Money and Bank Competition

Ping He Zehao Liu[*]

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Abstract

We examine the role of money and monetary policy when competitive banks face a systemic liquidity shock. Banks cannot provide a deposit contract contingent on the systemic liquidity shock, and introducing money to the economy allows banks to provide a nominal deposit contract that leads to state-contingent real consumption for depositors while resulting in over-competition among banks at the same time. Monetary authority can always improve the depositors' welfare through providing liquidity to the banks with liquidity shortage, however, an opposite operation of extracting liquidity from the banks with liquidity shortage can eliminate over-competition among banks ex-ante and thus help the economy to achieve the social optimal allocation.

Keywords: Systemic Liquidity Shock; Liquidity Intermediation; Competition; Monetary Policy

JEL Classification: E40, E50, G20

[*] He: School of Economics and Management, Tsinghua University. (heping@sem.tsinghua.edu.cn) Liu: School of Economics and Management, Tsinghua University. (liuzh.13@sem.tsinghua.edu.cn)

I. Introduction

The role of money in modern theory of finance is largely under-explored. Standard role of money as a medium of exchange studied by Kiyotaki and Wright (1989, 1993) seems to have little relevancy to financial activities. He et al. (2005, 2008) assume bank deposits are a safer medium of exchange compared with cash, and study the substitutability and complementarity between inside money and outside money while neglecting the other roles of banks. Pioneering work by Diamond and Rajan (2006) proposes a new role of money in a banking system that nominal deposits offer banks a hedge against real supply shortage, and in their model, banks do not have to liquidate long-term real investment as much as in the economy without money as the price level adjusts to absorb some excess demand. In this paper, we study a model of competitive banks facing systemic demand shocks. In our model, equilibrium prices adjustments yield state-contingent real consumptions for depositors, but banks over-compete for depositors, and this leads to welfare loss to depositors. To suppress over-competition among banks ex-ante, the optimal monetary policy might involve extracting liquidity from banks facing liquidity shortage.

We study the classic model of Diamond and Dybvig (1983) with systemic liquidity shock, that is, the portion of depositors with liquidity need is uncertain. Under the social optimal allocation, depositors consume different amount of real goods contingent on the realization of the systemic demand shock. However, the usual deposit contract cannot be state-contingent in reality, and banks have to make the same payment no matter whether the liquidity shock occurs or not. Therefore, the demand contract cannot achieve the social optimal allocation. At the same time, we assume banks cannot terminate long-term loan contracts or long-term real investment, and therefore, banks have to leave enough liquidity in their balance sheet to be solvent when they face systemic liquidity shocks.[1] With the solvency requirement, banks leave enough liquidity in the vault to meet the demand shock with less long-term investment, but banks make a profit when the liquidity shock occurs as there are fewer patient depositors whom banks need to pay more. We can show that competitive banks make non-zero positive profits as they cannot offer deposit contracts contingent on the liquidity shock. Positive bank profits further reduce the depositors' welfare.

When we introduce money in the economy, though the nominal payment of deposit contract is the same, the real goods consumed by depositors can be different depending on the realization of the liquidity shock. The price mechanism can change the goods allocation between depositors and banks, and smooth the goods allocation between the impatient depositors and the patient depositors. We call these effects "redistribution effect".

The separation of money and real goods make the banks less constrained in competing for depositors. Without money, if banks pay too much to the impatient depositors, banks have to cut down their real investment. With money, even if banks pay a large amount of money to the impatient depositors, banks can still get enough money to pay the patient depositors by selling proceeds from long term investment. Then introducing money entails over-competition in the economy, that is, in order to attract depositors, banks pay too much in nominal term to the impatient depositors who withdraw early. With over-competition, when the liquidity shock occurs, more impatient depositors compete for a limit amount of real goods despite that they get a large payment in nominal term while banks and fewer patient depositors consume the rest. Therefore, over-competition brings

[1] This liquidity requirement is consistent with the banking system regulation environment nowadays. Basel III introduces several new liquidity requirements. Among them, the liquidity coverage ratio (LCR) acts as a short-term liquidity requirement, which requires that banking organizations should maintain sufficient liquid assets to cover the liquidity need over a 30-day stress period. Other regulation requirements such as net stable funding ratio (NSFR) supplements LCR and propose other liquidity requirements.

unbalanced real goods consumption between the impatient depositors and the patient depositors and increases bank profits, thus causes welfare loss to depositors. Over-competition reduces the positive effect of money.

In general, money may not help the banking system with a systemic liquidity risk achieve the social optimal allocation, and we find that a marginal monetary policy operation in the conventional direction can improve the positive effect of money and thus can always improve the social welfare. Conventional direction means that the central bank injects liquidity into the economy when banks face liquidity shock, or extract liquidity from the economy when the liquidity of the banking system is abundant. However, this conventional monetary policy operation cannot achieve the social optimal allocation. Nevertheless, an unconventional monetary policy may promote the economy to the social optimal allocation. That is, the central bank extracts liquidity through a high interest rate from banks when they face liquidity shocks. This monetary policy works because the high interest rate makes banks pay less to the impatient depositors and deposit the extra cash at the central bank, which suppresses the over-competition problem in the economy.

The key difference between our paper and Diamond and Rajan (2006) is that there is no competition among banks and prices and interest rates are very much exogenous in their paper. We show in this paper that competition might kill the positive welfare effect of money, and the optimal monetary policy may be counter-factual. Another important difference is that banks in our model are "monetarized" as they take nominal deposits and only keep cash in the vault, and banks cannot store real goods or liquidate long-term investment. With this type of "monetarized" banks, the welfare effect of price mechanism is different from Diamond and Rajan (2006). In our paper, the price mechanism can adjust the goods allocation between depositors and banks, and between the impatient depositors and the patient depositors. In Diamond and Rajan (2006), the price mechanism can reduce early liquidation of long-term investment, thus improve economic efficiency.

Our paper is related to three strands of existing literature. The first one is about the role of money and monetary policy in the economy. Keynesian economists claim that money influences the economy mainly through affecting interest rate and real investment, (See Hicks (1937) and Hansen (1949, 1953)). The major view of later monetarism led by Friedman (1948, 1959, 1970) is that variation in the money supply has major influences on national output in the short run and the price level over longer periods. Samuelson (1958) declares that money is a tool to promote the transactions between generations. New classical economics pioneered by Lucas (1972, 1975) and Sargent and Wallace (1975, 1976) claims that the money's effect on real economy is originated from the wrong cognition resulted from information asymmetry. Some economists make efforts to incorporate money into the real business cycle model developed by Kydland and Prescott (1982), and they focus on the role of money in promoting transactions. These studies use the money–in-utility model first developed by Sidrauski (1967) and the cash-in-advance model (See, for example, Clower (1967), Lucas (1980) and Stockman (1981)). However, the role of money interacting with the financial system is not well acknowledged in these macro-economic models.

The second strand is the literature on the bank's role as liquidity intermediaries. In Diamond and Dybvig (1983), banks offer insurance against idiosyncratic preference shocks. Qi (1994) extends Diamond and Dybvig's model into an overlapping generation model to discover banks' liquidity creation role under a dynamic framework. Diamond and Rajan (2001) argue that banks can resolve the liquidity problems that arise in direct lending by enabling depositors to withdraw at low cost, as well as buffer firms from the liquidity needs of their investors. Kashyap et al. (2002) argues that the provision of liquidity on demand may be the key factor which ties together the commercial banking activities of deposit taking and lending. In this paper, we analyze the role of money and monetary policy interacting with the liquidity role of banks.

The third one is the literature of bank competition and the efficiency of banking system. Some are about the relationship between competition and credit supply. Peterson and Rajan (1995) shows creditors are more likely to finance credit-constrained firms when credit markets are concentrated, and competition may lead to inefficient relationship lending. Shaffer (1998) shows that since the screening technologies of banks may not be that accurate, the average quality of a bank's pool of borrowers declines as the number of competitor increases. Marquez (2002) argues that more competing banks reduces banks' screening ability, leading to an inefficiency as more low-quality borrowers get financed. Park and Pennacchi (2009) claims that market-extension mergers by large multi-market banks will promote loan competition while reduce retail deposit competition, which harms depositors but help borrowers. Acharya et al. (2012) studies the inefficiency originated by the imperfect competition in the inter-bank market. There are also debates about the relationship between competition and financial stability, traditional view is that competition reduces banks' franchise values and increases the risk-taking behavior (See, for example, Furlong and Keeley (1989), Keeley (1990)), while some recent literature claims that competition promotes borrowers' incentive to choose safer investments, so competition may lead to a safer banking system or have a U-shape relation with financial stability (See, for example, Boyd and De Nicoló (2005) and Martinez-Miera and Repullo (2010)). In our study, introducing money into the banking system exacerbates the competition among banks and reduce the depositors' welfare.

The rest of the paper is organized as follows. In section II, we illustrate the framework of the model. In section III, we characterize the social optimal allocation and the competitive equilibrium without money as the benchmarks. In section IV, we introduce money to the economy, and study the properties of competitive equilibrium. In section V, we examine the role of monetary policy. Section VI concludes.

II. The Model Setup

Consider an economy of three dates (0, 1 and 2) with a continuum of depositors and a continuum of banks. Each depositor is endowed with one unit of real goods and one unit of money at date 0. Banks have no endowments, and banks take money deposits from depositors with a promised payoff at date 1 and 2. Banks have a production technology where investing one unit of real goods at date 0 leads to a payoff of R unit of real goods at date 2 definitely with $R > 1$, and banks use the money to buy real goods to make the long-term investment.[1] If one unit of project is liquidated prematurely at date 1, then the bank can collect a residual value of νR, and we assume ν is very small, and banks will never liquidate any projects. Depositors have a saving technology for real goods with zero return, and they save their money in banks and get money payment when they withdraw. Banks invest all the goods they purchase, and, therefore, banks are "monetarized" in the sense that they do not keep any real goods in the vault.

Depositors are homogeneous at date 0, and each depositor faces a risk of being impatient type or patient type realized at date 1. At date 1, each depositor learns his type. The impatient depositors only get utility from consumption at date 1,

[1] This is equivalent to assuming that banks lend cash to firms, and firms make the long-term investment while banks get all the proceeds. However, to simplify the model setup, we assume the banks make the investment directly, we do not explicitly have firms in the model.

given by $U(C_1)$. The patient depositors only obtain utility from consumption at date 2, given by $U(C_2)$. $U(\cdot)$ is twice continuously differentiable, strictly increasing and strictly concave function and satisfies Inada conditions, that is, $U'(\cdot) > 0$ and $U''(\cdot) < 0$, $U'(0) = +\infty$ and $U'(\infty) = 0$. As in Diamond and Dybvig (1983), we also assume that the relative risk aversion is greater than 1, that is, $-\frac{CU''(C)}{U'(C)} > 1$. However, depostior preference shock has a systemic component, which is different from Diamond and Dybvig (1983) where there is only idiosyncratic preference shock to depositors. In our model, the probability of turning impatient for each depositor may suddenly increase at date 1. At date 0, it is public information that with probability $1-\theta$, there is no systemic liquidity shock, and the proportion of the impatient depositors is λ_L at date 1, and with probability θ, the systemic liquidity shock occurs, and the proportion of the impatient depositors is $\lambda_H > \lambda_L$ at date 1. We assume λ_H and λ_L are not very large, and this assumption is consistent with what we observe in the data and simplifies our analysis.

At date 0, depositors save their money endowment as well as the proceeds from selling some of their real goods endowment in banks.[1] The impatient depositors will consume their own un-sold real goods at date 1, and probably buy some goods from the patient depositors to consume. If the patient depositors own some goods at date 1, they can either sell them to the impatient depositors at date 1 or store them until date 2.

Depositors not only obtain utility from real goods consumption, but also they get utility directly from cash-holding after they consume. One unit of cash provides a fixed amount of utility of ε and the utility obtained from cash is linear. Therefore, holding M unit of cash can bring $M\varepsilon$ unit of utility. This residual value of money

[1] We only look at the equilibrium in which depositors deposit some money in banks, but there does exist some equilibrium in which depositors keep all their money in hand.

captures the role of money as a store of value, and this assumption is necessary for the role of money as a medium of exchange in our model of finite horizon, otherwise, no depositors would hold money ex-ante and trade will not happen. The linearity assumption on the value of money makes the total utility of the depositors derived from holding cash a constant, and we will only look at the welfare of the depositors derived from real goods consumption to compare with that of the social optimal allocation and in the economy without money. Thus the role of money is "instrumental" for welfare comparison.

Banks are risk neutral and get utility from real goods consumption, but they do not get utility from cash-holding. With this assumption, the equilibrium outcome would be that banks consume some real goods at date 2 with all the cash going back to the depositors. If a bank doesn't fulfill his contracts at any date, the bank will suffer a large welfare loss, and, as a consequence, banks will always prepare enough liquidity to fulfill all the payment obligations.

The time line of the economic activities is as follows.

Date 0: Banks offer deposit contracts to depositors, and depositors deposit y units of money in a bank with the best contract. Banks use money buy I unit of real goods from depositors at price P_0 and invest them into projects. At the end of date 0, each depositor holds $1 + P_0 I - y$ units of cash and $1 - I$ units of goods. Each bank has $y - P_0 I$ units of money and a project with I unit of investment.

Date 1: Depositors' preference shocks are realized. The impatient depositors withdraw their deposits from banks, consume their own real goods $1 - I$, and buy some real goods from the patient depositors. The impatient depositors get utility from the remaining money at the end of date 1, and they leave the economy after date 1. The patient depositors sell some of their real goods to the impatient depositors in exchange for money at date 1.

Date 2: The return of long-term investment is realized. The patient depositors withdraw their deposits from banks. Banks sell some of their real goods in the market and use the proceeds to make the nominal payment. The patient depositors consume the real goods and get utility from the remaining money at the end of date 2. The real goods left in banks become banks' profits and are consumed by banks. Then the economy closes.

Remarks:

(i) At date 0, depositors can get additional money by selling real goods to banks in the market, and, therefore, the total amount of deposit held by each depositor may exceed one.

(ii) At date 2, banks do no derive utility from holding money, therefore, they will sell just enough investment output to pay off their nominal liabilities to the depositors, and they consume the rest of the real goods.

Before we characterize the competitive equilibrium for the model with money, we first analyze the social optimal allocation of this economy and the equilibrium without money for welfare comparison. With our assumptions on money and banks, we will only compare the utility of depositors derived from the real goods consumption, and the role of money and banks are instrumental.

III. Social Optimal Allocation and Real Deposit Contract

A. Social Optimal Allocation

We will first analyze the social optimal allocation. The social planner allocates different amounts of goods to depositors in the normal state (λ_L) and the liquidity shortage state (λ_H). Denote the depositors' consumption bundle as $\{C_1^H, C_2^H, C_1^L, C_2^L\}$, where C_1^j denotes the real-goods consumption of the impatient depositors in λ_j case, and C_2^j denotes the real-goods consumption of the patient depositors in λ_j case, with $j = \{H, L\}$.

The social planner solves the following maximization problem:

$$\max_{\{I, C_1^H, C_2^H, C_1^L, C_2^L, \beta_H, \beta_L\}} \theta[\lambda_H U(C_1^H) + (1-\lambda_H) U(C_2^H)] + (1-\theta)[\lambda_L U(C_1^L) + (1-\lambda_L) U(C_2^L)]$$

$$s.t. \quad \lambda_H C_1^H \leq (1-\beta_H)(1-I)$$

$$(1-\lambda_H) C_2^H \leq RI + \beta_H (1-I)$$

$$\lambda_L C_1^L \leq (1-\beta_L)(1-I)$$

$$(1-\lambda_L) C_2^L \leq RI + \beta_L (1-I)$$

Here, β_H denotes the proportion of real goods saved at date 1 and transferred to date 2 in liquidity shortage state, and β_L denotes this proportion in the normal state. Later we will see, in the optimal allocation we have $\beta_H = 0$.

The following proposition characterizes the social optimal allocation.

PROPOSITION 1: Let $\overline{I_H} = (1-\lambda_H)/(\lambda_H R + 1 - \lambda_H)$, $\overline{I_L} = (1-\lambda_L)/(\lambda_L R + 1 - \lambda_L)$, then the social optimal investment I_{FB}^* is either the optimal investment in the region $[\overline{I_H}, \overline{I_L}]$ or the optimal investment in the region $[\overline{I_L}, 1]$, whichever gives a higher expected utility for depositors. Moreover, (i) if $\overline{I_H} < I_{FB}^* < \overline{I_L}$, the optimal allocation is $C_1^H = (1 - I_{FB}^*)/\lambda_H$, $C_2^H = RI_{FB}^*/(1-\lambda_H)$, $C_1^L = RI_{FB}^* + 1 - I_{FB}^*$, $C_2^L = RI_{FB}^* + 1 - I_{FB}^*$; (ii) if $\overline{I_L} < I_{FB}^* < 1$, the optimal allocation is $C_1^H = (1 - I_{FB}^*)/\lambda_H$, $C_2^H = RI_{FB}^*/(1-\lambda_H)$, $C_1^L = (1 - I_{FB}^*)/\lambda_L$, $C_2^L = RI_{FB}^*/(1-\lambda_L)$.

PROOF: See Appendix.■

Intuitively, if the investment level at date 0 is too high, then the impatient depositors will consume too little at date 1 in the liquidity shortage state as long-term investment cannot be liquidated, but if the investment level at date 0 is too low, then the high return investment opportunities are wasted even though we can save goods for the patient depositors to consume at date 2.

If the investment level is not too high and in the region $[\overline{I_H}, \overline{I_L}]$, then allocating all the real goods to the impatient depositors at date 1 in the normal state cannot be optimal as the impatient depositors consume more than the patient depositors. Therefore, it is optimal to save some real goods for the patient depositors to consume at date 2, and let the consumption of the impatient depositors and the patient ones be the same in the normal state. In the liquidity shortage state, it is optimal to allocate all the non-invested real goods to the impatient depositors at date 1 while allocating all the investment return to the patient depositors at date 2.

If the investment level is high and in the region $[\overline{I_L}, 1]$, then allocating all the real goods at date 1 directly to the impatient depositors always leads to a smaller C_1 than C_2, and the optimal allocation is distributing all the resources at date 1 to the impatient depositors and all the resources at date 2 to the patient depositors in both the liquidity shortage state and the normal state and no real goods will be saved from date 1 to date 2.

If the optimal investment is in the region $[\overline{I_H}, \overline{I_L}]$, then the gain from consumption smoothing between two type of depositors (the smoothing effect) is greater than the gain from investing more and increasing total amount of real goods (the wealth effect). If the optimal investment is in the region $[\overline{I_L}, 1]$, then the gain from the wealth effect is greater than the gain from the smoothing effect.

