Are Direct Investments by the Federal Reserve a Good Idea? A Corporate Finance Perspective

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Abstract

Due to the crisis of 2007-2009, financial friction macro models are being used to provide a theoretical foundation for the evaluation of 'unconventional policy'. In these models banks take deposits from households and lend to firms. Empirically, other financial channels that are missing in the models, such as corporate bonds and equity, are also important. This paper analyzes a model in which bank loans and equity are both feasible. Households have limited ability to enforce their claims. If either the bank or the equity market are undistorted the equilibrium is socially efficient. If both are distorted the equilibrium is inefficient. In that case government policy aimed at the bank or at the firm can be helpful. Suitably chosen equity injections, loans, or interest rate subsidies can all work. Interest rate subsidies have the advantage that they occur later and there is less concern about cheating. Equity injections have the advantage that they minimize the necessary level of tax imposed on households that is needed to achieve optimality. Optimal equity injections and optimal loan subsidies induce reductions in household savings ('crowding out'). Optimal interest rate subsidies induce increases in household savings ('crowding in').
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1 Introduction

During the financial crisis of 2007-2009 Federal Reserve policy went well beyond the usual control of interest rates or interactions with banks through the discount window. They provided direct financing to some individual firms. As observed by Bernanke (2013) this policy, "... required the Fed to invoke emergency authority. There is a clause, 13(3), in the Federal Reserve Act that says under unusual and exigent circumstances (basically, in an emergency), the Fed can lend to entities other than just banks. This authority had not been used by the Fed since the 1930s." (page 78) This stimulated a significant literature in which the emergency policy interventions are interpreted as correcting dysfunctional banks, as in Gertler and Karadi (2011), Gertler and Kiyotaki (2011), and Christiano and Ikeda (2011).

Although these models have provided significant insights into issues facing banks under stress conditions, they do not allow for empirically important alternative financial channels that are also available, and that actually functioned during the stress times. It is a simple fact that even during the stress period, many firms raised a significant amount of finance by issuing equity and corporate bonds. As a result it is of potential importance to consider the extent to which this alters the implications for optimal policy towards banks in stress periods. Accordingly we analyze a two-period model based on Christiano and Ikeda (2011) to examine the impact of allowing alternative financing mechanisms. The model is modified by the introduction of a market for new equity in the firm, and by a contract enforcement problem at the firm. This is perhaps the simplest setting that allows us to make our points.

In the model there are two dates denoted 0 and 1. There are households, banks and firms, as well as a government (or "social planner"). There are many households, and each has a unit measure of members. Some members work as bankers while other members work as firm managers. There is perfect insurance inside the household. The household, the bank, and the firm each have an initial endowment. The household uses its endowment in period 0 to consume, deposit at the bank or to invest in firm equity. Following Grossman and Hart (1980), equity is assumed to be subject to free rider problems and so new equity does not monitor the firm. The bank uses its endowment along with any household deposits to make loans to firms in period 0. Following much of the banking literature the bank can monitor and enforce loans perfectly.[1] The firm uses its endowment, along with any bank loan received and the proceeds from any equity sales, to invest in capital that generates revenue. In period 1 the firm uses the revenue to repay anything it owes to the bank. The remainder is paid to the household as a dividend on the equity. The bank pays the interest it owes on deposits and the remainder is paid to the household as a dividend. In all markets agents are price takers. Market clearing determines the interest rate on bank deposits, the interest rate on bank loans, and the equilibrium return on equity.

Following Christiano and Ikeda (2011) moral hazard is introduced at the bank.[2] The banker can decide to make the payment that is due to the depositors, or can default. In a default the depositor gets nothing. The bank takes some of the money and rest is lost. The same moral hazard problem is introduced at the firm so that the firm manager can decide whether or not to default. Requiring managers to have a large enough stake in the success of the firm (or the bank) that they manage imposes a limitation on the willingness of investors to provide money. Comparison is made between what happens when the moral hazard only affects the bank as in Christiano and Ikeda (2011), and what happens when there is moral hazard at both the bank and the firm. In all cases optimal government policy is studied.

