} = \frac{1}{1-\beta_1}\left(s_i - \frac{S}{N} + \frac{1-\beta_1}{N[1+(N-1)\beta_1]}S\right).$$

Then we have $\frac{\partial}{\partial\nu}x_i^{(s)} < 0$ as long as $\frac{\sigma_i \overline{x}_i}{L} \leq 1/N$. By the same proof, we have $\frac{\partial}{\partial\nu}x_i^{(n)} < 0$.

In the end, by using the last equation and Equation (A-31), we have

$$\begin{aligned}\gamma\sigma_i(\nu x_i^s) &= \frac{\gamma}{1-\beta_1}\left(\sigma_i\overline{x}_i - \frac{L}{N}\right)\frac{\nu}{m} \\ &\quad + \frac{1}{1+(N-1)\beta_1}(\nu S).\end{aligned}$$

It is straightforward to check that $\frac{\partial}{\partial\nu}(\nu S) > 0, \frac{\partial}{\partial\nu}\left(\frac{\nu}{m}\right) > 0$. Therefore, $\sigma_i\overline{x}_i \geq \frac{L}{N}$ yields $\frac{\partial}{\partial\nu}(\nu x_i^{(s)}) > 0$.

$\square$

Proof of Proposition 8.

For the first correlated structure with K classes, the inverse matrix of the correlation matrix is

$$(R_1)^{-1} = \begin{bmatrix} R(\rho_1)^{-1} & \cdot & \cdots & 0 \\ \cdot & R(\rho_2)^{-1} & \cdots & 0 \\ \cdot & \cdots & \cdots & \cdot \\ 0 & \cdot & \cdots & R(\rho_K)^{-1} \end{bmatrix}.$$

Moreover, from the portfolio choice perspective, the optimal correlation coefficient is, for each $k = 1, \cdots, K$, determined by Proposition 1.

For the second correlated structure, it is straightforward to derive (for $J = 1$)

$$(R_2)^{-1} = \begin{bmatrix} X & -\rho_2 X e \\ -\rho_2 e' X & 1 + \rho_2^2 e' X e \end{bmatrix}.$$

where X is the inverse matrix of the $M \times M$ matrix $R(\rho_1) - \rho_2^2 e e'$. By using the Sherman-Morrison formula, we have

$$X = \frac{1}{1-\rho_1}\left(I_M - \frac{\rho_1 - \rho_2^2}{1 + (M-1)\rho_1 - M\rho_2^2} e e'\right). \tag{A-38}$$

Therefore, the diversification benefits, $G(\rho_1, \rho_2) \equiv (s_1, \cdots, s_M, s_{M+1}) \cdot (R^{II})^{-1} \cdot (s_1, \cdots, s_M, s_{M+1})'$, with the correlation coefficient ρ_1 and ρ_2 in the second correlated structure, can be written as

$$G(\rho_1, \rho_2) = s' X s + 2\alpha t \sum_{i=1}^{M} s_i + t^2 \beta \tag{A-39}$$

where

$$s = (s_1, \cdots, s_M)', t = s_{M+1}, \alpha = -\frac{\rho_2}{1 + (M-1)\rho_1 - M\rho_2^2}, \beta = 1 + \frac{M\rho_2^2}{1 + (M-1)\rho_1}.$$

By simple algebra, we have

$$\frac{\partial G(\rho_1, \rho_2)}{\partial \rho_2} = \frac{2M\rho_2}{1 + (M-1)\rho_1} t^2 - \frac{2(1 + (M-1)\rho_1 + M\rho_2^2}{(1 + (M-1)\rho_1 - M\rho_2^2)^2} t \sum_{i=1}^{M} s_i. \tag{A-40}$$

Therefore, for a region of ρ_2 with small values compared with ρ_1, $G(\rho_1, \rho_2)$ is decreasing with respect to ρ_2. For the discussion on ρ_1, the proof is similar to the proof in Proposition

2. Therefore, the representative agent always chooses the highest possible correlations $\overline{\rho}_1 + \epsilon_1, \overline{\rho}_2 + \epsilon_2$ in equilibrium. $\square$

Appendix B: A Dispersion Measure

Given a time series of economic variables $X_1(t), \cdots, X_N(t), t = 1, 2, \cdots$, there are many approaches to estimate how these economic variables are correlated, interconnected and co-dependent. The literature is largely concerned with common factors in these economic variables. On the other hand, if these economic variables are fairly dispersed, it is not likely to have significant common factor. Therefore, a dispersion measure to some extent also measures the correlated information as we explain below.

A function $f : (X_1, \cdots, X_N) \in \mathbb{R}^N \to [0, 1]$ is a dispersion measure if it satisfies the following three properties:

1. (Positively homogeneous property) Given any $\lambda > 0$, $f(\lambda X_1, \cdots, \lambda X_N) = f(X_1, \cdots, X_N)$;
2. (Symmetric property) Given any translation $\sigma : \{1, \cdots, N\} \to \{1, \cdots, N\}$, $f(X_1, \cdots, X_N) = f(X_{\sigma(1)}, \cdots, X_{\sigma(N)})$;
3. (Majorization property) Assuming $(X_1, \cdots, X_N)$ weakly dominates $(Y_1, \cdots, Y_N)$, then $f(X_1, \cdots, X_N) \geq g(X_1, \cdots, X_N)$.