We can show that when the gap between λ_H and λ_L is small, the optimal investment is always in the region $[\overline{I_L}, 1]$. The extreme situation is $\lambda_H = \lambda_L$, and the model degenerates to the case with a single λ, which is identical to Diamond and Dybvig (1983). In their model, all the real goods $1-I$ at date 1 are allocated to the impatient depositors and all the long-term investment return are allocated to the patient depositors. In this paper, we will focus on the situation where λ_H and λ_L are close and the optimal investment is in the region $[\overline{I_L}, 1]$, that is, there is no real goods transfer between two dates in social optimal allocation so that our results can be comparable to previous literature. This assumption also simplifies our analysis. In addition, the economics and the major results when λ_H and λ_L are not close are basically the same.

B. Competitive Equilibrium with Real Deposit Contract

Now we begin to discuss the competitive economy without money as another benchmark case to compare with the economy with money.

Depositors deposit their real goods at banks, and banks directly invest goods in projects. Banks cannot offer a deposit contract with payments contingent on the demand shocks, and they have to pay the same amount of real goods in the liquidity shortage state and the normal state. For each unit of real goods deposits, depositors can get D_1^R unit of real goods payment if they withdraw at date 1, and get D_2^R unit of real goods payment if they withdraw at date 2. Denoting the real consumption bundle in this economy as $\{C_1^{HR}, C_2^{HR}, C_1^{LR}, C_2^{LR}\}$ where superscript R represents the economy with "real" deposit contract, we have $C_1^{HR} = C_1^{LR} = D_1^R$ and $C_2^{HR} = C_2^{LR} = D_2^R$.

The time line is as follows.

Date 0: Banks offer deposit contracts to depositors, depositors choose the bank that offers the best payoffs. If banks offer the same contracts, depositors randomly pick one to make deposits. Then banks decide their optimal investment I and keep the remaining goods $1 - I$ in hand to satisfy the possible liquidity need at date 1.

Date 1: Depositors' types are realized. The impatient depositors withdraw their deposits and consume.

Date 2: The patient depositors withdraw their deposits and consume. The remaining real goods held by banks after all the deposit contracts are paid off become the banks' profits.

Since the market is competitive, all the banks will offer the contracts which provide the highest expected utility at date 0 to depositors under the budget constraints. Then if some contracts provide the same expected utility to depositors, banks will choose the one that brings them the highest real-goods profit.

DEFINITION 1 (Competitive Equilibrium without Money): A competitive equilibrium without money is a deposit contract $\{D_1^R, D_2^R\}$ such that, (i) the deposit contract is feasible, that is, banks are solvent in both the normal state and the liquidity shortage state; and (ii) no banks have incentive to deviate to any other feasible deposit contract.

Now we characterize the equilibrium deposit contracts in the lemma below.

LEMMA 1: In any symmetric competitive equilibrium, the equilibrium deposit contract has the following properties:

(i) In the liquidity shortage state, the impatient depositors consume all the real goods that are not invested at date 1, and we have $\lambda_H D_1^R = 1 - I$.

(ii) The no bank-run condition needs to be satisfied, that is, $D_1^R \leq D_2^R$.

PROOF: See Appendix.■

The intuition behind Lemma 1 is quite simple. For part (i), if there are real goods left in the vault of banks at date 1 in the liquidity shortage state, banks can always improve the deposit contract by offering higher payoff at date 1 or offering higher payoff at date 2 and investing more. Part (ii) is the no bank-run constraint, that is, payment to the impatient depositors cannot be strictly larger than payment to the patient depositors.

With Lemma 1, we can now simplify the banks' optimization problem in a competitive equilibrium as follows:

$$\max_{\{I,\, D_1^R,\, D_2^R,\, \beta_L\}} \theta[\lambda_H U(D_1^R)+(1-\lambda_H)U(D_2^R)]+(1-\theta)[\lambda_L U(D_1^R)+(1-\lambda_L)U(D_2^R)]$$

$$\begin{aligned} s.t.\ \ & \lambda_H D_1^R = 1-I \\ & (1-\lambda_H)D_2^R \le RI \\ & \lambda_L D_1^R \le (1-\beta_L)(1-I) \\ & (1-\lambda_L)D_2^R \le RI+\beta_L(1-I) \\ & D_1^R \le D_2^R \end{aligned}$$

where β_L is the proportion of real goods transferred from date 1 to date 2 in the normal state.

Proposition 2 characterizes the competitive equilibrium without money.

PROPOSITION 2: In a symmetric competitive equilibrium without money, (i) the equilibrium deposit contract is given by:

$$D_1^R = \frac{1-I_R^*}{\lambda_H},\ \ D_2^R = \frac{RI_R^*}{1-\lambda_L}+\frac{(1-I_R^*)-\lambda_L D_1^R}{1-\lambda_L}$$

where the equilibrium investment, I_R^*, is given by the solution of the following maximization problem:

$$\max_{\{I\}} \theta[\lambda_H U(D_1^R)+(1-\lambda_H)U(D_2^R)]+(1-\theta)[\lambda_L U(D_1^R)+(1-\lambda_L)U(D_2^R)]$$

$$s.t.\ I \geq \frac{1-\lambda_H}{\lambda_H R + 1 - \lambda_H},\ D_1^R = \frac{1-I}{\lambda_H},\ D_2^R = \frac{RI}{1-\lambda_L} + \frac{(1-I) - \lambda_L D_1^R}{1-\lambda_L}$$

and I_R^* is lower than the social optimal level, I_{FB}^*, and (ii) banks make zero profits in the normal state but make positive profits in the liquidity shortage state.

PROOF: See Appendix.■

The intuition for positive bank profits is as follows. Banks leave abundant real goods in the vault to provide enough liquidity when there is a systemic liquidity shock, however, if the liquidity shock does not occur, there are more depositors withdraw at date 2 and the extra goods saved at date 1 become inefficient ex-post. It is easy to see that, the amount of total real goods banks pay to depositors in the liquidity shortage state, $\lambda_H D_1^R + (1-\lambda_H) D_2^R$, is strictly smaller than that in the normal state, $\lambda_L D_1^R + (1-\lambda_L) D_2^R$, as $D_1^R \leq D_2^R$ and $\lambda_L < \lambda_H$. Therefore, banks can only make profits in the liquidity shortage state.

Besides under-investment, bank profits further reduce the welfare of depositors. In the next section, we introduce money into the economy and the price mechanism makes the real consumptions of depositors contingent on the systemic liquidity shock, and this can reduce bank profits and change the equilibrium investment level. However, allowing for nominal deposit contracts also brings over-competition among banks, which reduces the welfare of depositors.

IV. Competitive Equilibrium with Money

A. Definition and Characterization of Competitive Equilibrium

By saving one unit of money in a bank at date 0, depositors can get D_1^N units of money if they withdraw at date 1, and they will get D_2^N units of money if they withdraw at date 2. Superscript N represents the economy with money (“nominal”). Denote the real-goods consumption bundle in the economy with money as $\{C_1^{HN}, C_2^{HN}, C_1^{LN}, C_2^{LN}\}$ where the consumption of the impatient depositors in the liquidity shortage state is C_1^{HN}, in the normal state is C_1^{LN}, and the consumption of the patient depositors in the liquidity shortage state is C_2^{HN}, in the normal stateis C_2^{LN}. Denote the prices of the real goods in the economy as $\{P_0, P_1^H, P_1^L, P_2^H, P_2^L\}$, where P_0 denotes the price of real goods at date 0, and P_m^j denotes the price of real goods in the λ_j case at date m with $j = \{H, L\}$ and $m = \{1,2\}$.

DEFINITION 2 (Competitive Equilibrium with Money): A competitive equilibrium with money is a collection of nominal deposit contract, $\{D_1^R, D_2^R\}$, real-goods consumptions, $\{C_1^{HN}, C_2^{HN}, C_1^{LN}, C_2^{LN}\}$, and prices of the real goods, $\{P_0, P_1^H, P_1^L, P_2^H, P_2^L\}$, such that, (i) the deposit contract is feasible, that is, given the set of prices, banks are solvent in both the normal state and the liquidity shortage state; (ii) no banks have incentive to deviate to any other feasible deposit contract; (iii) prices make the goods market clear at each date; (iv) no depositors have incentive to change the amount of their deposits at date 0.

Before we characterize the equilibrium, we first derive some features of the competitive equilibrium. We first claim that the price at date 1 will never be lower than that at date 2. Otherwise, the patient depositors will always hold the real goods to date 2 and no one will sell real goods at date 1, and cash will be useless at date 1, which cannot be an equilibrium. We also know that as in Lemma 1 no bank-run constraint needs to be satisfied in a competitive equilibrium. We summarize these two results in Lemma 2 below.

LEMMA 2: In any competitive equilibrium, we have (i) the price at date 1 will never be lower than that at date 2, that is, $P_1^j \geq P_2^j$, $j = \{H, L\}$; and (ii) $D_1^N \leq D_2^N$.

PROOF: Omitted. ■

In this paper, in order to address our main idea more clearly, we focus on the equilibrium in which $P_1^j > P_2^j$ and $D_1^N < D_2^N$. The condition $D_1^N < D_2^N$ is consistent with the fact that depositors suffer some loss they withdraw prematurely. The condition $P_1^j > P_2^j$ is consistent with the fact that the price first significantly goes up during each recession with liquidity shortage, and then goes down at the end of recession or after recession. Figure 1 shows the percentage change of CPI in the U.S. from January, 1970 to November, 2015.

[Insert Figure 1 Here]

Moreover, in any equilibrium with $P_1^H > P_2^H$ and $P_1^L > P_2^L$, our equilibrium characterization is much easier, and in any such equilibrium, the patient depositors will sell all of their goods at date 1 at a higher price, and the impatient depositors will consume all the real goods that were not invested. We summarize this result in the lemma below.

LEMMA 3: In any competitive equilibrium with $P_1^H > P_2^H$ and $P_1^L > P_2^L$, the impatient depositors will consume all the real goods that were not invested.

PROOF: Omitted.■

Lemma 4 characterizes the equilibrium prices as a function of equilibrium investment, real consumptions, deposit contract, and date-0 deposit.

LEMMA 4: Given banks' investment I, date-0 deposit y, deposit contract D_1^N and D_2^N, and real consumptions $\{C_1^{HN}, C_2^{HN}, C_1^{LN}, C_2^{LN}\}$, define $f_\varepsilon^H(I) = \frac{U'(C_1^{HN})(1-\lambda_H)(1-I)}{\lambda_H(1+P_0 I - y + D_1^N y)}$ and $f_\varepsilon^L(I) = \frac{U'(C_1^{LN})(1-\lambda_L)(1-I)}{\lambda_L(1+P_0 I - y + D_1^N y)}$, and the equilibrium prices are given by:

(i) At date 1, if $\varepsilon > f_\varepsilon^H(I)$, $P_1^H = \frac{U'(C_1^{HN})}{\varepsilon}$, and if $\varepsilon \le f_\varepsilon^H(I)$, $P_1^H = \frac{\lambda_H(1+P_0 I - y + D_1^N y)}{(1-\lambda_H)(1-I)} \le \frac{U'(C_1^{HN})}{\varepsilon}$; if $\varepsilon > f_\varepsilon^L(I)$, $P_1^L = \frac{U'(C_1^{LN})}{\varepsilon}$, and if $\varepsilon \le f_\varepsilon^L(I)$, $P_1^L = \frac{\lambda_L(1+P_0 I - y + D_1^N y)}{(1-\lambda_L)(1-I)} \le \frac{U'(C_1^{LN})}{\varepsilon}$;

(ii) At date 2, $P_2^j = \frac{U'\left(C_2^{jN}\right)}{\varepsilon}$, $j = \{H, L\}$;

(iii) At date 0, P_0 solves the following equation

$$\theta[\lambda_H U'(C_1^{HN})\frac{D_1^N}{P_1^H} + (1-\lambda_H)U'(C_2^{HN})\frac{D_2^N}{P_2^H}] + (1-\theta)[\lambda_L U'(C_1^{LN})\frac{D_1^N}{P_1^L} + (1-\lambda_L)U'(C_2^{LN})\frac{D_2^N}{P_2^L}]$$

$$= \theta[\lambda_H U'(C_1^{HN})\frac{1}{P_0} + (1-\lambda_H)U'(C_2^{HN})\frac{1}{P_0}\cdot\frac{P_1^H}{P_2^H}] + (1-\theta)[\lambda_L U'(C_1^{LN})\frac{1}{P_0} + (1-\lambda_L)U'(C_2^{LN})\frac{1}{P_0}\cdot\frac{P_1^L}{P_2^L}].$$

PROOF: See Appendix.■

The economic intuitions for Lemma 4 are as follows. Depositors trade in the market to balance between the real goods consumption and the cash holding. At date 1, when the value of money, ε, is high, the impatient depositors will not spend all their money and the marginal utility from real goods consumption is equal to the marginal utility from cash holding, and price equals the impatient depositors' marginal utility of real goods consumption; when the value of money, ε, is low, the impatient depositors will spend all their money, and price is given by the binding cash-in-advance constraint. However, at date 2, the patient depositors will never spend all their money as banks do not hold cash at the ending of date 2, and the price always equals the patient depositors' marginal utility of consumption goods. Date 0 price is solved from market clearance condition, and at this price depositors want to sell the amount of real goods that banks want to buy.

LEMMA 5: Generically, in any competitive equilibrium, depositors either deposit all of their money in banks or don't deposit any money in banks at date 0.

PROOF: See Appendix. ■

By Lemma 5, we know that, for any regular equilibrium in which depositors save at least some money in the bank, there is no money left in the depositors' hands at date 0. At the beginning of date 1, each bank has 1 unit of cash in the vault after spending some money buying the real goods from depositors for investment, but each depositor owns more than 1 unit of deposit as he has also deposited the proceeds from selling the real goods to banks in the market.

To further simplify our equilibrium characterization, we will focus on ε's range where there is a unique equilibrium with $P_1^j > P_2^j$, $j = \{H, L\}$, and $D_1^N < D_2^N$. Denote the equilibrium investment in the economy with money as I^E, and it is easy to check that the amount of deposit for each depositor at date 0 is then $y = 1 + P_0 I^E$. We now characterize the equilibrium deposit contract, real consumption, investment level and bank profits in Proposition 3.

PROPOSITION 3: There exists two cutoff values of ε, $\overline{\varepsilon}$ and $\underline{\varepsilon}$. For any ε in the range $[\underline{\varepsilon}, \overline{\varepsilon}]$, there is a unique equilibrium with $P_1^j > P_2^j$, $j = \{H, L\}$, $P_2^H > P_2^L$, and $D_1^N < D_2^N$, and we have in equilibrium:

(i) Banks offer deposit contract with $D_1^N = \frac{1}{\lambda_H y}$ and $D_2^N = \frac{P_2^L R I^E + 1 - \lambda_L/\lambda_H}{(1-\lambda_L)y}$;

(ii) The equilibrium consumptions satisfy: $C_1^{HN} = (1 - I^E)/\lambda_H$, $C_1^{LN} = (1 - I^E)/\lambda_L$, $C_2^{LN} = RI^E/(1 - \lambda_L)$, and C_2^{HN} being the solution of following equation:

$$U'(C_2^{HN})(1-\lambda_H)C_2^{HN} = \frac{1-\lambda_H}{1-\lambda_L}\left[U'\left(\frac{RI^E}{1-\lambda_L}\right)RI^E + \left(1-\frac{\lambda_L}{\lambda_H}\right)\varepsilon\right];$$

(iii) There are two exogenous cutoff values, ε^H and ε^L, corresponding to $f_\varepsilon^H(I^E)$ and $f_\varepsilon^L(I^E)$ respectively, and we have:

when $\varepsilon^H \leq \varepsilon \leq \bar{\varepsilon}$, equilibrium investment I^E is the solution of the following equation:

$$\left[\theta\frac{1-\lambda_H}{1-\lambda_L}+(1-\theta)\right]P_2^L R=\left[\theta\frac{1-\lambda_H}{1-\lambda_L}\left(1-\frac{\lambda_L}{\lambda_H}\right)+1\right]P_0;$$

when $\varepsilon^L \leq \varepsilon < \varepsilon^H$, I^E is the solution of the following equation:

$$\left[\theta\frac{1-\lambda_H}{1-\lambda_L}+(1-\theta)\right]P_2^L R=\left[\theta\frac{1-\lambda_H}{1-\lambda_L}\left(1-\frac{\lambda_L}{\lambda_H}\right)+\theta U'\left(\frac{1-I}{\lambda_H}\right)\frac{1}{P_1^H\varepsilon}+(1-\theta)\right]P_0;$$

when $\underline{\varepsilon} \leq \varepsilon < \varepsilon^L$, I^E is the solution of the following equation:

$$\left[\theta\frac{1-\lambda_H}{1-\lambda_L}+(1-\theta)\right]P_2^L R=\left\{\begin{matrix}\theta\frac{1-\lambda_H}{1-\lambda_L}\left(1-\frac{\lambda_L}{\lambda_H}\right)+\theta U'\left(\frac{1-I}{\lambda_H}\right)\frac{1}{P_1^H\varepsilon}+\\(1-\theta)\left[\frac{\lambda_L}{\lambda_H}U'\left(\frac{1-I}{\lambda_L}\right)\frac{1}{P_1^L\varepsilon}+\left(1-\frac{\lambda_L}{\lambda_H}\right)\right]\end{matrix}\right\}P_0;$$

(iv) Banks make zero profits in the normal state, and the banks' profit in the liquidity shortage state is given by:

$$\pi_B^H = RI^E-\frac{1-\lambda_H}{1-\lambda_L}\cdot\frac{P_2^L RI^E+1-\lambda_L/\lambda_H}{P_2^H}.$$

PROOF: See Appendix.■

One of the important results in Proposition 3 is that, the price at date 2 in the liquidity shortage state is higher than that in the normal case. The reason is that, the prices at date 2 are proportional to the marginal utility of the patient depositors, which is negatively related to the patient depositor's real goods consumption. There are fewer patient depositors in the liquidity shortage state, which increases the consumption for each patient depositor. However, banks make positive profits in the liquidity shortage state, and the total amount of real goods consumed by the patient depositors is smaller than that in the normal state, which decreases the consumption of each patient depositor. We show that when the relative risk aversion of depositors is greater than 1, the second effect is dominant, and the consumption for each patient depositor is lower in the liquidity shortage state. Here the banks' profits can be understood as an insurance fee paid by depositors for the preference shock. Therefore, if depositors are more risk averse, the insurance premium is higher and banks make more profits, which leads to a lower consumption for each patient depositor in the liquidity shortage state.

As we can see from Proposition 3 that, with money being introduced into the economy, depositors' real goods consumption are state-contingent as price adjust with the liquidity shock, and banks make profits (in real goods) in equilibrium at the cost of the depositors' consumptions. However, in the lemma below, we can actually show that, for any level of investment, we can always find a deposit contract such that banks make zero profits.

LEMMA 6: Given the equilibrium prices and investment level, there exists a deposit contract such that banks make zero profits.

PROOF: See appendix.■

The zero-profit contract in Lemma 6 will be used as a benchmark to demonstrate the over-competition problem which is addressed in the next proposition.