The main result of the analysis is that justification of unconventional monetary policy requires frictions in all financial channels at the same time. As long as one channel is able to operate without distortion, then in our model the equilibrium is socially efficient even without any special government policy. Justification for government policy thus requires that all financial markets are dysfunctional at the same time. At such times there is scope for suitable policy to improve the equilibrium. Direct loans by the Federal Reserve to individual firms are one method, but not the only one. We show that there are several

[1] See Diamond (1984), Williamson (1986), Calomiris and Kahn (1991), Boyd and De Nicolo (2005) and Freixas and Rochet (2008)

[2] This moral hazard problem has been used in many studies such as Calomiris and Kahn (1991) and Rampini and Viswanathan (2010).

alternative policies that are capable of attaining social efficiency. From the social planner's perspective these policies are equally desirable. However, these policies do have a number of different implications from each other for private sector actions.

The optimal policy can take the form of an equity injection, a loan, or an interest rate subsidy. Any of these can be used. Furthermore any of these can be targeted at the bank or targeted at the firm. For instance, a suitable policy aimed at the bank induces the bank not to want to default. When that happens the bank uses its monitoring ability to induce efficient production by the firm. Similarly, a suitable policy aimed at the firm will also work. In this case the government gives the firm enough extra value so that defaulting is unattractive to the management of the firm. Efficient production is then in the interest of both the firm and the economy as a whole.

Optimal government policy in the frictional equilibrium reduces early consumption and increase later consumption. As a result there is an increase in the resources that flow initially from the household to the firm. Part of this flow goes through the government and part of it goes through private investment. When optimal equity injections or optimal bank loans are used, household savings is reduced. When optimal interest rate subsidies are used, household savings increases. Equity injections can achieve social optimality with a lower level of tax imposed on households than are needed with subsidized loans or subsidized interest rates. Pure equity injections are directly targeted at the incentive to cheat.[3] However, it should be noted that interest rate subsidies may have an advantage if later taxation is preferred to earlier taxation for some reason that goes beyond our model.

How does our model relate to the idea that the policies are for emergencies? The perspective we adopt from Christiano and Ikeda (2011) is that in normal times neither friction is binding. Think of this as the 70 years that the Fed did not invoke the emergency power to justify unconventional policy. A crisis is the point at which suddenly one or more of the frictions creates a binding constraint. In our setting, in order to justify the unconventional policy it is necessary that the newly binding constraints affect all financial channels at the same time. If, for example, the corporate bond market continues to function normally

[3]Of course, there may be other drawbacks to equity injections that are outside of our model. For example if only some firms are to get equity injections, who selects the winners? Introducing such offsetting considerations would mitigate the attraction of equity injections and complicate the analysis.

the justification for unconventional Federal Reserve policy is more tenuous than it might seem in a model that had more restricted financing alternatives permitted.

Our paper is related to large literatures in macroeconomics, banking and corporate finance. Surveys are provided by Gertler and Kiyotaki (2011), Freixas and Rochet (2008), and Frank and Goyal (2008) respectively. In terms of the macro literature particularly closely related are the insightful papers of Gertler and Karadi (2011), Gertler and Kiyotaki (2011) and Christiano and Ikeda (2011) which provide the foundation for our analysis. The key difference is that we permit households and firms to interact without going through the bank. This difference is empirically relevant because these alternative channels actually exist and operat on a large scale. We show that this difference is significant for policy implications as well.

There are many banking papers[4], including studies of optimal bank capital ratios by Van den Heuvel (2008), Begenau (2014), and Nguyen (2013). Allen and Carletti (2013) assumes segmented deposit and bank equity markets when equity capital is more expensive. They find that bank lending differs depending on the amount of capital it has. These studies, like the macro literature have a general equilibrium flavor, but assume that firm finance happens through the bank. Thus the key difference is our focus on the presence of alternative financial channels.

There are starting to be studies with alternative channels in the form of shadow banks. Hanson et al. (2014) consider a model in which shadow banks compete with regular banks. Both banks and shadow banks provide liquidity to households. However they finance the liquidity provision differently. Deposit insurance plays a key role in their interpretation of banks. Their focus is on the type of asset that banks invest in. They do not really focus on firm incentive issues that play a large role in our approach.

To study credit crunches, Holmstrom and Tirole (1997) have risk-neutral investors,

[4]Traditionally the question was why banks exist. It could be due to the provision of monitoring services over borrowers, as in Diamond (1984). It could be due to the provision of liquidity to depositors as in Diamond (1997). There is also interest in why deposit taking and lending activities coexist in a bank, see Diamond and Rajan (2000), Diamond and Rajan (2001), and Kashyap et al. (2002). Gorton and Winton (1995) show that the coexistence of bank deposit taking and lending activities may induce the government to leave the banking sector risky. These banking studies are really about why a bank might exist in something akin to the modern form. These models are not designed to address the emergency Fed policy issues that we study.

intermediaries, and firms. There is a moral hazard problem at the firm and at the intermediary so that the managers may shirk. They show that the allocation of capital affects the equilibrium, a feature that also shows up in our analysis. They study managerial shirking behavior while we study limited contract enforceability. The policy implications are rather different as a result.