By a vector $a = (a_1, \cdots, a_N)$ weakly dominates $b = (b_1, \cdots, b_N)$ we mean that

$$\sum_{i=1}^{k} a_i^* \geq \sum_{i=1}^{k} b_i^*, k = 1, \cdots, N \tag{B-1}$$

where a_i^* is the element of a stored in decreasing order. When $f(Y) \leq f(X)$ for a dispersion measure we call Y is more dispersed than X under the measure f. One example is the portfolio weight in a financial market so the dispersion measure captures how one portfolio is more dispersed than another. Samuelson's famous theorem states that equally-weighted portfolio is always the optimal one for a risk-averse agent when the risky assets have IID return. Boyle et al (2012) also find that equally-weight portfolio beats many optimal asset allocation under parameter uncertainty. As another example, the dispersion measure can be used to investigate the diversification of optimal portfolio in a general setting. See Ibragimov, Jaffee and Walden (2011) and Hennessy and Lapan (2003).

This paper concerns with one example of dispersion measure.

Lemma 5 *Given a non-zero vector $X = (X_1, \cdots, X_N)$,*

$$f(X_1, \cdots, X_N) = \sqrt{\frac{1}{N-1}\left(N\frac{\sum X_i^2}{(\sum_i X_i)^2} - 1\right)}$$

is a dispersion measure.

Proof: Both the positive homogeneous property and the symmetric property are obviously satisfied. To prove the majorization property, we first assume that, because of the positively homogeneous property, $\sum X_i = \sum Y_i = 1$. Let $g(x_1, \cdots, x_N) = N(\sum_i x_i^2) - (\sum_i x_i)^2$. Notice that $g(\cdot)$ is a Schur convex function in the sense that

$$(x_i - x_j)\left(\frac{\partial g}{\partial x_i} - \frac{\partial g}{\partial x_j}\right) \geq 0, \forall i, j = 1, \cdots, N.$$

Then by the majorization theorem, see Marshall and Olkin (1979), $g(X) \geq g(Y)$. Then $f(X) \geq f(Y)$. $\square$

To illustrate the relation between the dispersion measure with comovement measures, we consider a CAPM model with assets $i = 1, \cdots, N$, $\mu_i - r_f = \beta_i(\mu_M - r_f)$, then

$$\frac{\mu_i - r_f}{\sigma_i} = \frac{\beta_i}{\sigma_i}(\mu_M - r_f).$$

Therefore, the dispersion measure of Sharpe ratios $s_1, \cdots, s_N$ is the same dispersion measure of risk-adjusted betas, $\frac{\beta_1}{\sigma_1}, \cdots, \frac{\beta_N}{\sigma_N}$, while the traditional approach is to compare those betas, $\beta_1, \cdots, \beta_N$. $\square$

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Figure 1: $K(\beta)$ with respect to β

This graph displays the property of the correlation boundary, $K(\beta)$, over the region $0 \leq \beta \leq \frac{1}{N-1}\left\{\frac{\nu}{1-\nu}\frac{\Omega(\sigma\overline{x})}{1-\Omega(\sigma\overline{x})}N-1\right\}$, As shown, $K(\beta)$ is always strictly larger than β, and $K(\beta)$ increases when β increases. The parameters are $N=3$, $\nu=0.6$, $\bar{a}_1=50$, $\sigma_1=9\%$, $\bar{x}_1=5$; $\bar{a}_2=60$, $\sigma_2=10\%$, $\bar{x}_2=3.7$; $\bar{a}_3=15$, $\sigma_3=12\%$, $\bar{x}_3=10.5$. Notice that $\Omega(\sigma\overline{x})=0.41$ and $\frac{1}{N-1}\left\{\frac{\nu}{1-\nu}\frac{\Omega(\sigma\overline{x})}{1-\Omega(\sigma\overline{x})}N-1\right\}=1.06$.