PROPOSITION 4 (Over-competition): In a competitive equilibrium, banks pay all the money in the vault to the impatient depositors at date 1 in the liquidity shortage state. Compared with the corresponding zero-profit deposit contract, the equilibrium deposit contract pays too much at date 1 but too little at date 2.

PROOF: See Appendix.■

With a zero profit deposit contract offered by banks, depositors will consume all the real goods. However, the zero profit deposit contract cannot be the equilibrium outcome, as with such a contract banks have some extra money left in the vault after making the payment at date 1 in both the normal state and the liquidity shortage state, and a bank can deviate by either paying more at date 1 or increasing investment at date 0 and offering a higher payment at date 2. Therefore, competition drives banks to pay all of their cash at date 1 to the impatient depositors in the liquidity shortage state. However, given the investment level, paying more money to the impatient depositors at date 1 will not get them consume more real goods at date 1, while the patient depositors getting paid less at date 2. Therefore, the over-competition allocates some real goods to banks at the cost of the patient depositors. If banks could pay less to the impatient depositors (the early-withdrawal actions), and pay more to the patient depositors, depositors will be better off ex-ante.

Before compare the nominal-cash economy and real-goods economy in this paper, we state some comparative statics results regarding the relationship between equilibrium investment and the value of money.

PROPOSITION 5: In the range of $\underline{\varepsilon} \leq \varepsilon \leq \bar{\varepsilon}$, the equilibrium investment is increasing with the value of money, and there is under-investment when the value of money is low.

PROOF: See Appendix.■

Proposition 5 tells us that with the value of money gets higher, equilibrium investment also increases. With a low value of money, the impatient depositors' have little incentive to keep some cash in hand at the end of date 1, and they demand more money to buy real goods at date 1. As a result, at date 0, banks will reduce his investment and hold more cash at date 1 to meet this demand.

However, there could be over-investment when the value of money is high. When the value of money is high, the impatient depositors do not spend all their money to buy real goods at date 1, and investing more will allow banks to pay more to the patient depositors with a relatively small cut of the payment to the impatient depositors.

B. Welfare Comparison

We now compare the depositors' welfare obtained from real goods consumption in the economy with money with that in the economy without money. Besides money brings over-competition as we discussed above, similar to Diamond and Rajan (2006), depositors get state-contingent real goods consumptions with the price adjustment while their nominal deposit payments are not state-contingent. However, the welfare effect of the price adjustment is uncertain.

First of all, introducing money into the economy can smooth the consumption of depositors across preference shocks, and thus improve their welfare. Social optimal allocation requires that the impatient depositors consume all the real goods at date 1 both in the liquidity shortage state and the normal state. But in the economy without money, real deposit contract cannot achieve state-contingent consumption for depositors, and in the normal state, some un-invested real goods at date 1 are saved and paid to the patient depositors at date 2. Therefore, in the normal state, the patient depositors consume too much at date 2 while the impatient depositors consume too little at date 1. After introducing money, impatient depositors can consume all the un-invested real goods even in the normal state, and the consumptions of the patient depositors and the impatient depositors are more balanced in the economy with money, which increases depositors' expected utility at date 0. This is a redistribution effect between patient and impatient depositors and this effect is positive.

At the same time, introducing money into the economy may change the real goods allocation between depositors and banks. Banks make zero profits in the normal state, however, there is possibility that banks' profits in the liquidity shortage state may be higher in the economy with money. On the one hand, as the price at date 2 is higher in the liquidity shortage state than in the normal state, banks can get higher income from selling investment output, which leads to a higher profit for banks. With the zero-profit deposit contract, banks always have zero profit, but with the competition among banks, the zero-profit deposit contract cannot sustain in a competitive equilibrium, and banks make positive profits.

The redistribution effect has two components, redistribution effect between the impatient depositors and the patient ones, and that between depositors and banks. The first component is welfare improving for depositors, but the effect of the latter one is could be negative for the welfare of depositors due to competition among banks. The net effect of money on the depositors' welfare may be positive or negative depending on the utility function and parameter value, and we will analyze these effects with some specific form of utility function.

Next we will use a specific utility function form to help us characterize the effect of money on the depositors' welfare derived from real goods consumptions.

PROPOSITION 6: When λ_H and λ_L are close enough to each other and the utility function of the depositors is $U(C) = T - C^{-1}$, money can improve the depositors' welfare obtained from real-goods consumptions if and only if

$$\frac{\lambda_L[R(1-\theta)-1][(\sqrt{R}-1)\lambda_L+1]}{R^{3/2}} - \theta\varepsilon > 0.$$

PROOF: See Appendix.■

The intuition for the condition in the above proposition is as follows. When the liquidity shortage state is more likely to appear, that is, θ is larger, the probability of banks making positive profits is higher as banks make positive profits only in the liquidity shortage state, and this leads to a lower expected payoff to the depositors. When money is more valuable, the equilibrium investment is higher. Banks have more profits with higher equilibrium investment as this leaves more income from investment and thus more profit for banks at date 2 in the liquidity shortage state. Therefore, introducing money is more likely to increase banks' profits and lower the depositors' welfare. When the investment return, R, is higher, the gap between the impatient depositors' consumption and the patient depositors' consumption is lager, the consumption smoothing effect across preference shocks of the price mechanism is larger, and money has a positive welfare effect.

For a clearer welfare comparison, we give a numerical example of our model. Let λ_H=0.16, λ_L=0.155, θ=0.07, R=1.15, ε=0.137 and $U(C) = 1 - C^{-1}$.

The social optimal investment, I^*_{FB}, is 0.83525, and the expected utility of the depositors is 0.11076. The equilibrium investment in the economy without money, I^*_R, is 0.83055, which is lower than the social optimal investment, and there is an under-investment problem. Banks' profit in the liquidity shortage state is 0.0004 units of goods, the expected utility of the depositors is 0.11028, which is worse than that from the social optimal allocation.

In the economy with money, the equilibrium investment, I^E, is 0.83530, which is higher than the social optimal investment, and there is an over-investment problem. Banks' profit in the liquidity shortage state is 0.0218 units of real goods, and the expected utility of the depositors is 0.11016, which is lower than the expected utility in real goods economy. Therefore, introducing money into the economy pushes up the investment, but banks' profit in the liquidity shortage state goes up significantly, and the welfare of the depositors goes down. The prices in equilibrium are P_0=6.575, P_1^H=6.991, P_1^L=6.561, P_2^H=5.797 and P_2^L=5.732. The equilibrium contracts are D_1^N=0.963 and D_2^N=1.010, and the total expected nominal payoff $E[D^N]$=1.0025.

V. Monetary Policy

A. Marginal Operation of Monetary Policy

After discussing money's positive and negative role, a natural question is whether we can use monetary policy to improve the welfare of the depositors? Here we focus on the most common tool of monetary policy, open market operation. We start the discussion of monetary policy by analyzing marginal operation. Marginal operation means that the amount and the interest rate of open market operation are both marginal. Under marginal monetary policy operation, the central bank also acts as a price taker, and the prices in the economy will not be influenced. The following proposition illustrates that what kind of marginal operation can improve depositors' welfare.

PROPOSITION 7: The central bank can improve the depositors' welfare derived from real-goods consumption with either one of the following marginal operations:

(i) In the liquidity shortage state, the central bank provides liquidity to banks through reverse repo operation at date 1, and banks pay back the principal and interest to the central bank at date 2;

(ii) In the normal state, the central bank extracts liquidity from banks through repo operation at date 1, and pay the principal and interest to banks at date 2.

Moreover, the operation in the liquidity shortage state is more effective.

PROOF: See Appendix.■

The monetary policy in the above proposition is consistent with the usual monetary policies in reality. In the liquidity shortage state, banks' liquidity at date 1 improves while their profit at date 2 reduces. In the normal state, banks' liquidity at date 1 reduces while their liquidity at date 2 increases, which relaxes their binding constraint at date 2, and competition will drive banks to offer higher deposit payments to the depositors, which reduces the banks' profits.

The marginal monetary policy operation in the liquidity shortage state has two effects to improve depositors' welfare. Firstly, banks need to pay off the loan from the central bank at date 2 in the liquidity shortage state, which reduces banks' profit. Secondly, banks have more liquidity at date 1in the liquidity shortage state, which allows banks offer a higher date 1 payment to the depositors, and this increases the bank liquidity in the normal state at date 2 and drives up the date 2 payment to the depositors while driving down the banks' profits. The marginal monetary policy operation in the normal state increases banks' liquidity at date 2 in the normal state so it only has the second effect.

If the central bank conducts open market operations only in the liquidity shortage state, the central bank has a net cash income. If the central bank conducts open market operations only in the normal state, the central bank has a net cash payment. Central bank can operate in both states to keep the expected money growth being zero.

B. Optimal Monetary Policy

Though marginal monetary operation can always improve the depositors' welfare derived from real goods consumptions, but it may not result in the social optimal allocation. In this section, we characterize the optimal monetary policy that leads to the social optimal policy.

PROPOSITION 8: The optimal monetary policy that leads to the social optimal allocation must have the following features:

(i) The central bank issues a bond to banks at date 1 and repay the bond at date 2 in the liquidity shortage state, and the central bank does nothing in the normal state;

(ii) The bond interest rate i_B in optimal monetary policyis the solution of the following equation.

$$\theta U'\left(\frac{RI_{FB}^*}{1-\lambda_H}\right)R+(1-\theta)U'\left(\frac{RI_{FB}^*}{1-\lambda_L}\right)R=(1-\theta)\widehat{P_0}\varepsilon+\theta\left[\widehat{P_0}(1+i_B)+\lambda_H i_B\hat{y}\cdot x_1\right]\varepsilon$$

and x_1 is the solution of the following equation system.

$$\begin{cases}\left[\lambda_H x_1+(1-\lambda_H)x_2\right]\hat{y}=-\widehat{P_0}(1+i_B)+\widehat{P_2^H}R-\lambda_H i_B\hat{y}\cdot x_1\\ \left[\lambda_L x_1+(1-\lambda_L)x_2\right]\hat{y}=-\widehat{P_0}+\widehat{P_2^L}R\end{cases}$$

where I_{FB}^* is the social optimal investment, $\widehat{P_0}$, $\widehat{P_2^H}$, and $\widehat{P_2^L}$ are the equilibrium prices under the social optimal investment, and $\hat{y}$ is the total nominal deposit at date 0, that is, $\hat{y}=1+\widehat{P_0}I_{FB}^*$.

PROOF: See Appendix.■

Notice that if the central bank wants to implement a monetary policy which can promote the economy to the social optimal allocation, the direction of monetary policy operation is different from the usually one in reality. In the competitive equilibrium without central bank intervention, as banks make positive profits in the liquidity shortage state and the impatient depositors consume the same amount of un-invested real goods in total at date 1 in both states, the patient depositors consume less real goods in total at date 2 in the liquidity shortage state than in the normal state. At date 2, we must have $P_2^H > P_2^L$ as the patient depositors own more money with more impatient depositors buying goods at date 1 while the real goods consumption is smaller in the liquidity shortage state. Under the social optimal allocation, banks make zero profits. Since the number of patient depositors is larger in the normal state, then each patient depositor consumes less real goods and the marginal utility of goods consumption is higher in the normal state, then we must have $\widehat{P_2^H} < \widehat{P_2^L}$. Therefore, under the social optimal allocation, banks have higher income from selling the investment output at date 2 in the normal state than in the liquidity shortage state, which is the opposite to the outcome of a competitive equilibrium. To achieve the social optimal allocation, the central bank needs to extract liquidity from banks at date 1 and give back liquidity to banks at date 2 in the liquidity shortage state.

Another important difference between the optimal monetary policy and the marginal monetary policy is that now the central bank can only operate in the liquidity shortage state. Operation in the normal state is invalid. If the central bank lend to banks at date 1 and banks pay back loans at date 2 in the normal state, banks have more liquidity at date 1 in the normal state, but banks cannot increase their date-1 payment, D_1^N, to the impatient depositors as the date-1 budget constraint for the banks will be violated in the liquidity shortage state. So banks cannot benefit from borrowing from the central bank in the normal state.

The interest rate of central bank intervention is derived from the banks' no-deviation condition. When the monetary policy operation is marginal, banks don't have incentive to deviate to change investment. The optimal monetary policy is not a marginal operation, and banks may have incentive to deviate. If the interest rate of bond is too high, banks may want to cut investment and hold more cash to buy the high interest rate bond at date 1. If the interest rate of bond is too low, banks may want to increase investment and borrow money from the central bank at date 1. With the interest rate given by proposition 8, banks have no incentive to deviate.

We now give a necessary and sufficient condition under which the social optimal allocation can be achieved with the optimal monetary policy stated in proposition 9.

PROPOSITION 9: Under the optimal monetary policy, banks will offer a deposit contract $\{\widehat{D_1^N}, \widehat{D_2^N}\}$ as follows:

$$\widehat{D_1^N} = \frac{i_B + \dfrac{\lambda_H - \lambda_L}{1-\lambda_L} - \dfrac{1-\lambda_H}{1-\lambda_L}\widehat{P_2^L}RI_{FB}^* + \widehat{P_2^H}RI_{FB}^*}{\left(\dfrac{\lambda_H - \lambda_L}{1-\lambda_L} + \lambda_H i_B\right)\hat{y}}, \quad \widehat{D_2^N} = \frac{\widehat{P_2^L}RI_{FB}^* + 1 - \lambda_L \widehat{D_1^N}\hat{y}}{(1-\lambda_L)\hat{y}}$$

where I_{FB}^* is the social optimal investment, $\widehat{P_0}$, $\widehat{P_2^H}$, and $\widehat{P_2^L}$ are the equilibrium prices under the social optimal investment, and$\hat{y}$is the total nominal deposit at date 0, that is, $\hat{y} = 1 + \widehat{P_0}I_{FB}^*$. The social optimal allocation can be achieved if and only if $\lambda_H\widehat{D_1^N}\hat{y} \geq \widehat{P_2^H}(1-\lambda_H)(1-I_{FB}^*)$ and $\lambda_L\widehat{D_1^N}\hat{y} \geq \widehat{P_2^L}(1-\lambda_L)(1-I_{FB}^*)$.

PROOF: See Appendix.■

To achieve the social optimal allocation, the optimal monetary policy needs to make the impatient depositors consume all the un-invested real goods, $1-I$. If $\widehat{D_1^N}$ is too small, the impatient depositors own too little money, and the patient depositors will not sell all their real goods to the impatient depositors. The two conditions in the above proposition ensures that $\widehat{D_1^N}$ is large enough for the impatient depositors to consume all the un-invested real goods in both the liquidity shortage state and the normal state.

The optimal monetary policy suppresses over-competition in the economy by extracting liquidity from the banking system at an attractive interest rate at date 1 in the liquidity shortage state, and it eliminates banks' profit and promotes the economy to the social optimal allocation. Now banks save some money at the central bank instead of paying all the money to the impatient depositors at date 1, and banks can pay more to the patient depositors at date 2.

One implication of the optimal monetary policy is that the aggregate money supply in the economy is growing. Since the central bank can only operate in the liquidity state, the expected money growth rate is positive. Another implication is that, of the yield curve is upward sloping as the interest rate between date 1 and date 2 is positive due to the central bank intervention in the liquidity shortage state.

Finally, we will analyze the optimal monetary policy with a numerical example. Using the same parameters as in the previous section, we can calculate now the equilibrium interest rate of the bond is 107.9%, which is very high. Banks will offer a deposit contract with $\widehat{D_1^N}$ = 0.908 and $\widehat{D_2^N}$ = 1.019. Depositors will deposit $\hat{y}$ =6.493. The equilibrium prices are $\widehat{P_0}$=6.577, $\widehat{P_1^H}$=6.811, $\widehat{P_1^L}$=6.559, $\widehat{P_2^H}$=5.666 and $\widehat{P_2^L}$=5.733. With $\lambda_H\widehat{D_1^N}\hat{y}$=0.943, $\lambda_L\widehat{D_1^N}\hat{y}$=0.913 and $\widehat{P_2^H}(1-\lambda_H)(1-I_{FB}^*)$ =0.784, $\widehat{P_2^L}(1-\lambda_L)(1-I_{FB}^*)$ =0.798, the conditions $\lambda_H\widehat{D_1^N}\hat{y} \geq \widehat{P_2^H}(1-\lambda_H)(1-I_{FB}^*)$ and $\lambda_L\widehat{D_1^N}\hat{y} \geq \widehat{P_2^L}(1-\lambda_L)(1-I_{FB}^*)$ are satisfied. So the social optimal allocation is achieved by such a monetary policy.

VI. Conclusion

This paper illustrates the role of money and monetary policy when banks are competing with each other given a systemic bank liquidity shock. We first show that depositors' welfare cannot be social optimal in the economy without money as the real deposit contract cannot be state-contingent. When we introduce money into the economy, the real goods consumptions become state-contingent as price adjusts, but there is also over-competition among banks which reduces the welfare of depositors. We then show that marginal monetary policy in the conventional direction can always improve depositors' welfare but cannot achieve the social optimal allocation. However, an unconventional monetary policy may be implemented to promote the economy to the social optimal allocation. Implementing the optimal monetary policy requires the central bank completely knows the parameters of the economy. However, marginal monetary policy can still always improve the depositors' welfare. Therefore, if the structure of the economy is not fully understood, maybe a more conservative marginal monetary policy in the conventional direction is appropriate.

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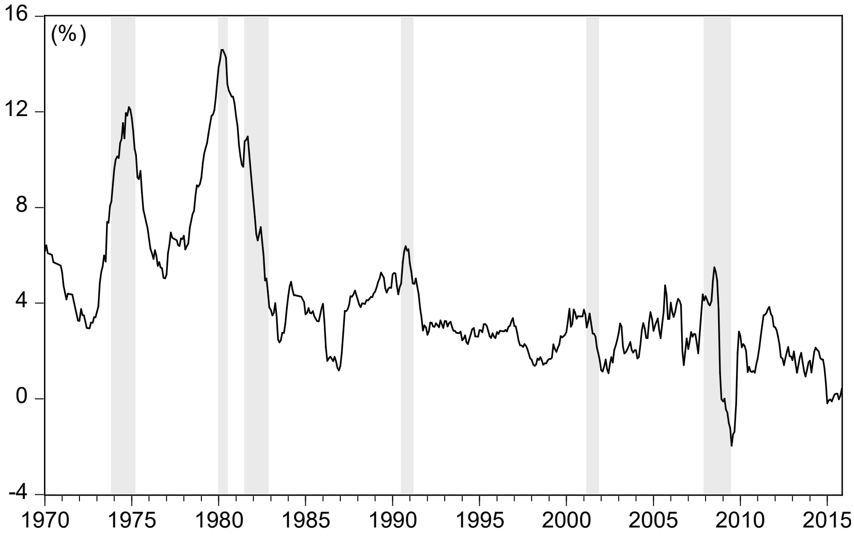

FIGURE 1. U.S. CONSUMER PRICE INDEX FOR ALL URBAN CONSUMERS, 1970-2015

SOURCE: US. BUREAU OF LABOR STATISTICS, FRED, NBER

Notes: The shadow areas represent recession periods of U.S. announced by NBER.