The corporate finance literature generally takes it for granted that firms obtain funds from corporate bonds and equity, as well as bank loans. It is also common to investigate the impact of incentive problems at the firm level, e.g. Jensen and Meckling (1976), Hart and Moore (1994), and Rampini and Viswanathan (2010). Limited commitment at the firm level is studied in a particularly rich setting by Ai et al. (2013). It is much less common for such papers to take an equilibrium perspective, or to evaluate the social welfare impacts of policy as we do.

Following Modigliani and Miller (1958) much traditional corporate finance does not distinguish between bank loans and corporate bonds. This is fine for many purposes. But it is inadequate for consideration of bank policy. For example classical papers such as Stiglitz (1969), and Miller (1977) take an equilibrium perspective, but do not explicitly model banks. Denis and Mihov (2003) do empirically study differences between various forms of debt. More recently Gornall and Strebulaev (2013) considers the interaction of capital structure at banks and at borrowers in a partial equilibrium setting.

The plan for the paper is as follows. The next section provides some motivating evidence to show that the bank loan channel coexists with other channels for household savings to reach firms. In Section 3 the friction free version of the economy is presented as a benchmark. The impact of bank and firm level frictions on the equilibrium are presented in Section 4. The social welfare properties are presented in Section 5. In Section 6 policies that can improve social welfare of an equilibrium with frictions are analyzed. The conclusion is in Section 7. A number of the proofs are collected in an appendix as Section 8.

2 Motivating Evidence

Fact 1 *Bank deposits are not all that big relative to other household financial asset acquisition.*

Household money that is not being used for consumption is allocated to many different assets, not just bank accounts. The Financial Accounts of the United States[5] for 2013, provides useful information. Personal disposable income was $12,476 billion from which personal outlays took $11,910 billion and $566 billion was personal savings. In other words consumption is much larger than savings.

Net flows of bank deposits come in several forms. Checkable deposits and currency were $82 billion, time and savings deposits were $185 billion and money market funds were -$26 billion. If we sum these three as being a rough estimate of net acquisition of bank deposits it says that there was a net acquisition of about $241 billion in bank deposits. Households acquisition of corporate and foreign bonds amounted to $104 billion.

Direct household net equity is routinely negative in recent years and in 2013 it was -$324 billion. However intermediated equity are large positive numbers. Mutual fund shares were $723 billion, and pension entitlements were $493 billion. If these three categories are summed they can be viewed as roughly the household net investments in equity, and it adds up to $892 billion. These numbers do not exactly add up largely due to holdings of Treasuries, and holdings of Agencies and GSE assets. These are often large values which are positive in some years and negative in other years. In 2013 both were negative.

Overall this shows that in 2013 net equity investments by households were bigger than net acquisition of bank accounts. This fact is not special to the year 2013.

Fact 2 *Bank loans are not the dominant source of financing flows to the corporate sector.*

The data is again from the Financial Accounts of the United States for 2013. It should be kept in mind that the interactions between banks and firms is complex because banks provide transaction services along with loans. Firms acquired $118 billion of checkable deposits and currency. We might suppose that the acquisition of checking deposits by

[5] http://www.federalreserve.gov/releases/z1/current/z1.pdf see Tables F100 and F101.

firms mainly reflect transaction motives rather than what we think of as a bank financing channel.

From the banks firms receive depository institution loans n.e.c. of \$95 billion, other loans and advances of \$55 billion, and mortgages of \$104 billion. Adding these three together, bank finance of nonfinancial business amounted to roughly \$253 billion. This bank financing was dwarfed by \$640 billion worth of corporate bonds. Net corporate equity issues were -\$384 billion, trade receivables were \$155 billion, trade payables were \$137 billion, and there are many other categories that are lumped together as miscellaneous liabilities in the amount of \$279 billion.