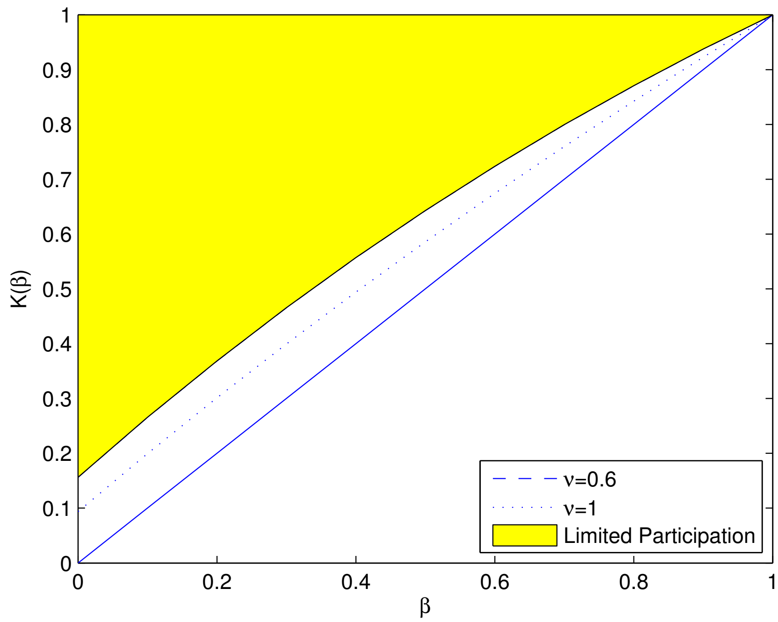

Figure 2: Sharpe ratios in a full participation equilibrium

This three dimensional figure shows the sensitivity of Sharpe ratios with respect to the highest plausible correlation coefficient $\beta 1$ and $\beta 2$ in a full participation equilibrium. The parameters are $N = 3$, $\nu = 0.3$, $\sigma_1 = 9\%$, $\bar{x}_1 = 1$; $\sigma_2 = 10\%$, $\bar{x}_2 = 5$; $\sigma_3 = 12\%$, $\bar{x}_3 = 10.5$. Notice that $\Omega(\sigma\overline{x}) = 0.56$ and $\frac{1}{N-1}\left\{\frac{\nu}{1-\nu}\frac{\Omega(\sigma\overline{x})}{1-\Omega(\sigma\overline{x})}N - 1\right\} = 0.3$.