Appendix

PROOF OF PROPOSITION 1

Note that all the constraints must bind in the optimal solution. Therefore, we have

$C_1^H = \frac{(1-\beta_H)(1-I)}{\lambda_H}$, $C_2^H = \frac{RI+\beta_H(1-I)}{1-\lambda_H}$, $C_1^L = \frac{(1-\beta_L)(1-I)}{\lambda_L}$, $C_2^L = \frac{RI+\beta_L(1-I)}{1-\lambda_L}$

Substituting the above equations into the optimization problem, we can derive the first order conditions with respect to *I*, β_H and β_L:

For *I*:

(A1)
$$\theta\left[-(1-\beta_H)U'\left(\frac{(1-\beta_H)(1-I)}{\lambda_H}\right)+(R-\beta_H)U'\left(\frac{RI+\beta_H(1-I)}{1-\lambda_H}\right)\right]+$$
$$(1-\theta)\left[-(1-\beta_L)U'\left(\frac{(1-\beta_L)(1-I)}{\lambda_L}\right)+(R-\beta_L)U'\left(\frac{RI+\beta_L(1-I)}{1-\lambda_L}\right)\right]=0$$

For β_H:

(A2)
$$\theta\left[-(1-I)U'\left(\frac{(1-\beta_H)(1-I)}{\lambda_H}\right)+(1-I)U'\left(\frac{RI+\beta_H(1-I)}{1-\lambda_H}\right)\right]=0$$

For β_L:

(A3)
$$(1-\theta)\left[-(1-I)U'\left(\frac{(1-\beta_L)(1-I)}{\lambda_L}\right)+(1-I)U'\left(\frac{RI+\beta_L(1-I)}{1-\lambda_L}\right)\right]=0$$

With the strict monotonicity of $U'(\cdot)$, (A2) and (A3) imply:

(A4)
$$\beta_j = \frac{(1-I)(1-\lambda_j)-\lambda_j RI}{1-I}, \; j=\{H,L\}$$

which is subject to the constraint $\beta_j \geq 0$ due to non-transferability from date 2 to date 1.

As β_j is decreasing in λ_j, we have $\beta_H < \beta_L$ and the following results:

A. When $0 < \beta_H < \beta_L$, we have $I < (1-\lambda_H)/(\lambda_H R + 1 - \lambda_H) \equiv \overline{I}_H$. With (A4), we know $C_1^H = C_2^H = RI + 1 - I$ and $C_1^L = C_2^L = RI + 1 - I$. At the investment level $\overline{I}_H$, $C_1^H = C_2^H = RI + 1 - I$ and $C_1^L = C_2^L = RI + 1 - I$. But since $R > 1$, C_1^H, C_2^H, C_1^L, C_2^L are all strictly greater than the consumption under all investment level $I < \overline{I}_H$. So any $I < \overline{I}_H$ will never be the optimal investment.

B. When $0 = \beta_H < \beta_L$, we have $\overline{I}_H \le I < (1-\lambda_L)/(\lambda_L R + 1 - \lambda_L) \equiv \overline{I}_L$. In the λ_L case, $C_1^L = C_2^L = RI + 1 - I$ holds. In the λ_H case, there is no transfer between two dates[1]. When $\overline{I}_H \le I < \overline{I}_L$, directly allocating all the goods at date 1 to the impatient depositors and all the goods at date 2 to the patient depositors implies $C_1^H < C_2^H$. Therefore, transferring goods from date 1 to date 2 will expand the gap between C_1^H and C_2^H, which makes the consumptions less smoothed. So we must have $C_1^H = (1-I)/\lambda_H$ and $C_2^H = RI/(1-\lambda_H)$.

C. When $\beta_H = \beta_L = 0$, we have $\overline{I}_L \le I < 1$, by analogy, both in the λ_H case and the λ_L case, any real goods transfer between two dates would make consumptions less smoothed. So the optimal allocation is $C_1^H = (1-I)/\lambda_H$, $C_2^H = RI/(1-\lambda_H), C_1^L = (1-I)/\lambda_L, C_2^L = RI/(1-\lambda_L)$.

Therefore, the optimal investment satisfies either $\overline{I}_H \le I < \overline{I}_L$ or $\overline{I}_L \le I < 1$, whichever gives a higher expected utility of the depositors ex ante.

PROOF OF LEMMA 1

[1] For simplicity, we also call the liquidity shortage state as "the λ_H case" and the normal state as "the λ_L case" in the proof hereafter.

(i) First, $\lambda_H D_1^R \le 1 - I$ must be satisfied to make the contract feasible. Now we consider a contract where there are some remaining goods under the λ_H case at date 1 ($\lambda_H D_1^R < 1 - I$). As $\lambda_L < \lambda_H$, there will also be some goods left under the λ_L case ($\lambda_L D_1^R < 1 - I$). A bank can increase investment from I to $I + \Delta I$ with the solvency requirement still satisfied. This bank can keep D_1^R unchanged but with a higher D_2^R. Then the real consumption at date 1, C_1^{HR} and C_1^{LR} remain the same and the real consumption at date 2, C_2^{HR} and C_2^{LR} are strictly higher. Thus any contract with $\lambda_H D_1^R < 1 - I$ cannot be optimal.

(ii) For any contract with $D_1^R > D_2^R$, a patient depositor can deviate by withdrawing his deposits at date 1 and hold the real goods to date 2 then consume them, he will be strictly better off comparing with withdrawing his deposit at date 2. So there is an incentive to deviate and bank run happens. This cannot be an equilibrium.

PROOF OF PROPOSITION 2

We prove this proposition by two steps. In step 1, we prove banks always make zero profits in the normal state and characterize the equilibrium investment. In step 2, we show the equilibrium investment is lower than the social optimal level.

Step 1: from the banks' optimization problem, we can get the banks' budget constraints for date 2's payment as follows.

$$D_2^R \le \frac{RI}{1-\lambda_H} \quad \text{and} \quad D_2^R \le \frac{RI}{1-\lambda_L} + \frac{(1-I) - \lambda_L D_1^R}{1-\lambda_L}$$

in the λ_H case and in the λ_L case, respectively. Combing with $D_1^R = (1-I)/\lambda_H$ obtained from Lemma 1, we know the second constraint is equivalent to:

(A5) $$D_2^R \leq \frac{RI}{1-\lambda_L} + \frac{(1-I)(1-\lambda_L/\lambda_H)}{1-\lambda_L}$$

These two constraints may not hold simultaneously. In the situation where banks' budget constraint is not binding, banks do not deliver all the goods to depositors and have profits.

The difference between $D_2^R \leq RI/(1-\lambda_H)$'s right-hand side and (A5)'s right-hand side is:

(A6) $$\frac{RI(\lambda_H - \lambda_L) - (1-\lambda_H)(1-I)(1-\lambda_L/\lambda_H)}{(1-\lambda_H)(1-\lambda_L)}$$

When $I \geq (1-\lambda_H)/(\lambda_H R + 1 - \lambda_H)$, we have (A6)$\geq 0$, which means the budget constraint in the λ_L case is tighter and payment at date 2 should be $D_2^R = \frac{RI}{1-\lambda_L} + \frac{(1-I)(1-\lambda_L/\lambda_H)}{1-\lambda_L}$. The constraint in the λ_H case does not bind in general, so banks have profits in the λ_H case.

When $I < (1-\lambda_H)/(\lambda_H R + 1 - \lambda_H)$, we have (A6)$<0$, which means the budget constraint in the λ_H case is tighter. So the payment at date 2 is $D_2^R = \frac{RI}{1-\lambda_H}$. Banks have profits in the λ_L case.

However, (A6) <0 will never occur in the equilibrium. Because when $I < (1-\lambda_H)/(\lambda_H R + 1 - \lambda_H)$, substituting I into $D_1^R = (1-I)/\lambda_H$ and $D_2^R = RI/(1-\lambda_H)$ leads to $D_1^R > D_2^R$, which contradicts with the no bank-run constraint $D_1^R \le D_2^R$. So this situation is precluded. Then the only possible situation in the economy without money is that the budget constraint in the λ_L case is tighter. Hence, the optimal investment and contract are determined by corresponding constraints.

Banks' profit in the liquidity shortage state is $1 - I + RI - \lambda_H D_1^R - (1-\lambda_H) D_2^R$, which can be further expressed as:

$$\frac{(\lambda_H - \lambda_L)[RI_R^* - \frac{1-\lambda_H}{\lambda_H}(1 - I_R^*)]}{1-\lambda_L}$$

Step 2: Now we compare the optimal investment in the economy without money (I_R^*) with the social optimal investment (I_{FB}^*). Depositors' utility in the liquidity shortage (λ_H) state ((A7)) and utility in the normal (λ_L) state ((A8)) can be written as:

(A7) $$\lambda_H U(D_1^R) + (1-\lambda_H) U(D_2^R)$$

(A8) $$\lambda_L U(D_1^R) + (1-\lambda_L) U(D_2^R)$$

Starting from I_{FB}^*, consider a marginal change of investment $\Delta I < 0$.

(a) After the change, the budget constraint in the λ_L case is still binding, so the total resources allocated to depositors in the λ_L case ($\lambda_L D_1^R + (1-\lambda_L) D_2^R$) remain the same. But at the point of social optimal investment, we have:

$$\frac{1 - I_{FB}^*}{\lambda_H} < \frac{1 - I_{FB}^*}{\lambda_L} < \frac{RI_{FB}^*}{1-\lambda_L} < \frac{RI_{FB}^*}{1-\lambda_L} + \frac{(1 - I_{FB}^*)(1 - \lambda_L / \lambda_H)}{1-\lambda_L}$$

which means the consumption in the λ_L case is no smoother than the social optimal case. The decrement of investment increases D_1^R and decreases D_2^R, which makes the consumption smoother and increase the utility of depositors.

(b) As for the utility change in the λ_H case in (A7), we know that the profit of banks in the liquidity shortage state decreases with the decrease of *I*, then a higher portion of the real goods are consumed by the patient depositors at date 2. Moreover, D_1^R increases and D_2^R decreases in this case, then the consumption is smoother.

Therefore, at the social optimal level of investment, the decrement of ΔI has two positive effects in the economy without money. So in the economy without money, at the point of social optimal investment, banks have incentive to decrease investment to attract depositors, and the equilibrium investment is less than the social optimal investment, that is, $I_R^* < I_{FB}^*$.

PROOF OF LEMMA 4

(i) Here we derive the price at date 1 in the liquidity shortage state, and then we can get prices at date 1 in the normal state by in a similar way.

At date 1, each impatient depositor withdraws $D_1^N y$ unit of cash from bank, buy some goods and get some residual utility from remaining cash to maximize his utility. The utility maximization problem is

$$\max_{\{d\}} U(d+1-I)+(1+P_0 I - y + D_1^N y - P_1^H d)\varepsilon \quad s.t.\ P_1^H d \le \frac{1}{\lambda_H}$$

where *d* denotes the demand for goods of this impatient depositor.

The Kuhn-Tucker conditions for this problem are:

$$U'(d+1-I)-P_1^H\varepsilon-\mu P_1^H=0$$
$$P_1^H d\le 1+P_0I-y+D_1^N y \quad \mu\ge 0$$
$$\mu[P_1^H d-(1+P_0I-y+D_1^N y)]=0$$

(a) First we know there are two cases where P_1^H has different forms according to the discussion of the Kuhn-Tucker conditions. If $P_1^H d<1+P_0I-y+D_1^N y$, then $\mu=0$ and $P_1^H=U'(d+1-I)/\varepsilon$. This case corresponds to the situation when the price of goods is relatively low, so this impatient depositor does not spend all his cash to buy goods and have some cash left. If $\mu>0$, then $P_1^H d=1+P_0I-y+D_1^N y$, and $P_1^H=\dfrac{1+P_0I-y+D_1^N y}{d}$. This case corresponds to the situation when this depositor uses all of his cash to buy goods. So the budget constraint of this impatient depositor is binding.

Total demand by all the impatient depositors is $\lambda_H d$. As for supply, the patient depositors want to sell all of their goods since $P_1^H>P_2^H$, so the total supply of goods is $(1-\lambda_H)(1-I)$. Market clearing condition is $\lambda_H d=(1-\lambda_H)(1-I)$, which implies $d=(1-\lambda_H)(1-I)/\lambda_H$. Substituting it into P_1^H, we have $P_1^H=U'(\dfrac{1-I}{\lambda_H})/\varepsilon=U'(C_1^{HN})/\varepsilon$ when impatient depositors use part of their cash to buy goods and keep some cash after purchase, and $P_1^H=\dfrac{\lambda_H(1+P_0I-y+D_1^N y)}{(1-\lambda_H)(1-I)}$ when impatient depositors use up all of their cash to buy goods.

(b) Then we find the ranges of the two cases. The Kuhn-Tucker conditions imply that, when ε is small, the price determined by $U'(d+1-I)/\varepsilon$ is large, and at this price $P_1^H d$ will exceed $1+P_0I-y+D_1^N y$, which means that the budget constraint of the impatient depositors will bind, and P_1^H turns to $\frac{\lambda_H(1+P_0I-y+D_1^N y)}{(1-\lambda_H)(1-I)}$. So there is a cutoff of ε given *I*, which we denote as $f_\varepsilon^H(I)$. When ε is greater than this cutoff value, price is relatively low and the impatient depositors have some cash left. Otherwise price is relatively high and impatient depositors will use up all their cash. At the cutoff point, we have:

$$U'(d+1-I)/\varepsilon = \frac{\lambda_H(1+P_0I-y+D_1^N y)}{(1-\lambda_H)(1-I)}$$

Solving this equation for $f_\varepsilon^H(I)$ and substituting *d* into it, we have:

$$f_\varepsilon^H(I) = \frac{U'(C_1^{HN})(1-\lambda_H)(1-I)}{\lambda_H(1+P_0I-y+D_1^N y)}$$

Therefore, when $\varepsilon > f_\varepsilon^H(I)$, $P_1^H = U'(C_1^{HN})/\varepsilon$, and when $\varepsilon \le f_\varepsilon^H(I)$, $P_1^H = \frac{\lambda_H(1+P_0I-y+D_1^N y)}{(1-\lambda_H)(1-I)}$.

(c) Finally we compare P_1^H with $U'(C_1^{HN})/\varepsilon$ when $\varepsilon \le f_\varepsilon^H(I)$. When $\varepsilon < f_\varepsilon^H(I)$, we have $\mu > 0$ and $U'(d+1-I) - P_1^H\varepsilon = \mu P_1^H > 0$, and substituting the value of *d*, we have $P_1^H < U'(\frac{1-I}{\lambda_H})/\varepsilon = U'(C_1^{HN})/\varepsilon$. At the cutoff point, $P_1^H = U'(C_1^{HN})/\varepsilon$. In summary, when $\varepsilon \le f_\varepsilon^H(I)$, we have $P_1^H \le U'(C_1^{HN})/\varepsilon$.

(ii) Now we derive the prices at date 2. Since banks sell goods and make payment of deposit contracts simultaneously, it is equivalent to the case that the patient depositors get cash corresponding to the amount of real goods sold by banks and use the cash to buy goods. Since the patient depositors hold some cash at the beginning of date 2, they will never run out of cash. Take the liquidity shortage state as example. Assume that in equilibrium each patient depositor consumes C_2^{HN}. These goods are sold by banks. The price P_2^H should clear the market with no deviation by depositors, and we have $P_2^H = U'(C_2^{HN}) / \varepsilon$. Then by analogy we know the price at date 2 in the normal state is $P_2^L = U'(C_2^{LN}) / \varepsilon$.

(iii) Price at date 0 is also determined by market clearing condition. P_0 is such that depositors just want to sell I amount of goods to banks at date 0.

Consider the benefit from selling more goods to get ΔM unit of extra cash. Depositors are either indifferent between saving and not saving or already saving all of their cash in banks and willing to save even more. With the extra money ΔM, depositors will save the cash in banks. Therefore, in the λ_H case, a patient depositor can get extra money of $\Delta M \cdot D_2^N$ and use the money to buy $\Delta M \cdot D_2^N / P_2^H$ unit of goods, and an impatient depositor gets extra money of $\Delta M \cdot D_1^N$ and spends the money at date 1 and gets $\Delta M \cdot D_1^N / P_1^H$ unit of goods. In the λ_L case, a patient depositor gets $\Delta M \cdot D_2^N / P_2^L$ goods and an impatient depositor gets $\Delta M \cdot D_1^N / P_1^L$ goods.

Consider the benefit from selling less goods with the cash income reduced by ΔM. Holding $\Delta M / P_0$ extra real goods, in the λ_H case, a patient depositor wants to sell all the $\Delta M / P_0$ goods at date 1 and then buy goods at date 2, so the benefit is $(\Delta M / P_0) \cdot (P_1^H / P_2^H)$, and an impatient depositor just consumes the extra goods, so the benefit is $(\Delta M / P_0) \cdot (P_1^H / P_1^H) = \Delta M / P_0$. In the λ_L case, the benefit for a patient depositor is $(\Delta M / P_0) \cdot (P_1^L / P_2^L)$, and the benefit for an impatient depositor is still $\Delta M / P_0$.

Price P_0 is such that depositors are indifferent between selling more and selling less, and we have:

$$\theta[\lambda_H\left(U(C_1^{HN}+\frac{\Delta M\cdot D_1^N}{P_1^H})-U(C_1^{HN})\right)+(1-\lambda_H)\left(U(C_2^{HN}+\frac{\Delta M\cdot D_2^N}{P_2^H})-U(C_2^{HN})\right)]+$$
$$(1-\theta)[\lambda_L\left(U(C_1^{LN}+\frac{\Delta M\cdot D_1^N}{P_1^L})-U(C_1^{LN})\right)+(1-\lambda_L)\left(U(C_2^{LN}+\frac{\Delta M\cdot D_2^N}{P_2^L})-U(C_2^{LN})\right)]$$
$$=\theta[\lambda_H\left(U(C_1^{HN}+\frac{\Delta M}{P_0})-U(C_1^{HN})\right)+(1-\lambda_H)\left(U(C_2^{HN}+\frac{\Delta M}{P_0}\cdot\frac{P_1^H}{P_2^H})-U(C_2^{HN})\right)]+$$
$$(1-\theta)[\lambda_L\left(U(C_1^{LN}+\frac{\Delta M}{P_0})-U(C_1^{LN})\right)+(1-\lambda_L)\left(U(C_2^{LN}+\frac{\Delta M}{P_0}\cdot\frac{P_1^L}{P_2^L})-U(C_2^{LN})\right)]$$

Dividing both sides by ΔM, and taking $\Delta M\to 0$, we have:

$$\theta\,[\lambda_H U'(C_1^{HN})\frac{D_1^N}{P_1^H}+(1-\lambda_H)U'(C_2^{HN})\frac{D_2^N}{P_2^H}]+$$
$$(1-\theta)\,[\lambda_L U'(C_1^{LN})\frac{D_1^N}{P_1^L}+(1-\lambda_L)U'(C_2^{LN})\frac{D_2^N}{P_2^L}]$$
$$=\theta\,[\lambda_H U'(C_1^{HN})\frac{1}{P_0}+(1-\lambda_H)U'(C_2^{HN})\frac{1}{P_0}\cdot\frac{P_1^H}{P_2^H}]+$$
$$(1-\theta)\,[\lambda_L U'(C_1^{LN})\frac{1}{P_0}+(1-\lambda_L)U'(C_2^{LN})\frac{1}{P_0}\cdot\frac{P_1^L}{P_2^L}]$$

PROOF OF LEMMA 5

Consider a depositor's deviation by saving an extra marginal unit of deposit, Δy. In the liquidity shortage state, the benefit of deviation is that if a depositor becomes impatient, he can get $\Delta y D_1^N$ and buy $\Delta y D_1^N / P_1^H$ extra unit of goods at date 1. If a depositor turns out to be patient, he can get $\Delta y D_2^N$ and buy $\Delta y D_2^N / P_2^H$ extra unit of goods at date 2.