Overall this data illustrates the fact that firms are not restricted to using bank loans as the main financing vehicle. Corporate bonds by themselves are larger than bank loans, and there are a large number of other firm financing channels in use. While the exact numbers do fluctuate from year-to-year, the orders of magnitude are not special to the year 2013. Firms are not typically dependent on banks as the sole form of financing. They really do have alternatives, and these alternatives are, if anything, a larger channel than bank loans for U.S. firms.

In the model presented in this paper the alternative channel of financing is called equity. However, given the rest of the modelling structure it might equally well have been called corporate bonds. In reality equity and corporate bonds are not fully interchangeable. Our purpose is simply to show that it is important to recognize that banks are not the only financing channel.

In this paper, we do not dig more deeply into a broader range of financing alternatives – a topic which is, of course a central focus in corporate finance. Colla et al. (2013) and Eckbo and Kisser (2013) provide a more extensive empirical analysis of corporate use of commercial paper, lines of credit, term loans, bonds, equity, and leases, along with other more minor types of financing. There is considerable variability among firms in the use of these alternatives. Denis and Mihov (2003) show that higher credit firms tend to use more non-bank debt. Since these firms also tend to be large, they have a major impact on the aggregate measures of financial flows in the US economy. Frank and Goyal (2009) observe that equity is particularly important to the financing of smaller firms.

3 Benchmark Model Economy

The economy consists of households, banks, and firms each of which starts with a non-negative initial endowment, denoted y_h, y_b, y_f respectively. Households must choose consumption in both periods (c_0 , c_1) and savings (S). Savings can be deposited at a bank (d_h), or supplied as equity (e_h) to a firm. Banks decide their demand for deposits (d_b), and their willingness to supply loans (ℓ_b) to firms. Firms have a demand for equity (e_f) and a demand for bank loans (ℓ_f). The firm uses its initial endowment plus any funds from equity issues or bank loans to produce revenue. Bank profits and firm profits are returned to the household for use in purchasing period 1 consumption. The bank deposit market interest rate is r_d. The return on new equity is r_e, and the rate on a bank loan to a firm is r_ℓ.

An equilibrium is a set $\{c_0, c_1, d, e, \ell, r_d, r_e, r_\ell\}$ such that: (i) the household, bank and firm problems are solved, (ii) the debt ($d_h = d_b \equiv d$), equity ($e_h = e_f \equiv e$) and loan ($\ell_b = \ell_f \equiv \ell$) markets clear, (iii) $\{c_0, c_1, d, e, \ell, r_d, r_e, r_\ell\} \geqslant 0$.

This definition guides the structure of the analysis. The first step is to solve for the quantity variables from the household $\{c_0, c_1, d_h, e_h\}$, the bank $\{d_b, \ell_b\}$, and the firm $\{e_f, \ell_f\}$. Once that is done, market clearing is imposed for the debt, equity and loan markets. Market clearing is used to obtain the set of market returns, $\{r_d, r_e, r_\ell\}$. Finally, all of these values are used to calculate the values of the objective functions for each sector.

It is worth considering the possible interpretation of $d < 0$ and $e < 0$. Analytically there is nothing wrong with negative values for d and e. Instead of depositing money at the bank, the household is borrowing money from the bank. Instead of making an equity investment in the firm, the firm is repurchasing equity from the household. So negative values do have a financial interpretation. However, in reality the market does not generally treat positive and negative values symmetrically. For instance bank loan to a firm might have a greater interest rate than a bank deposit received from a firm. This could reflect a range of costs or information asymmetries that go well beyond the scope of the current paper.

3.1 Household Preferences and Budget Constraints

The household structure directly follows the approach of Christiano and Ikeda (2011). There are a large number of identical households. Each has a unit measure of members. Some are investors while others are bankers of firms. The bank and the firm are assumed to maximize their own profits. However, the household model means that the consumption decisions include any profits generated by the bank and the firm as can be seen in the budget constraint of the household problem given by equation (3).