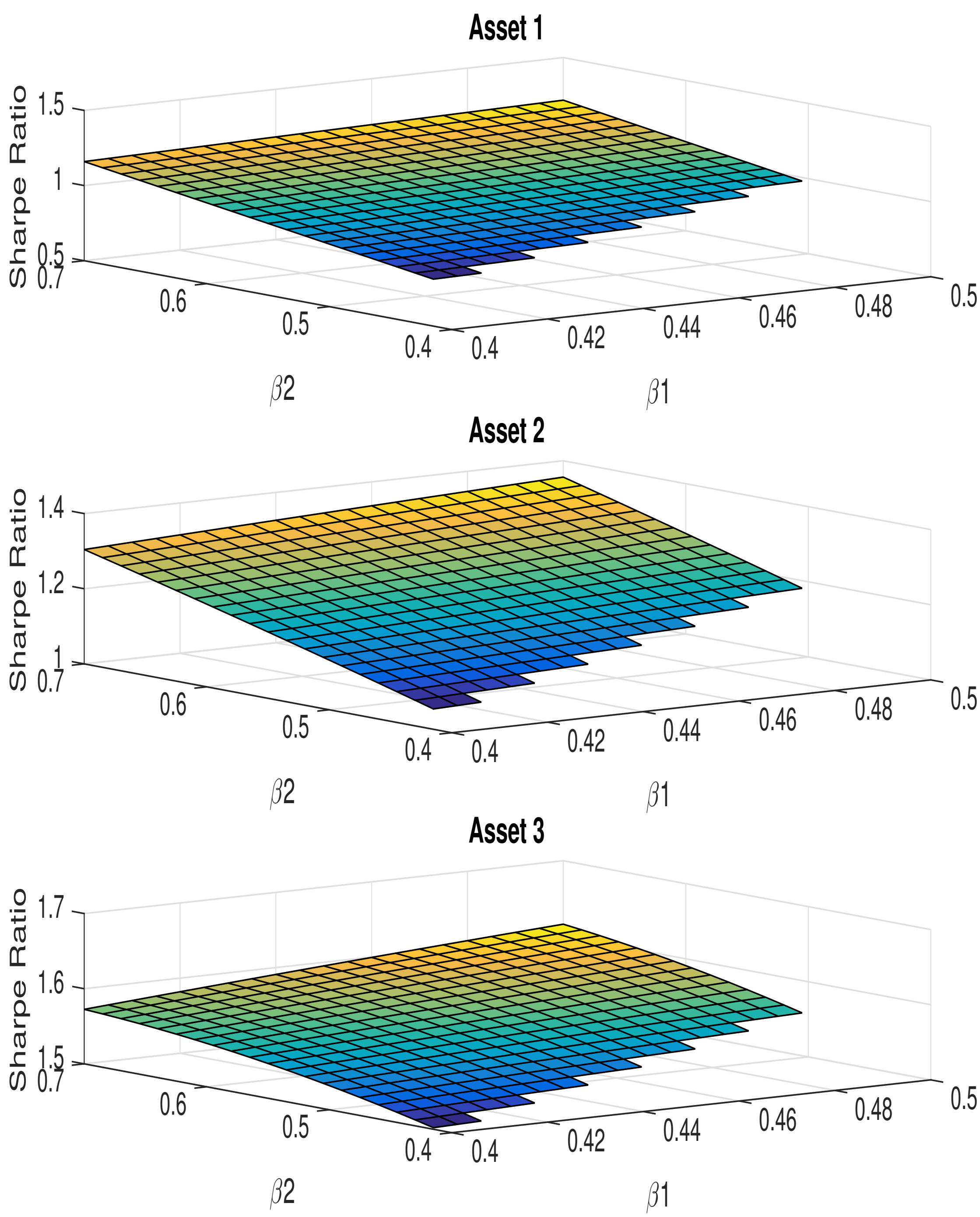

Figure 3: Dispersion of Sharpe Ratios

This three dimensional figure plots the dispersion, $\Omega(s)$, of Sharpe ratios in equilibrium A. As shown the dispersion decreases with increasing of β_1 and β_2, of sophisticated and naive respectively. The same decreasing property of the dispersion in the homogeneous environment and in Equilibrium B are reported in Table 3 and Table 4. The parameters in this figure are $N = 3$, $\nu = 0.3$, $\sigma_1 = 9\%$, $\bar{x}_1 = 1$; $\sigma_2 = 10\%$, $\bar{x}_2 = 5$; $\sigma_3 = 12\%$, $\bar{x}_3 = 10.5$. Notice that $\Omega(\sigma\bar{s}) = 0.56$ and $\frac{1}{N-1}\left\{\frac{\nu}{1-\nu}\frac{\Omega(\sigma\bar{x})}{1-\Omega(\sigma\bar{x})}N - 1\right\} = 0.3$.

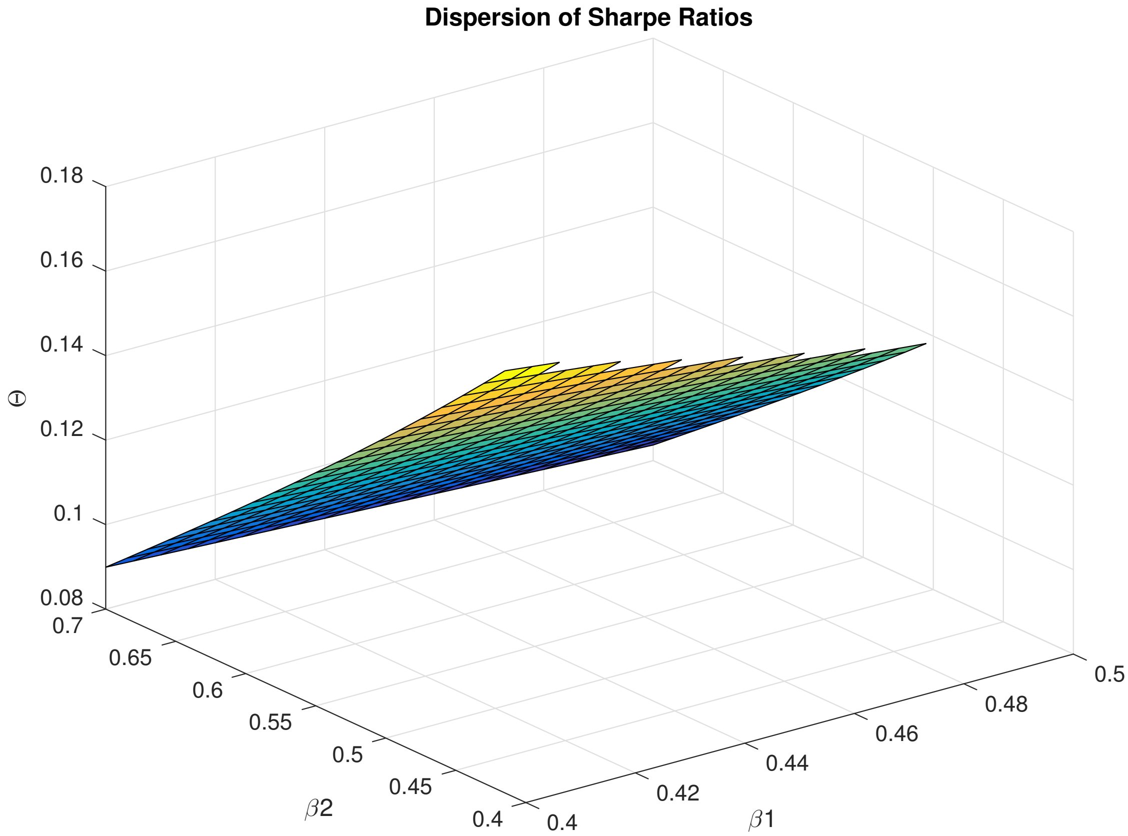

Figure 4: Optimal Holdings of Agents

This figure reports the optimal holdings of the sophisticated and naive agent on high quality and low quality asset respectively. The parameters in this figure are $N = 3$, $\nu = 0.3$, $\sigma_1 = 9\%$, $\bar{x}_1 = 1$; $\sigma_2 = 10\%$, $\bar{x}_2 = 5$; $\sigma_3 = 12\%$, $\bar{x}_3 = 10.5$. Notice that $\Omega(\sigma\overline{x}) = 0.56$ and $\frac{1}{N-1}\left\{\frac{\nu}{1-\nu}\frac{\Omega(\sigma\overline{x})}{1-\Omega(\sigma\overline{x})}N - 1\right\} = 0.3$. $\frac{\sigma_1\bar{x}_1}{L} = 0.048 < \frac{1}{N}$, $\frac{\sigma_1\bar{x}_1}{L} = 0.8582 > \frac{1}{N}$.