Consider a depositor's deviation by reducing his saving by Δy. As he is indifferent of buying and selling goods at date 0, let's say he just keeps the cash in hand. In the liquidity shortage state, if he becomes impatient, he can buy $\Delta y / P_1^H$ extra unit of goods at date 1. If he becomes patient, he can buy $\Delta y / P_2^H$ extra unit of goods at date 2. Combining with the current consumption, we have the utility gain of saving more and saving less. By analogy we can derive the corresponding utility gain in the normal state.

Here if both D_1^N and D_2^N are smaller than 1, depositors will not save any cash in banks. If both D_1^N and D_2^N are greater than 1, depositors will definitely save all cash in banks because saving is strictly superior to not saving. Otherwise, by equalizing the marginal gain of increasing and decreasing saving, we have:

$$
\begin{aligned}
&\theta[\lambda_H\left(U(C_1^{HN}+\frac{\Delta y\cdot D_1^N}{P_1^H})-U(C_1^{HN})\right)+(1-\lambda_H)\left(U(C_2^{HN}+\frac{\Delta y\cdot D_2^N}{P_2^H})-U(C_2^{HN})\right)]+\\
&(1-\theta)[\lambda_L\left(U(C_1^{LN}+\frac{\Delta y\cdot D_1^N}{P_1^L})-U(C_1^{LN})\right)+(1-\lambda_L)\left(U(C_2^{LN}+\frac{\Delta y\cdot D_2^N}{P_2^L})-U(C_2^{LN})\right)]\\
&=\theta[\lambda_H\left(U(C_1^{HN}+\frac{\Delta y}{P_1^H})-U(C_1^{HN})\right)+(1-\lambda_H)\left(U(C_2^{HN}+\frac{\Delta y}{P_2^H})-U(C_2^{HN})\right)]+\\
&(1-\theta)[\lambda_L\left(U(C_1^{LN}+\frac{\Delta y}{P_1^L})-U(C_1^{LN})\right)+(1-\lambda_L)\left(U(C_2^{LN}+\frac{\Delta y}{P_2^L})-U(C_2^{LN})\right)]
\end{aligned}
$$

Divide both sides by Δy, and take $\Delta y \to 0$. Combining with some results in proposition 3, we have y determined by

$$
\text{(A9)}\quad
\begin{aligned}
&\theta\,[\lambda_H U'(C_1^{HN})\frac{D_1^N}{P_1^H}+(1-\lambda_H)D_2^N\varepsilon]+(1-\theta)\,[\lambda_L U'(C_1^{LN})\frac{D_1^N}{P_1^L}+(1-\lambda_L)D_2^N\varepsilon]\\
&=\theta\,[\lambda_H U'(C_1^{HN})\frac{1}{P_1^H}+(1-\lambda_H)\varepsilon]+(1-\theta)\,[\lambda_L U'(C_1^{LN})\frac{1}{P_1^L}+(1-\lambda_L)\varepsilon]
\end{aligned}
$$

(a) When $P_1^H = \frac{U'(C_1^{HN})}{\varepsilon}$ and $P_1^L = \frac{U'(C_1^{LN})}{\varepsilon}$, (A9) is equivalent to

$$\text{(A10)} \qquad \theta\,[\lambda_H D_1^N + (1-\lambda_H) D_2^N] + (1-\theta)\,[\lambda_L D_1^N + (1-\lambda_L) D_2^N] = 1$$

We will see that (A10) is independent of the value of y. If the left hand side of (A10) is greater than 1, depositors will save all of their cash in banks because once the contract is given, the left hand side will not change with y and it will always be greater than 1. If (A10) holds, then depositors will be indifferent between saving and not saving. Since (A10) is independent of y, increasing y will not influence the equation. Then by the assumption of depositors will continue to save when indifferent, depositors will save all the cash in banks and they are still indifferent at this point. However, if the left hand side of (A10) is smaller than 1, since decreasing y doesn't influence (A10), then depositors will not save any cash in banks. Then the banks are abandoned by the market and it is not an equilibrium with banks. In summary, only the following equilibrium outcome is possible:

$$\text{(A11)} \qquad \theta\,[\lambda_H D_1^N + (1-\lambda_H) D_2^N] + (1-\theta)\,[\lambda_L D_1^N + (1-\lambda_L) D_2^N] \geq 1$$

(b) When either P_1^H or P_1^L doesn't have the simple forms in (a), (A10) no longer holds and we should come back to (A9). As depositors are price takers, (A9) is also independent of the value of y. Then in a non-degenerate equilibrium where banks play some roles, depositors will also save all the cash in banks. Specifically, given investment I, taking ε being smaller than both of $f_\varepsilon^H(I)$ and $f_\varepsilon^L(I)$ as example, and we have (A9) become:

$$\text{(A12)} \qquad \begin{aligned} &\theta\,[\lambda_H U'(\frac{1-I}{\lambda_H})\frac{D_1^N}{P_1^H} + (1-\lambda_H) D_2^N \varepsilon] + (1-\theta)\,[\lambda_L U'(\frac{1-I}{\lambda_L})\frac{D_1^N}{P_1^L} + (1-\lambda_L) D_2^N \varepsilon] \\ &= \theta\,[\lambda_H U'(\frac{1-I}{\lambda_H})\frac{1}{P_1^H} + (1-\lambda_H)\varepsilon] + (1-\theta)\,[\lambda_L U'(\frac{1-I}{\lambda_L})\frac{1}{P_1^L} + (1-\lambda_L)\varepsilon] \end{aligned}$$

For depositors, when they choose whether to deposit cash in banks, they take the deposit contract, prices, and banks' investment as given, then every variable in (A12) is fixed when depositors make the deposit decision. Since we have assumed that when depositors are indifferent between depositing cash and not depositing cash, they choose to continue to deposit cash in banks, depositors just compare the left hand side of (A12) and the right hand side of (A12), and in a non-degenerate equilibrium, the left hand side should be greater, and depositors will also save all the cash in banks.

PROOF OF PROPOSITION 3

Generally, there are three possible types of equilibria under the equilibrium prices: banks have incentive to invest more, banks are indifferent between investing more and investing less, and banks have incentive to invest less. In the following proof, we will discuss them respectively under different equilibrium conditions and show that, (i) there is no equilibrium when banks have incentive to invest less; (ii) there exits equilibria when banks have incentive to invest less and banks are indifferent between investing more and less; (iii) there exists a range of ε where there is a unique equilibrium, and banks have incentive to invest more in this unique equilibrium. We will also derive this range and characterize this unique equilibrium.

Part A. Equilibrium when banks have incentive to invest less

For the case with $\theta P_2^H R + (1-\theta) P_2^L R < P_0$, we can show that risk-neutral banks want to invest less. If a bank deviates by cutting ΔI investment, it will get extra cash of $P_0 \Delta I$ at date 0, and its loss in expectation is $[\theta P_2^H R + (1-\theta) P_2^L R] \Delta I$ unit of cash at date 2. Then the bank will always deviate as the benefit is strictly greater than loss. Then this situation will never appear in equilibrium.

Part B. Equilibrium when banks have incentive to invest more

For the case with $\theta P_2^H R + (1-\theta) P_2^L R > P_0$, risk-neutral banks always have incentive to invest more. However, different from the previous case, banks have a maximal investment constraint here. That is, a bank has to hold enough money to pay D_1^N. Therefore, if in equilibrium all the banks propose the contract that pay all the cash to the impatient depositors at date 1, no individual bank can deviate to invest more as any deviating bank that invests more at date 0 will not have enough liquidity to pay the impatient depositors at date 1. Therefore, in this case, the only possible type of equilibria is that banks pay all their 1 unit of cash to the impatient depositors at date 1.

Next we prove that there indeed exists such an equilibrium of this type, and characterize the properties of this equilibrium. The proof is divided in three steps. In step 1, we do some mathematical preparations. In step 2, we show that if there is such an equilibrium, banks' budget constraint is binding in the normal state, and we derive the deposit contract from which banks have no incentive to deviate under given investment. We also derive depositors' consumption in equilibrium. In step 3, we allow banks can deviate by adjusting the investment and deposit contract simultaneously, and derive the equilibrium investment which banks will not deviate from. Then any form of deviation will not happen and it is actually an equilibrium.

Step 1: Before starting the main part of the proof, we first prove some functions' monotonicity in Lemma A1.

LEMMA A1: (i) $U'(x)x$ is decreasing with x. (ii) $U'\left(\frac{x}{1-\lambda_H}\right)x$ is also decreasing with x. (iii) $U'\left(\frac{c}{x}\right)\frac{1-x}{x}$ is increasing with x where c is a constant and $x = \lambda_H$ or λ_L.

PROOF:

(i) The derivative is $U''(x)x+U'(x)$. CRRA>1 implies $-\frac{U''(x)}{U'(x)}x>1$ and $U''(x)x+U'(x)<0$. So $U'(x)x$ is decreasing with x.

(ii) Let $t=\frac{x}{1-\lambda_H}$, then $x=(1-\lambda_H)t$, and $U'\left(\frac{x}{1-\lambda_H}\right)x=(1-\lambda_H)U'(t)t$. Since $1-\lambda_H>0$, t is increasing with x, and $U'(t)t$ is decreasing with t, we know $U'\left(\frac{x}{1-\lambda_H}\right)x$ is decreasing with x.

(iii) Taking derivative of this function, we get $-\frac{1}{x^2}\left[U''(\frac{c}{x})\frac{c}{x}(1-x)-U'(\frac{c}{x})\right]$. By $U''(x)x<-U'(x)$, $U''\left(\frac{c}{x}\right)\frac{c}{x}<-U'\left(\frac{c}{x}\right)$, and $x>0$, we have

$$-\frac{1}{x^2}\left[U''(\frac{c}{x})\frac{c}{x}(1-x)-U'(\frac{c}{x})\right]>\frac{1}{x^2}\left[U'(\frac{c}{x})(1-x)+U'(\frac{c}{x})\right]=\frac{1}{x^2}U'(\frac{c}{x})(2-x)$$

Note that $x=\lambda_H$ or λ_L, then $x<1$, and $2-x>0$. By $U'(\cdot)>0$, we know $\frac{1}{x^2}U'(\frac{c}{x})(2-x)>0$ and the derivative is greater than 0, which implies $U'\left(\frac{c}{x}\right)\frac{1-x}{x}$ is increasing with x.■

Step 2: Since there are more impatient depositors in the liquidity shortage state, and the payment of D_1^N to each depositor is the same in the liquidity shortage state and the normal state, the total payment at date 1 is higher in the liquidity shortage state. Then $\lambda_H D_1^N y = 1$, and $D_1^N = 1/\lambda_H y$. The budget constraint of banks in the liquidity shortage state is:

(A13) $$[\lambda_H D_1^N + (1-\lambda_H) D_2^N] y \le P_2^H RI + 1$$

The budget constraint of banks in the normal state is:

(A14) $$[\lambda_L D_1^N + (1-\lambda_L) D_2^N] y \le P_2^L RI + 1$$

(a) We first prove banks' budget constraint is binding in the normal state. Denote the total consumption by the patient depositors as G_2^H in the liquidity shortage state and G_2^L in the normal state. Then $G_2^H = (1-\lambda_H) C_2^{HN}$ and $G_2^L = (1-\lambda_L) C_2^{LN}$. We know $[\lambda_H D_1^N + (1-\lambda_H) D_2^N] y = P_2^H G_2^H + 1$ and $[\lambda_L D_1^N + (1-\lambda_L) D_2^N] y = P_2^L G_2^L + 1$ always hold no matter which constraint is binding.

Firstly, assume that banks' budget constraint is binding in the liquidity shortage state, which means (A13) is binding. Substituting $y = 1/\lambda_H D_1^N$ into (A13), we have $(1-\lambda_H) D_2^N / (\lambda_H D_1^N) = P_2^H RI$. Then by $\lambda_L D_1^N y = \lambda_L / \lambda_H$, we know in the normal state

(A15) $$\frac{(1-\lambda_L) D_2^N}{\lambda_H D_1^N} - (1 - \frac{\lambda_L}{\lambda_H}) = P_2^L G_2^L$$

Solving D_2^N / D_1^N from $(1-\lambda_H)D_2^N / (\lambda_H D_1^N) = P_2^H RI$ and substituting it into (A15), and also substituting $P_2^H = \frac{U'(RI/(1-\lambda_H))}{\varepsilon}$ and $P_2^H = \frac{U'(G_2^L/(1-\lambda_L))}{\varepsilon}$ into (A15), we get

$$\text{(A16)} \qquad U'\left(\frac{G_2^L}{1-\lambda_L}\right)\frac{G_2^L}{1-\lambda_L} - U'\left(\frac{RI}{1-\lambda_H}\right)\frac{RI}{1-\lambda_H} = -\frac{1}{1-\lambda_L}(1-\frac{\lambda_L}{\lambda_H})\varepsilon < 0$$

which implies $U'\left(\frac{G_2^L}{1-\lambda_L}\right)\frac{G_2^L}{1-\lambda_L} < U'\left(\frac{RI}{1-\lambda_H}\right)\frac{RI}{1-\lambda_H}$. By Lemma A1 (i), we have $\frac{G_2^L}{1-\lambda_L} > \frac{RI}{1-\lambda_H}$, and then $G_2^L > RI$ by $1-\lambda_L > 1-\lambda_H$, which contradicts with the budget constraint (A14). That is, at date 2, banks only have *RI* real goods, it's impossible for the patient depositors to consume more goods than *RI* in the normal state. Therefore, banks' budget constraint cannot be binding in the liquidity shortage state.

Then banks' budget constraint can only bind in the normal state, and we have $(1-\lambda_L)D_2^N / (\lambda_H D_1^N) - (1-\lambda_L/\lambda_H) = P_2^L RI$. Then solving D_2^N / D_1^N and substituting D_2^N / D_1^N, P_2^H, and P_2^L into the liquidity shortage state's constraint, we get

$$\text{(A17)} \qquad \frac{1-\lambda_H}{1-\lambda_L}\left[U'\left(\frac{RI}{1-\lambda_L}\right)RI + (1-\frac{\lambda_L}{\lambda_H})\varepsilon\right] = U'\left(\frac{G_2^H}{1-\lambda_H}\right)G_2^H$$

Subtracting both sides by $U'\left(\frac{RI}{1-\lambda_H}\right)RI$, we have

(A18)
$$U'\left(\frac{G_2^H}{1-\lambda_H}\right)G_2^H - U'\left(\frac{RI}{1-\lambda_H}\right)RI =$$
$$(1-\lambda_H)\left[U'\left(\frac{RI}{1-\lambda_L}\right)\frac{RI}{1-\lambda_L} - U'\left(\frac{RI}{1-\lambda_H}\right)\frac{RI}{1-\lambda_H} + \frac{1}{1-\lambda_L}(1-\frac{\lambda_L}{\lambda_H})\varepsilon\right]$$

Since $\frac{RI}{1-\lambda_L} < \frac{RI}{1-\lambda_H}$, and by monotonicity of $U'(x)x$, we know

$$U'\left(\frac{RI}{1-\lambda_L}\right)\frac{RI}{1-\lambda_L} > U'\left(\frac{RI}{1-\lambda_H}\right)\frac{RI}{1-\lambda_H}$$

In addition, as $\frac{1}{1-\lambda_L}(1-\frac{\lambda_L}{\lambda_H})\varepsilon > 0$, the right hand side of (A18) is greater than 0, and

$$U'\left(\frac{G_2^H}{1-\lambda_H}\right)G_2^H > U'\left(\frac{RI}{1-\lambda_H}\right)RI$$

By Lemma A1 (ii), now we have $G_2^H < RI$, which satisfies the budget constraint of banks. We know the budget constraint of banks binding in the normal state is feasible. Then $D_1^N = \frac{1}{\lambda_H y}$ and $D_2^N = \frac{P_2^L RI^E + 1 - \lambda_L/\lambda_H}{(1-\lambda_L)y}$.

(b) The equilibrium consumptions are $C_1^{HN} = (1-I)/\lambda_H$, $C_1^{LN} = (1-I)/\lambda_L$, and $C_2^{LN} = RI/(1-\lambda_L)$, and by (A17) and $G_2^H = (1-\lambda_H)C_2^{HN}$, C_2^{HN} is the solution of following equation.

$$U'(C_2^{HN})(1-\lambda_H)C_2^{HN} = \frac{1-\lambda_H}{1-\lambda_L}\left[U'\left(\frac{RI}{1-\lambda_L}\right)RI + \left(1-\frac{\lambda_L}{\lambda_H}\right)\varepsilon\right]$$

(c) The final thing of this step is to check banks don't have incentive to deviate from the contract stated in (a) under a given investment level. To prove this result, we first prove Lemma A2 which illustrates the relationship between prices in the liquidity shortage state and the normal state.

LEMMA A2: The price of goods in date 1 and date 2 are both higher in the liquidity shortage case. That is, $P_1^H > P_1^L$ and $P_2^H > P_2^L$.

PROOF OF LEMMA A2:

(i) We first prove $P_2^H > P_2^L$. Manipulating (A17), we get

$$\text{(A19)} \qquad U'\left(\frac{G_2^H}{1-\lambda_H}\right)\frac{G_2^H}{1-\lambda_H} - U'\left(\frac{RI}{1-\lambda_L}\right)\frac{RI}{1-\lambda_L} = \frac{1}{1-\lambda_L}(1-\frac{\lambda_L}{\lambda_H})\varepsilon > 0$$

which implies

$$U'\left(\frac{G_2^H}{1-\lambda_H}\right)\frac{G_2^H}{1-\lambda_H} > U'\left(\frac{RI}{1-\lambda_L}\right)\frac{RI}{1-\lambda_L}$$

By Lemma A1(i), we know $\frac{G_2^H}{1-\lambda_H} < \frac{RI}{1-\lambda_L}$. Since $U''(x)$ is negative, $U'(x)$ is decreasing with *x*. We know $U'\left(\frac{G_2^H}{1-\lambda_H}\right)/\varepsilon > U'\left(\frac{RI}{1-\lambda_L}\right)/\varepsilon$, which means $P_2^H > P_2^L$.