The representative household consumes both in period 0 and in period 1, with perfect internal insurance. The household enters period 0 with an endowment $y_h > 0$, consumes c_0 at time 0, and c_1 at time 1. The part of the endowment that is not consumed can be invested either as a bank deposit d_h, or as an equity investment in the firm e_h. The promised interest on bank deposit is r_d, and the expected return on the equity (dividends plus capital gains) is denoted r_e. Let π_b be the bank profit, and let π_f be the firm profit. The period utility function is $u(c)$, with $u' > 0$, $u'' < 0$. We assume power utility,

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}, 0 < \gamma \leqslant 1.$$

The household's problem is

$$\begin{aligned}
\max_{d_h \geqslant 0, e_h \geqslant 0} \quad & U(c_0, c_1) = u(c_0) + \beta u(c_1) && (1)\\
\text{s.t.} \quad & c_0 = y_h - (d_h + e_h) && (2)\\
& c_1 = (1 + r_d)d_h + (1 + r_e)e_h + \pi_b + \pi_f && (3)\\
& c_0 \geqslant 0, c_1 \geqslant 0. && (4)
\end{aligned}$$

If the initial endowment is large enough then the household saves. Savings can be invested either in bank deposits or in the equity market.[6] Let $S = d_h + e_h$ be the total savings and let $r_{\max} = \max\{r_e, r_d\}$. Then, by Lemma 17 from the Appendix, the optimal

[6] An alternative idea is that bank deposits are useful as money in transactions. In that case the model must face the classic issue of why money is valued. For instance, Begenau (2014) models this idea by directly placing bank deposits in the household utility function. In our model there is no extra benefit associated with bank deposits, although adding a simple version would amount to a rescaling.

savings is

$$S = \frac{y_h \beta^{1/\gamma} - (\pi_b + \pi_f)(1 + r_{\max})^{-1/\gamma}}{\beta^{1/\gamma} + (1 + r_{\max})^{(\gamma-1)/\gamma}}, \tag{5}$$

provided $S \geqslant 0$. This Lemma, along with many other proofs, are collected in Appendix 8. Whether $S \geqslant 0$ is given by the condition $(\pi_b + \pi_f) \leqslant y_h(\beta(1 + r_{\max}))^{1/\gamma}$. The quantity $y_h(\beta(1 + r_{\max}))^{1/\gamma}$ is the value of the returns to the household endowment if invested. Hence, the condition $S \geqslant 0$ is that the saving is worthwhile.

When the saving is worthwhile (i.e., $S \geqslant 0$), whether savings take the form of debt or of equity depends on the returns to each investment. If $r_d < r_e$, then $d_h = 0$ and $e_h = S$; if $r_d > r_e$, then $d_h = S$ and $e_h = 0$. If $r_d = r_e$, then $d_h + e_h = S$ for any nonnegative d_h and e_h; the household is equally happy holding any mix of debt and equity as long as they add up to the amount that the household wants to save. Optimal household consumption is

$$c_0 = \frac{y_h(1 + r_{\max})^{(\gamma-1)/\gamma} + (\pi_b + \pi_f)(1 + r_{\max})^{-1/\gamma}}{\beta^{1/\gamma} + (1 + r_{\max})^{(\gamma-1)/\gamma}}, \tag{6}$$

$$c_1 = \frac{\beta^{1/\gamma}(y_h(1 + r_{\max}) + \pi_b + \pi_f)}{\beta^{1/\gamma} + (1 + r_{\max})^{(\gamma-1)/\gamma}} = (\beta(1 + r_{\max}))^{1/\gamma} c_0. \tag{7}$$

If saving is not worthwhile (i.e. $(\pi_b + \pi_f) > y_h(\beta(1 + r_{\max}))^{1/\gamma}$), the household will still get any profits earned by the bank and the firm ($\pi_b + \pi_f$). In that case, the household consumes $c_0 = y_h, c_1 = \pi_b + \pi_f$.

To summarize, there are two cases. 1) If the household has too small an endowment, then it does not wish to save at all. It consumes the full endowment in period 0. In period 1 it consumes the sum of the bank profit and the firm profit. 2) If the household has a sufficiently large endowment, it will save, and the household quantity decisions are in effect provided by equations (5)-(7). All saving is invested in equity or debt depending on which has the higher return. If the returns are equal then the household treats them equally and is indifferent as the mix of debt and equity that it holds. This is spelled out more completely in the Appendix 8.

3.2 Bank Problem - No Friction

Banks are a major channel that takes money from households and uses it to provide funds to firms. After the firm produces revenue it repays the bank with interest. The bank uses that money to repay the household with interest. Any bank profits are paid to the bank owner that happens to be the household. So a bank starts with an endowment (y_b), has a demand for deposits (d_b), and supplies loans to the firm (ℓ_b). The interest rate on deposits is r_d, and the rate on loans is r_ℓ. The deposit and loan rates are determined by market clearing.