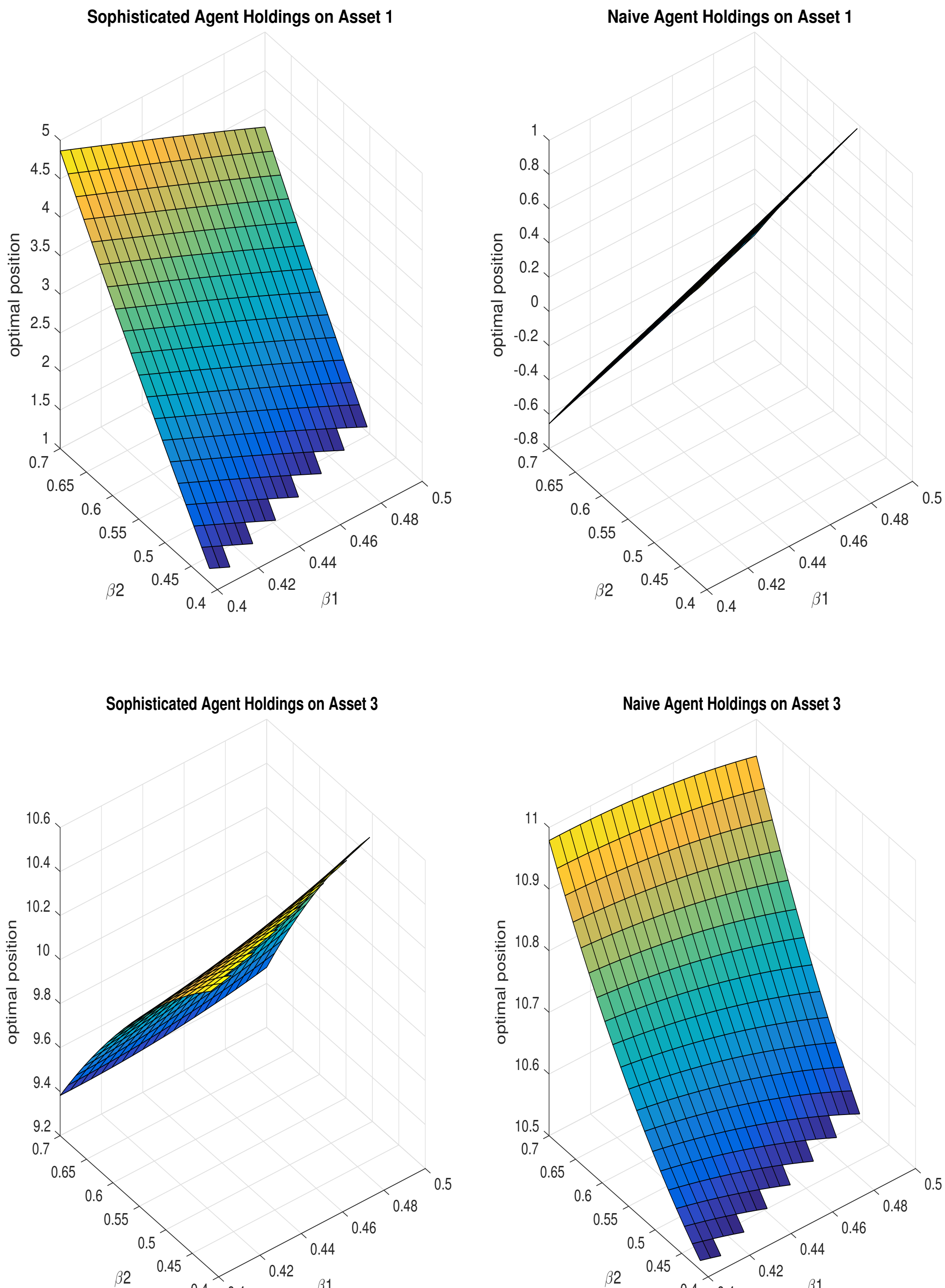

Figure 5: Optimal Portfolio Risks of Agents

This figure demonstrates that the variance (risk) of the optimal portfolio the sophisticated agent is always larger than the variance (risk) of the naive agent's optimal portfolio in a full participation equilibrium. It shows that a higher correlation uncertainty leads to a higher risk averse behavior. The same pattern is reported in Table 5 in equilibrium B. The parameters in this figure are $N = 3$, $\nu = 0.3$, $\sigma_1 = 9\%$, $\bar{x}_1 = 1$; $\sigma_2 = 10\%$, $\bar{x}_2 = 5$; $\sigma_3 = 12\%$, $\bar{x}_3 = 10.5$. In this situation $\Omega(\sigma\overline{x}) = 0.56$ and $\frac{1}{N-1}\left\{\frac{\nu}{1-\nu}\frac{\Omega(\sigma\overline{x})}{1-\Omega(\sigma\overline{x})}N-1\right\} = 0.3$.