(ii) The comparison between P_1^H and P_1^L is relatively easier. Comparing the marginal utility of goods or comparing the total cash withdrawn by the impatient depositors divided by total goods supply by the patient depositors both lead to $P_1^H > P_1^L$.■

In (A13) and (A14), since $P_2^H > P_2^L$, the total nominal resources in the liquidity shortage state is larger ($P_2^H RI + 1 > P_2^L RI + 1$). Due to the market's competition, given *I*, banks should propose the best contracts to depositors subject to the budget constraints. Then since $\lambda_H > \lambda_L$, only contracts with $D_1^N > D_2^N$ can make (A13) and (A14) hold simultaneously and make banks pay all the real goods to depositors. As a result, since $D_1^N < D_2^N$ in equilibrium, under a given investment level, banks want to continue to increase D_1^N to offer even better contracts to depositors in a competitive market. However, now banks already pay all of the cash at date 1 to the impatient depositors and thus D_1^N cannot increase any more. In other words, given the equilibrium investment, each bank will not deviate to adjust the deposit contract, because he has no incentive to decrease D_1^N and is incapable of increasing D_1^N. In summary, given the equilibrium investment, banks will not deviate from the current deposit contract.

Step 3: In this step, we allow banks can deviate by adjusting the investment and deposit contract simultaneously. Then after deviating by adjusting the investment level, a bank will also adjust the deposit contract under the new investment. We derive the equilibrium investment which banks will not deviate from. Then under the equilibrium investment and deposit contract, any form of deviation will not happen and we obtain the equilibrium.

In different ranges of ε, the equilibrium investment has different functional forms because the prices at date 1 are different. There will be two cutoff values of ε, which are corresponding to $f_\varepsilon^H(I)$ and $f_\varepsilon^L(I)$, we call them ε^H and ε^L. We prove $f_\varepsilon^H(I) \geq f_\varepsilon^L(I)$ and we will get $\varepsilon^H \geq \varepsilon^L$. By Lemma A1 (iii), with $\lambda_H > \lambda_L$, we know $U'\left(\frac{1-I}{\lambda_H}\right)\frac{1-\lambda_H}{\lambda_H} > U'\left(\frac{1-I}{\lambda_L}\right)\frac{1-\lambda_L}{\lambda_L}$, then it's easy to see,

$$U'\left(\frac{1-I}{\lambda_H}\right)(1-\lambda_H)(1-I) = f_\varepsilon^H(I) > f_\varepsilon^L(I) = U'\left(\frac{1-I}{\lambda_L}\right)(1-\lambda_L)(1-I)\frac{\lambda_H}{\lambda_L}$$

Now we can divide ε into three ranges and discuss the equilibrium investment in each range respectively. The equilibrium investment is the point where banks are indifferent between investing more unit of goods and investing less.

(a) When $\varepsilon \geq \varepsilon^H$, consider a bank's deviation of increasing investment by ΔI.

Firstly, we need to characterize how the deposit contract changes along with the change of investment. This bank's cash at date 0 reduces by $P_0\Delta I$, and the goods produced from projects increase by $R\Delta I$. By no-deviation condition of D_1^N and D_2^N, this bank still needs to pay all of his cash ($1-P_0\Delta I$) to the impatient depositors at date 1. Then D_1^N decreases by $\Delta D_1^N = P_0\Delta I/(\lambda_H y)$. In the liquidity shortage state, this bank pays out of all the cash, while in the normal state, the bank's total cash payment only decreases by $\lambda_L \Delta D_1^N y = \lambda_L P_0 \Delta I / \lambda_H$, which is smaller than $P_0\Delta I$. Therefore, the bank's date 2's liquidity in the normal state decreases by $P_0\Delta I(1-\lambda_L/\lambda_H)$. With the bank' profits in the liquidity shortage state denoted as π_H, the constraints for D_2^N are as follows:

(A20) $$(1-\lambda_H)\Delta D_2^N y \leq P_2^H R\Delta I + \pi_H$$

(A21) $$(1-\lambda_L)\Delta D_2^N y \leq P_2^L R\Delta I - P_0\Delta I(1-\lambda_L/\lambda_H)$$

Since $P_2^H > P_2^L$, $\pi_H > 0$, $P_0\Delta I(1-\lambda_L/\lambda_H)$ >0 and $1-\lambda_H < 1-\lambda_L$, we know (A21) is binding, which implies

$$\Delta D_2^N = \frac{P_2^L R\Delta I - P_0\Delta I(1-\frac{\lambda_L}{\lambda_H})}{(1-\lambda_L)y}$$

Secondly, we compare the gain and loss of this deviation. In the liquidity shortage state, the consumption of the patient depositors from this bank can be increased by

$$\Delta C_2^{HN} = \frac{\Delta D_2^N y}{P_2^H} = \frac{\frac{P_2^L}{P_2^H} R\Delta I - \frac{P_0}{P_2^H}\Delta I(1-\frac{\lambda_L}{\lambda_H})}{1-\lambda_L}$$

In the normal state, the budget constraint of the deviating is still binding, and this bank still pay out all his cash and the patient depositors' consumption can be increased by $\Delta C_2^{LN} = R\Delta I/(1-\lambda_L)$.

The expected welfare gain from the change of real goods consumption at date 2 is,

$$\begin{aligned} &\theta(1-\lambda_H)\left[U\left(C_2^{HN}+\Delta C_2^{HN}\right)-U\left(C_2^{HN}\right)\right]+ \\ &(1-\theta)(1-\lambda_L)\left[U\left(C_2^{LN}+\Delta C_2^{LN}\right)-U\left(C_2^{LN}\right)\right] \end{aligned}$$

Taking $\Delta I \to 0$, and substituting $P_2^H = U'(C_2^{HN})/\varepsilon$ and $P_2^L = U'(C_2^{LN})/\varepsilon$ into the above expression, we have the welfare gain

$$\left\{\theta\frac{1-\lambda_H}{1-\lambda_L}\left[P_2^L R - P_0(1-\frac{\lambda_L}{\lambda_H})\right]+(1-\theta)P_2^L R\right\}\Delta I\varepsilon$$

Now the impatient depositors keep cash after purchasing goods at date 1 in both the liquidity shortage state and the normal state, the expected welfare loss by cash is $P_0\Delta I\varepsilon$. The welfare gain and loss by reducing marginal unit of investment is symmetric. Therefore, the equilibrium investment can be obtained by equalizing the gain and the loss calculated above, and we get the equilibrium investment as the solution to the equation below.

(A22) $$\left[\theta\frac{1-\lambda_H}{1-\lambda_L}+(1-\theta)\right]P_2^L R=\left[\theta\frac{1-\lambda_H}{1-\lambda_L}(1-\frac{\lambda_L}{\lambda_H})+1\right]P_0$$

We claim that under the equilibrium investment, $P_2^L R > P_0$ and $P_2^H R > P_0$, which corresponds to the situation $\theta P_2^H R+(1-\theta)P_2^L R > P_0$. By (A22) we know,

$$\frac{P_2^L R}{P_0}=\frac{\theta\frac{1-\lambda_H}{1-\lambda_L}+1-\theta\frac{1-\lambda_H}{1-\lambda_L}\frac{\lambda_L}{\lambda_H}}{\theta\frac{1-\lambda_H}{1-\lambda_L}+1-\theta}$$

With $1-\lambda_H<1-\lambda_L$ and $\lambda_L<\lambda_H$, we have $\frac{1-\lambda_H}{1-\lambda_L}\frac{\lambda_L}{\lambda_H}<1$. Then we know $P_2^L R/P_0>1$ and $P_2^L R>P_0$. Due to $P_2^H > P_2^L$, we know $P_2^H R>P_0$ also holds.

(b) When $\varepsilon^L \leq \varepsilon < \varepsilon^H$, the welfare gain from goods consumption increase at date 2 has exactly the same function form as in (a). However, the welfare loss from cash reduction is different, because now in the liquidity shortage state, the impatient depositors' utility of purchasing an extra unit of goods is larger than keeping cash. So in the liquidity shortage state, date 1's cash reduction entails a consumption goods loss of $\Delta D_1^N y/P_1^H$. In the normal state, the utility loss stays the same, which is $P_0\Delta I\varepsilon$, while the expected utility loss is

$$\theta\left[\lambda_H U\left(\frac{1-I}{\lambda_H}-\frac{\Delta D_1^N y}{P_1^H}\right)-U\left(\frac{1-I}{\lambda_H}\right)\right]+(1-\theta)P_0\Delta I\varepsilon$$

With $\Delta I\to 0$, the welfare loss now is

$$\left[\theta\lambda_H U'\left(\frac{1-I}{\lambda_H}\right)\frac{P_0}{\lambda_H P_1^H}+(1-\theta)P_0\varepsilon\right]\Delta I$$

By equalizing the welfare gain and loss, the equilibrium investment solves the equation below:

$$\text{(A23)}\left[\theta\frac{1-\lambda_H}{1-\lambda_L}+(1-\theta)\right]P_2^L R=\left[\theta\frac{1-\lambda_H}{1-\lambda_L}(1-\frac{\lambda_L}{\lambda_H})+\theta U'\left(\frac{1-I}{\lambda_H}\right)\frac{1}{P_1^H\varepsilon}+(1-\theta)\right]P_0$$

Since now $U'\left(\frac{1-I}{\lambda_H}\right)>P_1^H\varepsilon$, the coefficient of the right hand side of (A23) is greater than that of (A22), then $P_2^L R$ is greater than P_0. Therefore, we have $P_2^L R>P_0$ and $P_2^H R>P_0$.

(c) When $\varepsilon<\varepsilon^L$, the welfare loss now is,

$$\left[\theta U'\left(\frac{1-I}{\lambda_H}\right)\frac{P_0}{P_1^H}+(1-\theta)\frac{\lambda_L}{\lambda_H}U'\left(\frac{1-I}{\lambda_L}\right)\frac{P_0}{P_1^L}+(1-\theta)P_0(1-\frac{\lambda_L}{\lambda_H})\varepsilon\right]\Delta I$$

The equilibrium investment decision equation is

$$\text{(A24)}\quad\left[\theta\frac{1-\lambda_H}{1-\lambda_L}+(1-\theta)\right]P_2^L R=\begin{Bmatrix}\theta\frac{1-\lambda_H}{1-\lambda_L}\left(1-\frac{\lambda_L}{\lambda_H}\right)+\theta U'\left(\frac{1-I}{\lambda_H}\right)\frac{1}{P_1^H\varepsilon}+\\(1-\theta)\left[\frac{\lambda_L}{\lambda_H}U'\left(\frac{1-I}{\lambda_L}\right)\frac{1}{P_1^L\varepsilon}+\left(1-\frac{\lambda_L}{\lambda_H}\right)\right]\end{Bmatrix}P_0$$

$U'\left(\frac{1-I}{\lambda_L}\right)>P_1^L\varepsilon$, thus the coefficient of the right hand side of (A24) is larger than that of (A23). As $P_2^L R$ is larger than P_0, we also have $P_2^H R>P_0$ when $\varepsilon<\varepsilon^L$.

Since $P_2^L R>P_0$ and $P_2^H R>P_0$ both hold in all three ranges of ε, we have $P_2^L RI^E+1>y=P_0 I^E+1$ and $P_2^H RI^E+1>y=P_0 I^E+1$, which implies

$\lambda_H D_1^N + (1-\lambda_H) D_2^N > 1$ and $\lambda_L D_1^N + (1-\lambda_L) D_2^N > 1$, and (A11) holds and depositors want to save all their cash in banks.

At the end of step 3, we characterize the expression of cutoff values ε^H and ε^L, which can be obtained by substituting the equilibrium investment in the corresponding parameter region in $f_\varepsilon^H(I)$ and $f_\varepsilon^L(I)$. Specifically, ε^H is the solution of ε in following equation system in (A25).

$$\text{(A25)} \qquad \begin{cases} \left[\theta \dfrac{1-\lambda_H}{1-\lambda_L} + (1-\theta)\right] P_2^L R = \left[\theta \dfrac{1-\lambda_H}{1-\lambda_L}\left(1-\dfrac{\lambda_L}{\lambda_H}\right) + 1\right] P_0 \\ \varepsilon = f_\varepsilon^H(I) = U'\left(\dfrac{1-I}{\lambda_H}\right)(1-\lambda_H)(1-I) \end{cases}$$

and ε^L is the solution of ε in following equation system in (A26).

$$\text{(A26)} \begin{cases} \left[\theta \dfrac{1-\lambda_H}{1-\lambda_L} + (1-\theta)\right] P_2^L R = \left[\theta \dfrac{1-\lambda_H}{1-\lambda_L}(1-\dfrac{\lambda_L}{\lambda_H}) + \theta U'\left(\dfrac{1-I}{\lambda_H}\right)\dfrac{1}{P_1^H \varepsilon} + (1-\theta)\right] P_0 \\ \varepsilon = f_\varepsilon^H(I) = U'\left(\dfrac{1-I}{\lambda_L}\right)(1-\lambda_L)(1-I)\dfrac{\lambda_H}{\lambda_L} \end{cases}$$

Till now, we have fully characterized this equilibrium with $\theta P_2^H R + (1-\theta) P_2^L R > P_0$, and prove that banks will not deviate. In this equilibrium banks' profits in the liquidity shortage state is given by:

$$RI^E - \frac{1-\lambda_H}{1-\lambda_L} \cdot \frac{P_2^L RI^E + 1 - \lambda_L / \lambda_H}{P_2^H}$$

Part C. Equilibrium when banks are indifferent between investing more and investing less

For the case with $\theta P_2^H R + (1-\theta) P_2^L R = P_0$, banks have no incentive to either increase investment or decrease investment. Increasing investment does not affect the total cash-holding of a bank. At this point, deposit contracts where banks don't pay all their 1 unit of cash to impatient depositors at date 1 can sustain as an equilibrium outcome. Different from the previous case when banks have incentive to invest more, now banks have no incentive to increase the investment. Then under the equilibrium investment, if there is a feasible contract which makes banks have zero profit, all the banks must choose this contract and have zero profit.

The following proof will proceed in three steps. In step 1, we derive the deposit contract when banks have zero profit. In step 2, we illustrate at what situation there is a unique equilibrium. In step 3, we specifically characterize the range of ε where there is unique equilibrium with $P_1^j > P_2^j$, $j = \{H, L\}$, and $D_1^N < D_2^N$.

Step 1: Zero profit means that the budget constraints (A13) and (A14) hold simultaneously

(A27) $$[\lambda_H D_1^N + (1-\lambda_H) D_2^N] y = P_2^H RI + 1$$

(A28) $$[\lambda_L D_1^N + (1-\lambda_L) D_2^N] y = P_2^L RI + 1$$

which implies

(A29) $$D_1^N = \frac{(\lambda_H - \lambda_L) + (1-\lambda_L) P_2^H RI - (1-\lambda_H) P_2^L RI}{(\lambda_H - \lambda_L) y}$$

Step 2: We claim that in this case any range of ε which leads to $P_1^H = P_2^H$ and $P_1^L = P_2^L$ cannot be equilibrium outcome.

With $\theta P_2^H R + (1-\theta) P_2^L R = P_0$, we have

$$P_0 = \theta P_2^H R + (1-\theta) P_2^L R = \theta P_1^H R + (1-\theta) P_1^L R > \theta P_1^H + (1-\theta) P_1^L$$

However, $P_0 > \theta P_1^H + (1-\theta)P_1^L$ will not be equilibrium outcome as selling goods at date 0 is always desirable. See $P_1^H = \frac{U'(C_1^{HN})}{\varepsilon}$ and $P_1^L = \frac{U'(C_1^{LN})}{\varepsilon}$ as an example, by Lemma 4 (iii), depositors are indifferent between selling more goods and less goods at date 0 when $P_0 = [\theta P_1^H + (1-\theta)P_1^L]/E[D]$, with $E[D] = \theta[\lambda_H D_1^N + (1-\lambda_H)D_2^N] + (1-\theta)[\lambda_L D_1^N + (1-\lambda_L)D_2^N]$. We know E[$D$]≥ 1 by the proof of Lemma 5, then $P_0 > \theta P_1^H + (1-\theta)P_1^L$ means $P_0 > [\theta P_1^H + (1-\theta)P_1^L]/E[D]$ and depositors want to sell all of their real goods to banks for investment and there will be no goods being consumed at date 1, which cannot be an equilibrium outcome.

Step 3: Next we characterize the range of ε in which we have $P_1^H = P_2^H$ and $P_1^L = P_2^L$. Denote the investment solved by $\theta P_2^H R + (1-\theta)P_2^L R = P_0$ as $\tilde{I}$, where $P_2^H = U'\left(\frac{RI}{1-\lambda_H}\right)/\varepsilon$, $P_2^L = U'\left(\frac{RI}{1-\lambda_L}\right)/\varepsilon$, $P_1^H = \frac{\lambda_H D_1^N y}{(1-\lambda_H)(1-I)}$, $P_1^L = \frac{\lambda_L D_1^N y}{(1-\lambda_L)(1-I)}$, P_0 is determined by Lemma 4, and D_1^N is determined by (A29).

Then, the cash amount withdrawn at date 1 in the liquidity shortage state is

$$\lambda_H D_1^N y = \lambda_H \frac{(\lambda_H - \lambda_L) + (1-\lambda_L)P_2^H R\tilde{I} - (1-\lambda_H)P_2^L R\tilde{I}}{\lambda_H - \lambda_L}$$

If the cash amount is just enough to buy $1-\tilde{I}$ unit of goods, $P_1^H = P_2^H$ gives

$$\tilde{\varepsilon}^H = \frac{U'\left(\frac{R\tilde{I}}{1-\lambda_H}\right)(\lambda_H - \lambda_L)(1-\lambda_H)(1-\tilde{I})}{\lambda_H[(\lambda_H - \lambda_L) + (1-\lambda_L)P_2^H R\tilde{I} - (1-\lambda_H)P_2^L R\tilde{I}]}$$

Similar, calculation in the normal state gives

$$\tilde{\varepsilon}^L = \frac{U'\left(\frac{R\tilde{I}}{1-\lambda_H}\right)(\lambda_H-\lambda_L)(1-\lambda_L)(1-\tilde{I})}{\lambda_L[(\lambda_H-\lambda_L)+(1-\lambda_L)P_2^H R\tilde{I}-(1-\lambda_H)P_2^L R\tilde{I}]}$$

Therefore, when $\varepsilon \le \min(\tilde{\varepsilon}^H, \tilde{\varepsilon}^L)$, the constraints $P_1^H = P_2^H$ and $P_1^L = P_2^L$ are both binding, and there is no equilibrium with $\theta P_2^H R + (1-\theta)P_2^L R = P_0$. Then there only exists one equilibrium with $\theta P_2^H R + (1-\theta)P_2^L R > P_0$ as we stated in *Part B*. When $\varepsilon > \min(\tilde{\varepsilon}^H, \tilde{\varepsilon}^L)$, there are two equilibria in the economy, which we will not discuss in detail in this paper. In summary, when there is a unique equilibrium, this equilibrium must be the equilibrium we stated in *Part B*.