The bank problem is,

$$\max_{\ell_b \geqslant 0, d_b \geqslant 0} \quad \pi_b = y_b + d_b + (1 + r_\ell)\ell_b - (1 + r_d)d_b - \ell_b \tag{8}$$

$$\text{s.t.} \quad \ell_b \leqslant y_b + d_b \tag{9}$$

The coefficient on ℓ_b is positive in the objective function given by expression (8), so the constraint given by inequality (9) must be binding. Accordingly, the bank objective function can be written as $\pi_b = (1 + r_\ell)y_b + (r_\ell - r_d)d_b$.

The bank has a linear objective function, and the markets are assumed to be perfectly competitive. So the optimal solutions are a mix of corner solutions and indifference conditions. Whether the solution is at a corner or not depends, of course, on the rates of return on bank deposits and bank loans. If the loan rate is below the deposit rate, the bank has no demand for bank deposits. If the loan rate is above the deposit rate rate the bank would like to have an infinite quantity of deposits.

These observations can be summarized as follows. The solution to the bank problem (8) depends on the interest rates as follows. Case 1. If the loan rate is below the deposit rate ($r_\ell < r_d$) then the bank has no demand for deposits and supplies its endowment as loans ($d_b = 0, \ell_b = y_b$) with profits of $\pi_b = (1 + r_\ell)y_b$. Case 2. If the loan rate equals the deposit rate ($r_\ell = r_d$) then the bank is equally willing to take any level of deposits ($d_b \in [0, +\infty)$) it lends out those deposits along with its endowment ($\ell_b = y_b + d_b$) with profits of $\pi_b = (1 + r_\ell)y_b$. Case 3. If the loan rate is above the deposit rate ($r_\ell > r_d$) the bank would demand an infinite amount of deposits to lend out ($d_b = +\infty, \ell_b = +\infty$)

which would generate infinite profits ($\pi_b = +\infty$). Infinite deposits and loans are not consistent with an equilibrium. So it is clear that in an equilibrium, $r_\ell \leqslant r_d$.

3.3 Firm Problem - No Friction

Firms take in funds which are used to generate revenue. The revenue is used to pay the bank loan and to meet commitments to new equity. Old equity gets the remainder which is then passed along to the household that owns the firm. This can also be interpreted as old and new equity being treated equally, but new equity buying in at a rate necessary to induce participation.

The firm starts with an endowment $y_f > 0$, and it may raise new funds in the form of an equity issue e_f or in the form of a bank loan ℓ_f. All of these funds are converted into capital k with $k = \ell_f + e_f + y_f$. Capital is productive and it generates revenue of Ak, $A > 1$. The bank loan is a promise to pay the bank $(1 + r_\ell)\ell_f$. The rate on such a loan is determined by market clearing in the loan market.

New equity is treated the same as old equity, after it has been issued. When the firm issues e_f it is selling a fraction λ of the firm to new equity. The money available to be split between new and old equity is $Ak-(1+r_\ell)\ell_f$. So old equity gets $(1-\lambda)(Ak-(1+r_\ell)\ell_f)$, and new equity gets $\lambda(Ak-(1+r_\ell)\ell_f)$. How is λ determined? Consider an equity investor who invests an amount e. That money could instead have been invested elsewhere earning $(1 + r_e)e$. The required rate of return on equity is denoted r_e, so

$$(1 + r_e)e = \lambda(Ak - (1 + r_\ell)\ell).$$

Accordingly,

$$\lambda = \frac{(1 + r_e)e}{Ak - (1 + r_\ell)\ell}.$$

Hence, the old equity/firm objective function can be expressed as $(1-\lambda)(Ak-(1+r_\ell)\ell_f) = A(\ell_f + e_f + y_f) - (1 + r_\ell)\ell_f - (1 + r_e)e_f$. This form of the maximization problem shows that what we are calling new equity can also be given other interpretations. In particular it could be a corporate bond, in which case r_e is the required return on a corporate bond.

The problem can also be rewritten to isolate the coefficients on the choice variables,

$$\max_{\ell_f \geqslant 0, e_f \geqslant 0} \pi_f = Ay_f + (A - (1 + r_\ell))\ell_f + (A - (1 + r_e))e_f.$$

An interior solution for bank loans requires $A-(1+r_\ell) = 0$, and an interior solution for equity requires $A - (1 + r_e) = 0$. Otherwise, for each security $i = \ell, e$, the firm's demand is zero if $A < 1 + r_i$, and the firm's demand is infinite if $A > 1 + r_i$. If $A = 1 + r_i$, then any finite value will do, and $\pi_f = Ay_f$. Thus the key case arises when $r_e = r_\ell = A - 1$. In this case the firm is willing to borrow any nonnegative finite quantity from the bank and it also regards all nonnegative finite quantities of new equity as equally desirable.