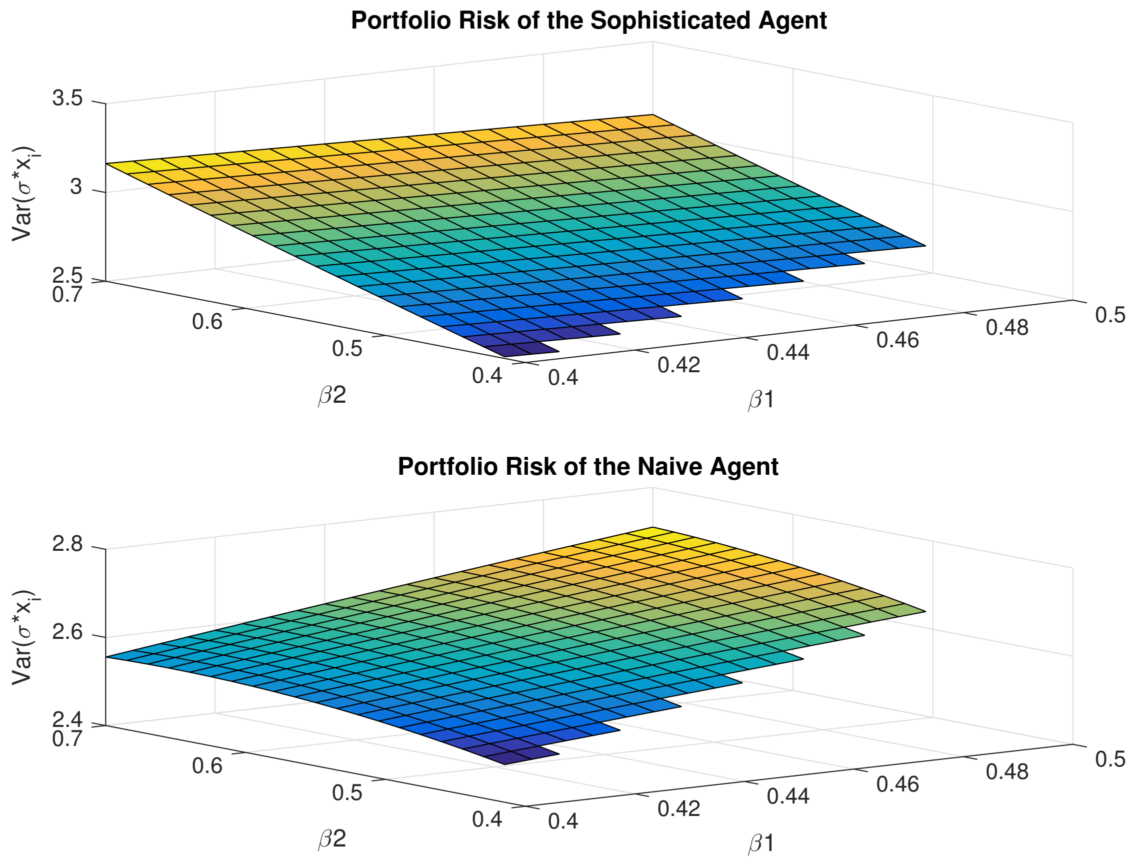

Figure 6: Dispersion of Optimal Portfolios of Agents

This figure demonstrates the dispersion of the optimal portfolio the sophisticated and naive agent respectively in a full participation equilibrium. Clearly, the dispersion of the sophisticated agent's optimal portfolio is smaller than the dispersion of the naive agent's optimal portfolio for all possible values of β_1 and β_2. The same pattern is reported in Table 3 in a limited participation equilibrium. The parameters in this figure are $N = 3$, $\nu = 0.3$, $\sigma_1 = 9\%$, $\bar{x}_1 = 1$; $\sigma_2 = 10\%$, $\bar{x}_2 = 5$; $\sigma_3 = 12\%$, $\bar{x}_3 = 10.5$. Note that $\Omega(\sigma\overline{x}) = 0.56$ and $\frac{1}{N-1}\left\{\frac{\nu}{1-\nu}\frac{\Omega(\sigma\overline{x})}{1-\Omega(\sigma\overline{x})}N - 1\right\} = 0.3$.