Finally, we calculate the range of ε where there is a unique equilibrium with $P_1^j > P_2^j$, $j = \{H, L\}$, and $D_1^N < D_2^N$. The condition of unique equilibrium provides one upper bound of ε. Denote this upper limit as $\bar{\varepsilon}^1$, then $\bar{\varepsilon}^1 = \min(\tilde{\varepsilon}^H, \tilde{\varepsilon}^L)$. The point where D_1^N just equals D_2^N in the unique equilibrium provides the other upper bound. Denote the second upper limit as $\bar{\varepsilon}^2$. Since at the point of ε^H, D_1^N is smaller than D_2^N, we know $\bar{\varepsilon}^2$ is greater than ε^H, and $\bar{\varepsilon}^2$ is the solution of ε in following equation system

(A30)
$$\begin{cases} \left[\theta\frac{1-\lambda_H}{1-\lambda_L}+(1-\theta)\right]P_2^L R = \left[\theta\frac{1-\lambda_H}{1-\lambda_L}\left(1-\frac{\lambda_L}{\lambda_H}\right)+1\right]P_0 \\ \frac{1}{\lambda_H} = \frac{P_2^L RI + 1 - \lambda_L/\lambda_L}{1-\lambda_L} \end{cases}$$

The final upper bound is the smaller one of the two upper bounds, that is, $\bar{\varepsilon} = \min(\bar{\varepsilon}^1, \bar{\varepsilon}^2)$.

The lower bound $\underline{\varepsilon}$ is the point at which either $P_1^H = P_2^H$ or $P_1^L = P_2^L$ in the unique equilibrium stated in *Part B*, whichever larger, and is lower than ε^L. We will solve the cutoff ε which leads to $P_1^H = P_2^H$ and $P_1^L = P_2^L$ respectively in the unique equilibrium and choose the larger one of these two cutoffs. Denote the solution of ε in equation system (A31) as $\underline{\varepsilon}^1$ and the solution of ε in equation system (A32) as $\underline{\varepsilon}^2$, then $\underline{\varepsilon} = \max(\underline{\varepsilon}^1, \underline{\varepsilon}^2)$.

$$\text{(A31)} \quad \begin{cases} \left[\theta\dfrac{1-\lambda_H}{1-\lambda_L} + (1-\theta)\right] P_2^L R = \left\{ \begin{array}{l} \theta\dfrac{1-\lambda_H}{1-\lambda_L}\left(1-\dfrac{\lambda_L}{\lambda_H}\right) + \theta U'\left(\dfrac{1-I}{\lambda_H}\right)\dfrac{1}{P_1^H\varepsilon} + \\ (1-\theta)\left[\dfrac{\lambda_L}{\lambda_H}U'\left(\dfrac{1-I}{\lambda_L}\right)\dfrac{1}{P_1^L\varepsilon} + \left(1-\dfrac{\lambda_L}{\lambda_H}\right)\right] \end{array} \right\} P_0 \\ \dfrac{1}{(1-\lambda_H)(1-I)} = \dfrac{U'(C_2^{HN})}{\varepsilon} \end{cases}$$

$$\text{(A32)} \quad \begin{cases} \left[\theta\dfrac{1-\lambda_H}{1-\lambda_L} + (1-\theta)\right] P_2^L R = \left\{ \begin{array}{l} \theta\dfrac{1-\lambda_H}{1-\lambda_L}\left(1-\dfrac{\lambda_L}{\lambda_H}\right) + \theta U'\left(\dfrac{1-I}{\lambda_H}\right)\dfrac{1}{P_1^H\varepsilon} + \\ (1-\theta)\left[\dfrac{\lambda_L}{\lambda_H}U'\left(\dfrac{1-I}{\lambda_L}\right)\dfrac{1}{P_1^L\varepsilon} + \left(1-\dfrac{\lambda_L}{\lambda_H}\right)\right] \end{array} \right\} P_0 \\ \dfrac{1}{(1-\lambda_L)(1-I)} = U'\left(\dfrac{RI}{1-\lambda_L}\right) / \varepsilon \end{cases}$$

PROOF OF LEMMA 6

At the equilibrium investment I^E, banks have zero profits, which implies

$[\lambda_H D_1^{N\prime} + (1-\lambda_H) D_2^{N\prime}]y = P_2^{H\prime} R I^E + 1$ and $[\lambda_L D_1^{N\prime} + (1-\lambda_L) D_2^{N\prime}]y = P_2^{L\prime} R I^E + 1$

We can solve for $D_1^{N\prime}$

$$D_1^{N'} = \frac{(\lambda_H - \lambda_L) + (1-\lambda_L)P_2^{H'}RI^E - (1-\lambda_H)P_2^{L'}RI^E}{(\lambda_H - \lambda_L)y}$$

The prices now are the prices when all the real goods are consumed by depositors and banks have no profits. That is, $P_2^{H'} = U'\left(\frac{RI^E}{1-\lambda_H}\right)/\varepsilon$ and $P_2^{L'} = U'\left(\frac{RI^E}{1-\lambda_L}\right)/\varepsilon$.

Then $D_2^{N'} = \frac{P_2^{L'}RI^E + 1 - \lambda_L D_1^{N'}y}{(1-\lambda_L)y}$.

PROOF OF PROPOSITION 4

First we prove $D_1^{N'} < D_1^N$. From proposition 3, we know $D_1^N = 1/(\lambda_H y)$. We only need to compare $1 + \frac{(1-\lambda_L)P_2^{H'}RI^E - (1-\lambda_H)P_2^{L'}RI^E}{\lambda_H - \lambda_L}$ with $\frac{1}{\lambda_H}$. Since $\frac{1}{\lambda_H} > 1$, as long as $\frac{(1-\lambda_L)P_2^{H'}RI^E - (1-\lambda_H)P_2^{L'}RI^E}{\lambda_H - \lambda_L} < 0$, we will have $D_1^{N'} < D_1^N$. It is equivalent to prove $(1-\lambda_L)P_2^H RI^E < (1-\lambda_H)P_2^L RI^E$, which is equivalent to $U'\left(\frac{RI^E}{1-\lambda_H}\right)\frac{RI^E}{(1-\lambda_H)} < U'\left(\frac{RI^E}{1-\lambda_L}\right)\frac{RI^E}{(1-\lambda_L)}$ with $P_2^{H'}$ and $P_2^{L'}$ being substituted. Since $U'(x)x$ is decreasing with x and $\frac{RI^E}{1-\lambda_H} > \frac{RI^E}{1-\lambda_L}$, we have $U'\left(\frac{RI^E}{1-\lambda_H}\right)\frac{RI^E}{(1-\lambda_H)} < U'\left(\frac{RI^E}{1-\lambda_L}\right)\frac{RI^E}{(1-\lambda_L)}$, which indicates that $\frac{(1-\lambda_L)P_2^{H'}RI^E - (1-\lambda_H)P_2^{L'}RI^E}{\lambda_H - \lambda_L} < 0$. Therefore, we have $D_1^{N'} < D_1^N$.

Next we prove $D_2^{N'} > D_2^N$. Since $D_1^{N'} < D_1^N$ and $P_2^{L'} = P_2^L$, we know

$$D_2^{N'} = \frac{P_2^{L'} R I^E + 1 - \lambda_L D_1^{N'} y}{(1-\lambda_L) y} < \frac{P_2^L R I^E + 1 - \lambda_L D_1^N y}{(1-\lambda_L) y} = D_2^N$$

However, with D_1^N banks pay all of 1 unit of cash at date 1, then if banks pay $D_1^{N'}$, they will have some cash left. Under equilibrium investment I^E, every bank has an incentive to deviate by investing more and reducing the cash left at date 1. Then $D_1^{N'}$ and $D_2^{N'}$ cannot be equilibrium outcome.

PROOF OF PROPOSITION 5

Denote the equilibrium investment with $\varepsilon \geq \varepsilon_C^H$ as I_1^E, the equilibrium investment with $\varepsilon_C^L \leq \varepsilon < \varepsilon_C^H$ as I_2^E and the equilibrium investment with $\varepsilon < \varepsilon_C^L$ as I_3^E. We need to prove $I_3^E < I_2^E < I_1^E$.

I_1^E is solved from (A22), I_2^E is solved from (A23) and I_3^E is from (A24). The left hand sides of (A22), (A23) and (A24) are the same, and in order to compare the solution we only need to compare the right hand sides. Since $U'\left(1 - I/\lambda_H\right)/(P_1^H \varepsilon) > 1$, the coefficient of P_0 at (A23)'s right hand side is greater than that of (A22). If (A22) holds, then the left hand side of (A23) is smaller than the right hand side. In order to make (A23) satisfied, we need to increase $P_2^L R$ or decrease P_0. We know that P_2^L increases and P_0 decreases as the equilibrium *I* decreases. Therefore the solution of (A23) is smaller than the solution of (A22). That is, $I_2^E < I_1^E$.

At the same time, as $U'\left(\frac{1-I}{\lambda_L}\right)\frac{1}{P_1^L\varepsilon}>1$, we know $\frac{\lambda_L}{\lambda_H}U'\left(\frac{1-I}{\lambda_L}\right)\frac{1}{P_1^L\varepsilon}+\left(1-\frac{\lambda_L}{\lambda_H}\right)>1$, which means the coefficient of P_0 at (A24)'s right hand side is greater than that of (A23). By analogy, we have $I_3^E < I_2^E$.

PROOF OF PROPOSITION 6

Step 1 of this proof compares depositors' expected utility in the economy without money with that in the economy with money by Taylor expansions and derive the conditions under which money can improve the depositors' welfare. Step 2 gives some comparative statics when money is desirable.

Step 1: The consumptions in the economy without money (real-goods economy) are

$$C_1^{HR}=\frac{1-I}{\lambda_H},\quad C_2^{HR}=\frac{RI}{1-\lambda_L}+\frac{(1-I)(1-\lambda_L/\lambda_H)}{1-\lambda_L},$$

$$C_1^{LR}=\frac{1-I}{\lambda_H},\quad C_2^{LR}=\frac{RI}{1-\lambda_L}+\frac{(1-I)(1-\lambda_L/\lambda_H)}{1-\lambda_L}$$

Under this specific utility function, by proposition 3, we can derive the patient depositors' consumption in the liquidity shortage state in the economy with money (nominal economy) as

$$C_2^{HN}=\frac{1}{\frac{1-\lambda_L}{RI}+\frac{\lambda_H-\lambda_L}{\lambda_H(1-\lambda_L)}\varepsilon}$$

Other consumptions in nominal economy are

$$C_1^{HN} = \frac{1-I}{\lambda_H},\ C_1^{LN} = \frac{1-I}{\lambda_L},\ C_2^{LN} = \frac{RI}{1-\lambda_L}$$

When $\lambda_H \to \lambda_L$, denote $\lambda_H = \lambda_L + \Delta\lambda$ and $\Delta\lambda \to 0$. The equilibrium investment in real-goods economy and nominal economy are both approaching the social optimal investment. That is, $I^E \to I_{FB}^*$ and $I_R^* \to I_{FB}^*$. Comparing the expected utility in the economy with money with that in the real-goods economy, we get sufficient and necessary condition under which introducing money improves depositors' expected utility obtained from real-goods consumption (we call it money is desirable hereafter).

$$\begin{aligned}&\theta\{\lambda_H[U(C_1^{HN}) - U(C_1^{HR})] + (1-\lambda_H)[U(C_2^{HN}) - U(C_2^{HR})]\} \geq \\ &(1-\theta)\{\lambda_L[U(C_1^{LR}) - U(C_1^{LN})] + (1-\lambda_L)[U(C_2^{LR}) - U(C_2^{LN})]\}\end{aligned}$$

We eliminate the first term of the left hand side as $C_1^{HN} = C_1^{HR}$. After we take the limit of $\lambda_H \to \lambda_L$, the sufficient and necessary condition will turn to a sufficient condition by eliminating equality.

$$\text{(A33)}\qquad \begin{aligned}&\theta(1-\lambda_L-\Delta\lambda)[U(C_2^{HN}) - U(C_2^{HR})] + \\ &(1-\theta)\{\lambda_L[U(C_1^{LN}) - U(C_1^{LR})] + (1-\lambda_L)[U(C_2^{LN}) - U(C_2^{LR})]\} > 0\end{aligned}$$

With Taylor expansion for $C_1^{LR} = (1-I)/\lambda_H$ at the point of $C_1^{LN} = (1-I)/\lambda_L$, we get

$$U(C_1^{LR}) = U\left(\frac{1-I}{\lambda_L}\right) + U'\left(\frac{1-I}{\lambda_L}\right)\left[\frac{\Delta\lambda(1-I)}{\lambda_L(\lambda_L+\Delta\lambda)}\right] + O(\Delta\lambda^2)$$

With Taylor expansion for $C_2^{HR} = C_2^{LR}$ at the point of $C_2^{LN} = RI/(1-\lambda_L)$, we get

$$U(C_2^{HR}) = U(C_2^{LR}) = U\left(\frac{RI}{1-\lambda_L}\right) + U'\left(\frac{RI}{1-\lambda_L}\right)\left[\frac{(1-I)\Delta\lambda/(\lambda_L+\Delta\lambda)}{1-\lambda_L}\right] + O(\Delta\lambda^2)$$

With Taylor expansion C_2^{HN} at the point of $C_2^{LN}=RI/(1-\lambda_L)$, we get

$$U(C_2^{HN}) = U\left(\frac{RI}{1-\lambda_L}\right) + U'\left(\frac{RI}{1-\lambda_L}\right)\left(-\frac{\frac{RI}{1-\lambda_L}\frac{\Delta\lambda}{(\lambda_L+\Delta\lambda)(1-\lambda_L)}\varepsilon}{\frac{1-\lambda_L}{RI}+\frac{\Delta\lambda}{(\lambda_L+\Delta\lambda)(1-\lambda_L)}\varepsilon}\right) + O(\Delta\lambda^2)$$

Then (A33) turns to

(A34)

$$\begin{aligned}&\theta\left[(1-\lambda_L-\Delta\lambda)U'\left(\frac{RI}{1-\lambda_L}\right)\left(-\frac{\frac{RI}{1-\lambda_L}\frac{\Delta\lambda}{(\lambda_L+\Delta\lambda)(1-\lambda_L)}\varepsilon}{\frac{1-\lambda_L}{RI}+\frac{\Delta\lambda}{(\lambda_L+\Delta\lambda)(1-\lambda_L)}\varepsilon}-\frac{(1-I)\frac{\Delta\lambda}{\lambda_L+\Delta\lambda}}{1-\lambda_L}\right)\right]\\ &+(1-\theta)\left[\lambda_L U'\left(\frac{1-I}{\lambda_L}\right)\frac{\Delta\lambda(1-I)}{\lambda_L(\lambda_L+\Delta\lambda)}-(1-\lambda_L)U'\left(\frac{RI}{1-\lambda_L}\right)\frac{(1-I)\frac{\Delta\lambda}{\lambda_L+\Delta\lambda}}{1-\lambda_L}\right]+O(\Delta\lambda^2)>0\end{aligned}$$

Since $\Delta\lambda>0$, dividing both side of (A34) by $\Delta\lambda$ and then taking $\Delta\lambda\to 0$, we have (A34) equivalent to

(A35)

$$\begin{aligned}&\theta\left[(1-\lambda_L)U'\left(\frac{RI}{1-\lambda_L}\right)\left(-\frac{\frac{RI}{1-\lambda_L}\frac{1}{\lambda_L(1-\lambda_L)}\varepsilon}{\frac{1-\lambda_L}{RI}}-\frac{(1-I)\frac{1}{\lambda_L}}{1-\lambda_L}\right)\right]\\ &+(1-\theta)\left[\lambda_L U'\left(\frac{1-I}{\lambda_L}\right)\frac{1-I}{\lambda_L^2}-(1-\lambda_L)U'\left(\frac{RI}{1-\lambda_L}\right)\frac{\frac{1-I}{\lambda_L}}{1-\lambda_L}\right]>0\end{aligned}$$

Since $\lambda_H \to \lambda_L$, we know asymptotically,

(A36)

$$U'\left(\frac{1-I}{\lambda_L}\right)\to RU'\left(\frac{RI}{1-\lambda_L}\right)$$

with which, (A35) is equivalent to

(A37)
$$[(1-\theta)R-1](1-I)\frac{(1-\lambda_L)^2}{(RI)^2} > \theta\varepsilon$$

By (A36), we have $I \rightarrow \dfrac{1-\lambda_L}{1-\lambda_L+\sqrt{R}\lambda_L}$, and substitute this into (A37), we can get the sufficient condition for money is desirable.

(A38)
$$\frac{\lambda_L[R(1-\theta)-1][(\sqrt{R}-1)\lambda_L+1]}{R^{3/2}} - \theta\varepsilon > 0$$

By analogy, we can get the sufficient condition for money is not desirable,

(A39)
$$\frac{\lambda_L[R(1-\theta)-1][(\sqrt{R}-1)\lambda_L+1]}{R^{3/2}} - \theta\varepsilon < 0$$

Step 2: We next calculate the left hand side of (A38) and (A39)'s derivatives with respect to θ, ε and R.

The derivative with respect to θ is $\dfrac{(-1+\lambda_L)\lambda_L-\sqrt{R}(\varepsilon+\lambda_L{}^2)}{\sqrt{R}}$. Since $\lambda_L < 1$, it is smaller than zero and thus the left hand side of (A38) and (A39) is decreasing with θ. We know money is desirable when left hand side is larger than 0, therefore a larger θ make money less desirable.

The derivative with respect to ε is $-\theta$. Since $\theta > 0$, it is smaller than zero and thus the left hand side is decreasing with ε. Therefore a larger ε makes money less desirable.

The derivative with respect to *R* is $\dfrac{\lambda_L\{[(3-R(1-\theta)](1-\lambda_L)+2\sqrt{r}\lambda\}}{2R^{5/2}}$. Then when $3-R(1-\theta)>0$, it is greater than zero and thus the left hand side is increasing with R. Therefore a larger R expands the parameter space in which money is desirable.

PROOF OF PROPOSITION 7

(i) First consider the open market operation in the liquidity shortage state. Denote the marginal interest rate as Δi and the marginal amount of loan (or amount of reverse repo operation) as ΔM. Since the operation is marginal, equilibrium investment will not change. Then in the liquidity shortage state, date 1's nominal payment increases to $D_1^N + \Delta D_1^N$, where $\Delta D_1^N = \frac{\Delta M}{\lambda_H y}$. In the normal state, since in the original equilibrium, banks don't pay all of their cash at date 1, but they are still able to pay $D_1^N + \Delta D_1^N$ at date 1 with the marginal operation. As prices don't change, the banks' budget constraint in the normal state is still binding. So the change of date 2' nominal payment is $\Delta D_2^N = -\frac{(\lambda_L / \lambda_H)\Delta M}{(1-\lambda_L)y}$, and the contract payment is $D_2^N + \Delta D_2^N$, which is smaller.

In the liquidity shortage state, banks also need to pay the interest of the loan. Since in the original equilibrium banks make profit in the liquidity shortage state and the operation is marginal, banks have enough liquidity to fulfill the payments in the liquidity shortage case. The change of total payment of banks at date 2 in the liquidity shortage state is $-\frac{1-\lambda_H}{1-\lambda_L}\frac{\lambda_L}{\lambda_H}\Delta M + \Delta M(1+\Delta i)$, which is strictly positive as $1-\lambda_H < 1-\lambda_L$ and $\lambda_L < \lambda_H$. That is, banks have to pay more in the liquidity shortage state, and the profit will decrease by $\left[-\frac{1-\lambda_H}{1-\lambda_L}\frac{\lambda_L}{\lambda_H}\Delta M + \Delta M(1+\Delta i)\right] / P_2^H$. As the central bank only allows banks to pay back the loan by cash, banks need to sell more goods to the patient depositors to collect more cash. The real goods consumed by every patient depositor in the liquidity shortage increases by $\left[\Delta M \dfrac{\lambda_H - \lambda_L}{\lambda_H(1-\lambda_L)} + \Delta M \cdot \Delta i\right] / \left[(1-\lambda_H)P_2^H\right]$ while the consumption in the normal state stays the same. This monetary policy makes depositors' expected utility obtained from real goods consumption strictly increase.

(ii) Next consider the case in which the central bank implements open market operation at date 1 in the normal state. The central bank implements a repo operation or issue a bond to extract liquidity from economy. Now banks can use the remaining cash to buy the bond because the operation is marginal and the remaining cash is always enough with ΔD_1^N not changed. We know the banks' budget constraint is still binding at date 2 as the operation is marginal. The change of D_2^N is $\Delta D_2^N = \dfrac{\Delta M \cdot \Delta i}{(1-\lambda_L)y}$. In the liquidity shortage state, banks have to pay extra $\dfrac{1-\lambda_H}{1-\lambda_L}(\Delta M \cdot \Delta i)$ nominal resources to fulfill the deposit contracts and the real goods consumed by every patient depositor in the liquidity shortage state increases by $\dfrac{1}{1-\lambda_L}\dfrac{\Delta M \cdot \Delta i}{P_2^H}$ and the consumption in the normal state stays the same. Then this monetary policy also makes depositors' expected utility obtained from goods consumption strictly higher.

(iii) Finally, compare the two types of monetary policies in (i) and (ii), we know

$$\frac{\Delta M \dfrac{\lambda_H - \lambda_L}{\lambda_H(1-\lambda_L)} + \Delta M \cdot \Delta i}{(1-\lambda_H)P_2^H} > \frac{\Delta M \cdot \Delta i}{(1-\lambda_H)P_2^H} > \frac{\Delta M \cdot \Delta i}{(1-\lambda_L)P_2^H}$$

We can see that the operation in the liquidity shortage state increases each patient depositor's consumption more than operation in the normal state does. For the two types of operations, utility improvement both occurs in the liquidity shortage state, thus the probability of operation taking effect is the same. Then reverse repo operation in the liquidity shortage state is more effective.

PROOF OF PROPOSITION 8

The proof proceeds in three steps. In step 1, we prove that the only feasible monetary policy to achieve the social optimal allocation is to issue a bond at date 1 in the liquidity shortage state and repay the bond at date 2. In step 2, we derive the deposit contract under a given interest rate of the bond. In step 3, we characterize the interest rate of the central bank bond which can leads to the social optimal allocation in equilibrium.

Step 1: If the social optimal allocation can be achieved, banks make no profits and all the real goods at date 2 are allocated to depositors. From now on, every variable with " ^ " denotes corresponding variables under social optimal investment, and we have $\widehat{P_2^H} = U'\left(\frac{RI_{FB}^*}{1-\lambda_H}\right)/\varepsilon$ and $\widehat{P_2^L} = U'\left(\frac{RI_{FB}^*}{1-\lambda_L}\right)/\varepsilon$. We have $\widehat{P_2^H} < \widehat{P_2^L}$ as $1-\lambda_H < 1-\lambda_L$, which implies that there are more total liquidity in the normal state, that is, $\widehat{P_2^H}RI + 1 < \widehat{P_2^L}RI + 1$.

From the proof of proposition 3, we know when $\varepsilon < \overline{\varepsilon}$, any zero-profit contract will lead to $P_1^H = P_2^H$ and $P_1^L = P_2^L$, which means the banks' payment for impatient depositors is insufficient to buy all the $1-I$ unit of goods at date 1. [1]

[1] Actually, $P_1 = P_2$ occurs because there are not enough cash at date 1. If impatient depositors can buy all the $1-I$ unit of goods, P_1 will be too low and $P_1 < P_2$. Then in equilibrium, patient depositors will sell less goods at date 1 (only part

Therefore, if the impatient depositors are able to buy all the $1-I$ unit of goods at date 1, banks must provide a higher D_1^N, which leads to a higher total payment in the liquidity shortage state as $\lambda_H > \lambda_L$. Therefore, given the zero-profit prices at date 2, $P_2^H = U'\left(\frac{RI}{1-\lambda_H}\right)/\varepsilon$, $P_2^L = U'\left(\frac{RI}{1-\lambda_L}\right)/\varepsilon$, if impatient depositors can buy all the $1-I$ unit of goods at date 1, the banks' liquidity at date 2 in the liquidity shortage state is not enough to fulfill the contract. So at such prices, the central bank must provide liquidity to banks in the liquidity shortage state to implement the zero-profit deposit contracts with which the impatient depositors are able to buy all the goods at date 1.

The feasible monetary policy is that the central bank issues a bond at date 1 in the liquidity shortage state, banks lower their D_1^N to have some cash left after payment and save the remaining cash in central bank, then at date 2 the central bank repay the principal and interest of the bond and banks get more cash to pay the patient depositors.

The social optimal allocation cannot be achieved if the central bank intervenes in the normal state. If the central bank wants to reduce the banks' total liquidity in the normal state, for example, providing a central bank loan to banks at date 1, banks cannot increase D_1^N since the budget constraint will be violated in the liquidity shortage state. Therefore, banks will not buy the central bank loan even they have some cash left after payment at date 1 in the normal state, and the monetary policy is invalid. In summary, the only feasible monetary policy is to issue a bond at date 1 in the liquidity shortage state and repay the bond at date 2.

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of $1-I$ unit of goods), and P_1 will increase and finally we have $P_1 = P_2$. In summary, $P_1 = P_2$ means impatient depositors can't buy all the $1-I$ unit of goods.

Step 2: We characterize the deposit contract under a given interest rate of the bond in this step.

(a) Given an interest rate of the bond (i_B), since the market is competitive, with the central bank' monetary intervention, banks must offer the deposit contract that implements the social optimal allocation.

Now banks' budget constraint in the liquidity shortage state is:

(A40) $$[\lambda_H \widehat{D_1^N} + (1-\lambda_H)\widehat{D_2^N}]\hat{y} \leq \widehat{P_2^H} RI_{FB}^* + 1 + (1-\lambda_H \widehat{D_1^N}\hat{y})i_B$$

Banks' budget constraint in the normal state is:

(A41) $$[\lambda_L \widehat{D_1^N} + (1-\lambda_L)\widehat{D_2^N}]\hat{y} \leq \widehat{P_2^L} RI_{FB}^* + 1$$

In the social optimal allocation, banks make zero profit, which means that (A40) and (A41) should both be equality, that is,

$$\begin{cases}[\lambda_H \widehat{D_1^N} + (1-\lambda_H)\widehat{D_2^N}]\hat{y} = \widehat{P_2^H} RI_{FB}^* + 1 + (1-\lambda_H \widehat{D_1^N}\hat{y})i_B \\ [\lambda_L \widehat{D_1^N} + (1-\lambda_L)\widehat{D_2^N}]\hat{y} = \widehat{P_2^L} RI_{FB}^* + 1\end{cases}$$

We can solve $\widehat{D_1^N}$ and $\widehat{D_2^N}$ as:

$$\widehat{D_1^N} = \frac{i_B + \dfrac{\lambda_H - \lambda_L}{1-\lambda_L} - \dfrac{1-\lambda_H}{1-\lambda_L}\widehat{P_2^L} RI_{FB}^* + \widehat{P_2^H} RI_{FB}^*}{\left(\dfrac{\lambda_H - \lambda_L}{1-\lambda_L} + \lambda_H i_B\right)\hat{y}}, \quad \widehat{D_2^N} = \frac{\widehat{P_2^L} RI_{FB}^* + 1 - \lambda_L \widehat{D_1^N}\hat{y}}{(1-\lambda_L)\hat{y}}$$

where $\hat{y} = 1 + \widehat{P_0} I_{FB}^*$.

(b) Now we prove that banks have no incentive to deviate from this deposit contract under the social optimal investment.

Assume that a bank deviates by cutting $\widehat{D_1^N}$ by $\Delta\widehat{D_1^N}$. In the liquidity shortage state, $\widehat{D_2^N}$ can increase by $\lambda_H \Delta\widehat{D_1^N}(1+i_B)/(1-\lambda_H)$, and in the normal state, $\widehat{D_2^N}$ can increase by $\lambda_L \Delta\widehat{D_1^N}/(1-\lambda_L)$. We know $\frac{\lambda_H(1+i_B)}{1-\lambda_H} > \frac{\lambda_L}{1-\lambda_L}$ since $\lambda_L < \lambda_H$ and $i_B > 0$, and this implies that $\widehat{D_2^N}$ increase less in the normal state. So the increase of $\widehat{D_2^N}$ is determined by $\lambda_L \Delta\widehat{D_1^N}/(1-\lambda_L)$. Since there is no operation in the normal state, depositors still get residual utility from 1 unit of cash altogether. While in the liquidity shortage state, the impatient depositors' cash decreases by $\lambda_H \Delta\widehat{D_1^N}\hat{y}$ at date 1, and patient depositors' cash increases by $\lambda_H \Delta\widehat{D_1^N}\hat{y}(1+i_B)$ at date 2. However, now banks make non-zero profits. The total real goods allocated to the patient depositors decreases by $\Delta\widehat{D_1^N}\hat{y}[\lambda_H(1+i_B)-\lambda_L]/\widehat{P_2^H}$. We need to compare the utility gain of net increase of cash $\lambda_H \Delta\widehat{D_1^N} i_B \hat{y}$ and the utility loss from real-goods decrease. The utility gain of net increase of cash is $\lambda_H \Delta\widehat{D_1^N} i_B \hat{y}\varepsilon$. The utility loss from real-goods decrease is:

$$(1-\lambda_H)U'\left(\frac{RI_{FB}^*}{1-\lambda_H}\right)\frac{\Delta\widehat{D_1^N}\hat{y}[\lambda_H(1+i_B)-\lambda_L]}{\widehat{P_2^H}(1-\lambda_H)} = \Delta\widehat{D_1^N}\hat{y}[\lambda_H(1+i_B)-\lambda_L]\varepsilon .$$

Since $\lambda_H(1+i_B)-\lambda_L > \lambda_H i_B$, the utility loss from real-goods decrease is larger, which means starting from the social optimal allocation point, banks will not deviate by decreasing $\widehat{D_1^N}$.

Increasing $\widehat{D_1^N}$ will decrease the real goods allocated to the impatient depositors and lower the total amount of cash held by depositors at the end of the economy, thus is always harmful to depositors and will never be optimal. In summary, starting from the social optimal point, banks will not deviate by changing the deposit contract $\widehat{D_1^N}$ and $\widehat{D_2^N}$.

The liquidity provided to banks can be expressed as

$$\frac{\lambda_H-\lambda_L}{1-\lambda_L}\widehat{D_1^N}-\frac{\lambda_H-\lambda_L}{1-\lambda_L}+\frac{1-\lambda_H}{1-\lambda_L}\widehat{P_2^L}RI_{FB}^*-\widehat{P_2^H}RI_{FB}^*$$

Step 3: Finally we characterize the interest rate of the central bank bond in equilibrium.

Now the operation is not marginal thus banks may have an incentive to deviate by increasing or decreasing investment at the social optimal investment. When the interest rate of bond is too high, banks may want to cut investment to hold more cash and invest more in the central bank bond with a high interest rate. When the interest rate of bond is too low, banks may want to increase investment and invest less in the central bank bond with a low interest rate. So the interest rate should balance these two concerns and banks have no incentive to deviate.

Consider a bank deviates by increasing ΔI unit of investment. We first characterize how the deposit contract will change with the change of investment. Total liquidity of this bank at date 1 decreases by $\widehat{P_0}\Delta I$. Then at date 2, in the liquidity shortage state, the bank's total liquidity decreases by $\widehat{P_0}\Delta I(1+i_B)-\widehat{P_2^H}R\Delta I$ attributed to the adjustment of investment, while the total liquidity of the bank at date 2 increases by $\widehat{P_2^L}R\Delta I-\widehat{P_0}\Delta I$ in the normal state. Since there is no monetary policy operation in the normal state, if $\widehat{D_1^N}$ stays constant, then $\widehat{D_2^N}$ has to be lowered, which is not desirable. Therefore, subject to the market competition, this bank will adjust $\widehat{D_1^N}$ and $\widehat{D_2^N}$ to ensure that its depositors can still get all the real goods. Denote the change of $\widehat{D_1^N}$ and $\widehat{D_2^N}$ as $\Delta\widehat{D_1^N}$ and $\Delta\widehat{D_2^N}$. The following equation system can ensure that this bank still has zero profit.

$$\text{(A42)} \quad \begin{cases} \left[\lambda_H \Delta\widehat{D_1^N} + (1-\lambda_H)\Delta\widehat{D_2^N}\right]\hat{y} = -\widehat{P_0}\Delta I(1+i_B) + \widehat{P_2^H} R\Delta I - \lambda_H i_B \hat{y}\Delta\widehat{D_1^N} \\ \left[\lambda_L \Delta\widehat{D_1^N} + (1-\lambda_L)\Delta\widehat{D_2^N}\right]\hat{y} = -\widehat{P_0}\Delta I + \widehat{P_2^L} R\Delta I \end{cases}$$

The last term in the first equation, $-\lambda_H i_B \hat{y}\Delta\widehat{D_1^N}$ means that this bank's total liquidity will also change with $\widehat{D_1^N}$ since more $\widehat{D_1^N}$ means this bank saves less cash in the central bank at date 1 thus get less interest income.

Then we derive the equilibrium interest rate of bond. In terms of the real goods, at date 2, since the social optimal allocation can be achieved with the central bank's intervention, and the deviating bank makes zero profit and all the real goods are allocated to depositors. Each patient depositor's real goods consumption increases by $R\Delta I/(1-\lambda_H)$ in the liquidity shortage state and $R\Delta I/(1-\lambda_L)$ in the normal state. With respect to cash, the total cash held by impatient depositors and patient depositors together decreases by $\widehat{P_0}\Delta I(1+i_B) + \lambda_H i_B \hat{y}\Delta\widehat{D_1^N}$ in the liquidity shortage state, and decreases by $\widehat{P_0}\Delta I$ in the normal state. In order to make banks have no incentive to deviate, the total welfare change should be no greater than 0. Deviation by decreasing one unit of investment is a symmetric situation and the welfare change of decreasing one unit of investment should also be no greater than 0. Then the equilibrium interest rate is given by equalizing the gain from the increase in real goods consumption and the loss due to the decrease of residual value of cash. That is,

$$\begin{aligned}&\theta(1-\lambda_H)\left[U'\left(\frac{RI_{FB}^*}{1-\lambda_H}+\frac{R\Delta I}{1-\lambda_H}\right)-U'\left(\frac{RI_{FB}^*}{1-\lambda_H}\right)\right]\\&+(1-\theta)(1-\lambda_L)\left[U'\left(\frac{RI_{FB}^*}{1-\lambda_L}+\frac{R\Delta I}{1-\lambda_L}\right)-U'\left(\frac{RI_{FB}^*}{1-\lambda_L}\right)\right]\\&=(1-\theta)\widehat{P_0}\Delta I\varepsilon+\theta\left[\widehat{P_0}\Delta I(1+i_B)+\lambda_H i_B\hat{y}\Delta\widehat{D_1^N}\right]\varepsilon\end{aligned}$$

Dividing both sides by ΔI and taking $\Delta I \to 0$, we get

$$\text{(A43)}\ \theta U'\left(\frac{RI_{FB}^*}{1-\lambda_H}\right)R+(1-\theta)U'\left(\frac{RI_{FB}^*}{1-\lambda_L}\right)R=(1-\theta)\widehat{P_0}\varepsilon+\theta\left[\widehat{P_0}(1+i_B)+\lambda_H i_B\hat{y}\frac{\Delta\widehat{D_1^N}}{\Delta I}\right]\varepsilon$$

Dividing both sides of (A42) by ΔI, we can solve $\Delta\widehat{D_1^N}/\Delta I$,

$$\text{(A44)}\quad\begin{cases}\left[\lambda_H\dfrac{\Delta\widehat{D_1^N}}{\Delta I}+(1-\lambda_H)\dfrac{\Delta\widehat{D_2^N}}{\Delta I}\right]\hat{y}=-\widehat{P_0}(1+i_B)+\widehat{P_2^H}R-\lambda_H i_B\hat{y}\dfrac{\Delta\widehat{D_1^N}}{\Delta I}\\ \left[\lambda_L\dfrac{\Delta\widehat{D_1^N}}{\Delta I}+(1-\lambda_L)\dfrac{\Delta\widehat{D_2^N}}{\Delta I}\right]\hat{y}=-\widehat{P_0}+\widehat{P_2^L}R\end{cases}$$

Substituting $\dfrac{\Delta\widehat{D_1^N}}{\Delta I}=x_1$ and $\dfrac{\Delta\widehat{D_2^N}}{\Delta I}=x_2$ in (A43) and (A44), we get the result in proposition 8 and the equilibrium interest rate of the bond issued by the central bank.

PROOF OF PROPOSITION 9

We have already characterized the expressions of $\widehat{D_1^N}$ and $\widehat{D_2^N}$ under the equilibrium interest rate of the bond in the proof of proposition 8. Whether social optimal allocation can be achieved by monetary policy is equivalent to whether under the contract $\widehat{D_1^N}$ and $\widehat{D_2^N}$ the impatient depositors can buy $1-I$ unit of real goods at date 1. When $\widehat{D_1^N}$ is small, if the impatient depositors are still able to buy all the $1-I$ unit of real goods, P_1 would be too low and could not be greater than P_2. In equilibrium, we would have $P_1 = P_2$ and the patient depositors will not sell all of $1-I$ unit of real goods, and the impatient depositors' consumption at date 1 will be smaller than $1-I$ unit of goods, which cannot be social optimal. Therefore, we can get the conditions in proposition 9 by making the price at date 1, P_1, which is equal to the amount of cash withdrawn by the impatient depositors under $\widehat{D_1^N}$ divided by $1-I$ unit of goods, larger than the date 2 price P_2.