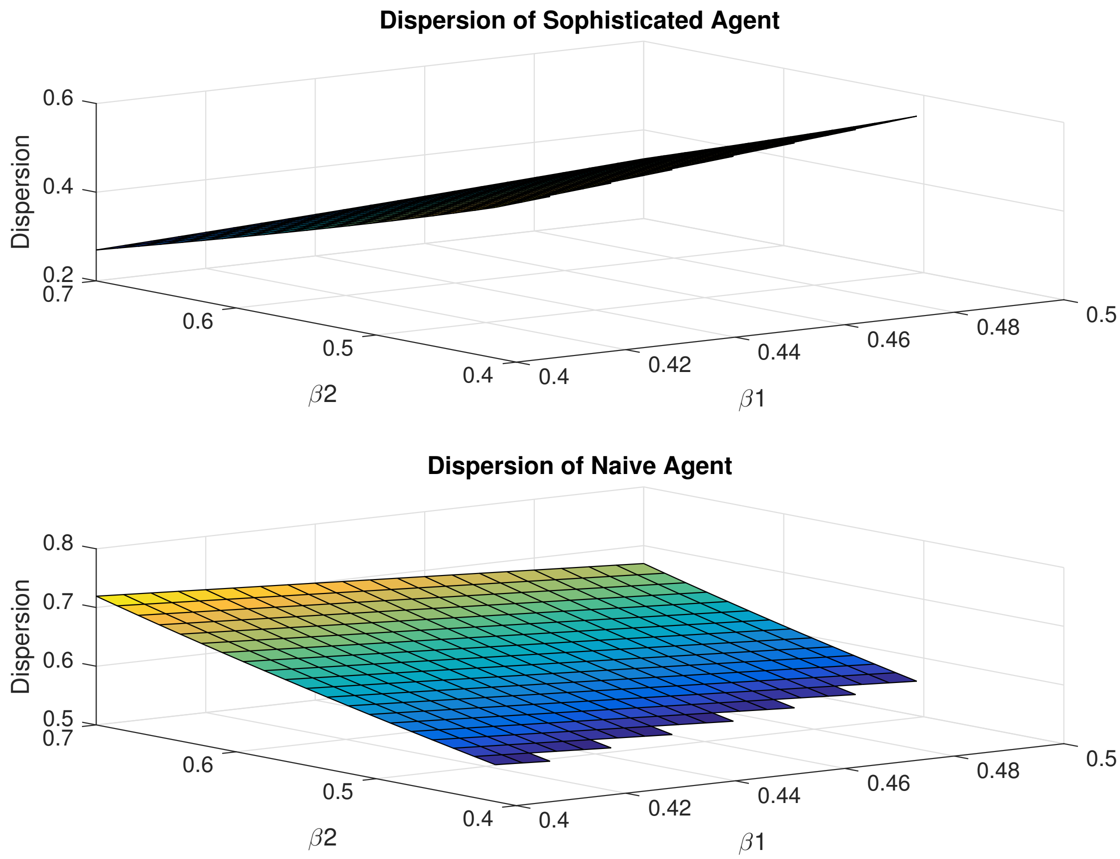

Table 1: Sharpe Ratios and Correlation Uncertainty Premium in a Homogeneous Model

This table reports the Sharpe ratios, the dispersion of all Sharpe ratios and the correlation uncertainty premium in homogeneous environment. The parameters are: $N = 3, \nu = 0.3, \sigma_1 = 9\%, \overline{x}_1 = 1, \sigma_2 = 10\%, \overline{x}_2 = 5, \sigma_3 = 12\%, \overline{x}_3 = 10.5$. Notice that the dispersion of the risks is $\Omega(\sigma\overline{x}) = 0.5558$. The reference correlation coefficient is $\overline{\rho} = 40\%$. By the "premium column" we mean the percentage of the correlation uncertainty premium over the Sharpe ratio without the correlation correlation uncertainty, that is, for the reference correlation coefficient.

ϵ	s_1	s_2	s_3	$\Omega(s)$	Premium Asset 1	Premium Asset 2	Premium Asset 3
0	0.79	1.49	2.70	0.54	NA	NA	NA
0.02	0.83	1.55	2.80	0.51	4.43%	4.14%	3.95%
0.04	0.86	1.61	2.91	0.48	8.87%	8.27%	7.90%
0.06	0.90	1.67	3.02	0.46	13.30%	12.41%	11.85%
0.08	0.93	1.74	3.12	0.43	17.73%	16.54%	11.85%
0.1	0.97	1.80	3.23	0.40	22.17%	20.68%	19.76%
0.12	1.01	1.86	3.34	0.38	26.60%	24.81%	23.71 %
0.14	1.04	1.92	3.44	0.38	31.03%	28.95%	27.66%
0.16	1.08	1.98	3.55	0.34	35.47%	33.08%	31.61%
0.18	1.11	2.04	3.66	0.31	39.90%	37.22%	35.56%
0.2	1.15	2.11	3.76	0.29	44.33%	41.36%	39.52%

Table 2: Sharpe ratios in a limited participation equilibrium

This table reports all Sharpe ratios in a limited participation equilibrium including the Sharpe ratios, the dispersion of all Sharpe ratios, the agent's optimal position and the corresponding dispersion of the optimal portfolio. The parameters are: $N = 3, \nu = 0.6, \sigma_1 = 2\%, \overline{x}_1 = 0.5, \sigma = 8\%, \overline{x}_2 = 3, \sigma_3 = 12\%, \overline{x}_3 = 10.5$. Notice that the dispersion of the risks is $\Omega(\sigma\overline{x}) = 0.7631$. The reference correlation coefficient is $\overline{\rho} = 40\%$.

β_1	$K(\beta_1)$	s_1	s_2	s_3	S	s_1/S	s_2/S	s_3/S	$\Omega(s)$
0.400	0.557	0.70	0.82	1.36	2.89	0.244	0.285	0.471	0.210
0.405	0.561	0.71	0.83	1.36	2.90	0.245	0.286	0.469	0.207
0.410	0.566	0.72	0.84	1.36	2.92	0.246	0.287	0.467	0.204
0.415	0.570	0.73	0.84	1.36	2.93	0.247	0.287	0.465	0.201
0.420	0.575	0.73	0.85	1.37	2.95	0.249	0.288	0.463	0.198
0.425	0.579	0.74	0.86	1.37	2.96	0.250	0.289	0.461	0.195
0.430	0.583	0.75	0.86	1.37	2.98	0.251	0.289	0.460	0.192
0.435	0.588	0.75	0.87	1.37	2.99	0.252	0.290	0.458	0.190
0.440	0.592	0.76	0.87	1.37	3.01	0.253	0.291	0.456	0.187
0.445	0.596	0.77	0.88	1.37	3.02	0.255	0.291	0.454	0.184
0.450	0.601	0.78	0.89	1.37	3.04	0.256	0.292	0.452	0.181
0.455	0.605	0.78	0.89	1.37	3.05	0.257	0.292	0.452	0.179
0.460	0.609	0.79	0.90	1.38	3.06	0.258	0.293	0.449	0.176
0.465	0.613	0.80	0.90	1.38	3.08	0.259	0.294	0.447	0.174
0.470	0.618	0.80	0.91	1.38	3.09	0.260	0.294	0.446	0.171
0.475	0.622	0.81	0.92	1.38	3.11	0.261	0.295	0.444	0.169
0.480	0.626	0.82	0.92	1.38	3.12	0.262	0.295	0.442	0.166
0.485	0.630	0.83	0.93	1.38	3.14	0.263	0.296	0.441	0.164
0.490	0.635	0.83	0.93	1.38	3.15	0.264	0.297	0.439	0.161
0.495	0.639	0.84	0.94	1.38	3.16	0.265	0.297	0.438	0.159
0.500	0.643	0.85	0.95	1.39	3.18	0.266	0.298	0.436	0.156

Table 3: Optimal portfolios in a limited participation equilibrium

This table reports a limited participation equilibrium including the agent's optimal position and the corresponding dispersion of the optimal portfolio. The parameters are: $N = 3, \nu = 0.6, \sigma_1 = 2\%, \overline{x}_1 = 0.5, \sigma_2 = 8\%, \overline{x}_2 = 3, \sigma_3 = 12\%, \overline{x}_3 = 10.5$. Notice that the dispersion of the risks is $\Omega = 0.7631$. The reference correlation coefficient is $\overline{\rho} = 40\%$. For each $i = 1, 2, 3$, $x_i^{(n)}$ and $x_i^{(s)}$ denote the optimal holding of the naive agent and the sophisticated agent on asset i. $\Omega^{(n)}$ represents the dispersion of the naive agent's optimal portfolio while $\Omega^{(s)}$ is the dispersion of the sophisticated agent's optimal portfolio.

β_1	$K(\beta_1)$	$(x_1^{(n)}, x_1^{(s)})$	$(x_2^{(n)}, x_2^{(s)})$	$(x_3^{(n)}, x_3^{(s)})$	$\Omega^{(n)}$	$\Omega^{(s)}$
0.400	0.557	(-6.478, 5.152)	(11.277, 9.982)	(1.791, 3.806)	1.000	0.629
0.405	0.561	(-6.484, 5.156)	(11.287, 9.976)	(1.792, 3.805)	1.000	0.629
0.410	0.566	(-6.489, 5.159)	(11.297, 9.969)	(1.794, 3.804)	1.000	0.628
0.415	0.570	(-6.495, 5.163)	(11.306, 9.962)	(1.796, 3.803)	1.000	0.628
0.420	0.575	(-6.500, 5.167)	(11.316, 9.956)	(1.797, 3.802)	1.000	0.628
0.425	0.579	(-6.506, 5.171)	(11.326, 9.949)	(1.799, 3.801)	1.000	0.628
0.430	0.583	(-6.512, 5.175)	(11.336, 9.943)	(1.800, 3.800)	1.000	0.628
0.435	0.588	(-6.517, 5.178)	(11.346, 9.936)	(1.802, 3.799)	1.000	0.627
0.440	0.592	(-6.523, 5.182)	(11.356, 9.930)	(1.803, 3.798)	1.000	0.627
0.445	0.596	(-6.529, 5.186)	(11.365, 9.923)	(1.805, 3.797)	1.000	0.627
0.450	0.601	(-6.534, 5.190)	(11.375, 9.917)	(1.807, 3.796)	1.000	0.627
0.455	0.605	(-6.540, 5.193)	(11.385, 9.910)	(1.808, 3.795)	1.000	0.627
0.460	0.609	(-6.546, 5.197)	(11.395, 9.903)	(1.810, 3.794)	1.000	0.626
0.465	0.613	(-6.551, 5.201)	(11.405, 9.897)	(1.811, 3.793)	1.000	0.626
0.470	0.618	(-6.557,5.205)	(11.415, 9.890)	(1.813, 3.791)	1.000	0.626
0.475	0.622	(-6.563, 5.208)	(11.424,9.884)	(1.814, 3.790)	1.000	0.626
0.480	0.626	(-6.568. 5.212)	(11.434, 9.877)	(1.816, 3.789)	1.000	0.626
0.485	0.630	(-6.574, 5.216)	(11.444,9.871)	(1.817, 3.788)	1.000	0.626
0.490	0.635	(-6.580, 5.220)	(11.454, 9.864)	(1.819, 3.787)	1.000	0.626
0.495	0.639	(-6.585, 5.223)	(11.464, 9.858)	(1.821, 3.786)	1.000	0.625
0.500	0.643	(-6.591, 5.227)	(11.474,9.851)	(1.822, 3.785)	1.000	0.625