3.4 Competitive Equilibrium - No Friction

Recall that an equilibrium is a set $\{c_0, c_1, d, e, \ell, r_d, r_e, r_\ell\}$ such that: (i) the household, bank and firm problems are solved, (ii) the debt ($d_h = d_b \equiv d$), equity ($e_h = e_f \equiv e$) and loan ($\ell_b = \ell_f \equiv \ell$) markets clear, (iii) $\{c_0, c_1, d, e, \ell, r_d, r_e, r_\ell\} \geqslant 0$. To verify that there is an equilibrium we follow the list of items in the definition. So far the quantity solutions have been obtained. The next step is to get the corresponding returns that are consistent with equilibrium.

It is helpful to clarify definitions. First, in an **equilibrium**, $d \equiv d_b = d_h$, $\ell = \ell_b = \ell_f$, $e \equiv e_h = e_f$. The bank deposit market, the bank loan market, and the new equity market must all clear. Second, we say that an equilibrium is **strictly interior** provided that at the equilibrium, $d > 0$, $\ell > 0$, and $e > 0$. Third, an equilibrium is said to have a (**positive) interest rate spread** provided it is also the case that $r_\ell > r_d$ with $d > 0$ and $\ell > 0$. **A positive interest rate difference** $r_\ell > r_d$ might or might not be a strictly interior equilibrium depending on what is happening in the equity market. If either $d = 0$ or $\ell = 0$, then we would not say that there is a positive interest rate spread because there is no trading in one or both of these markets. A key question is whether there is a positive interest rate spread, or not.

To characterize the equilibrium we start by observing the restrictions on the rates of return that must hold. These come from the necessity that the firm and the bank solutions must be finite. To have an interior solution the desired trades in the financial markets must be positive as well.

The firm is maximizing a linear objective function. If productivity is too high (relative to the cost of funds) the firms will want to raise an infinite amount of financing. Equilibrium requires a finite amount of production. For the firm problem to have a finite solution in an equilibrium it must be the case that $r_e \geqslant A - 1$ and $r_\ell \geqslant A - 1$.

Consider the bank's problem. For the bank problem to have a finite solution in an equilibrium it must be the case that $r_\ell \leqslant r_d$. Otherwise ($r_\ell > r_d$) the bank would issue an infinite amount of loans. Suppose that this inequality is made strict, so that $r_\ell < r_d$. Then the bank has no demand for deposits and so in an equilibrium it must be the case that $d = 0$. It follows directly that if there are positive bank deposits in an equilibrium ($d > 0$) it must be the case that $r_\ell = r_d$.

The next step is to consider the impact of comparing the return that the firm must pay on bank loans and the return that the firm must promise to new equity. In an equilibrium, 1) if the return on equity is strictly greater than the return on loans ($r_e > r_\ell$), then the return on equity then must exceed productivity ($r_e > A - 1$), and so the firm issues no new equity ($e = 0$). 2) if the return on bank loans strictly exceeds the return on equity ($r_\ell > r_e$), then the return on loans must exceed productivity ($r_\ell > A - 1$) and there will not be any bank loans ($\ell = 0$). In order to have an equilibrium with both equity issued and bank loans, it must be the case that $r_e = r_\ell$.

Recall that $S = d + e$ is the amount of savings by the household. Let S^* denote the equilibrium amount of savings. Then an interior equilibrium will have both $d > 0$ and $e > 0$, and so $S^* \geqslant 0$. Then an equilibrium is characterized as follows.

Proposition 3 *There is no equilibrium that has a positive interest spread. Under the condition,*

$$\beta^{1/\gamma} y_h - A^{(\gamma-1)/\gamma}(y_b + y_f) > 0, \tag{10}$$

there exists an interior equilibrium with: (i) The equilibrium interest rates are given by $r_d = r_e = r_\ell = A - 1 \equiv r$. *(ii) The equilibrium deposit* $d \geqslant 0$ *and the equilibrium equity investment* $e \geqslant 0$

satisfy

$$d + e = S^* \equiv y_h - \frac{A^{(\gamma-1)/\gamma}(y_h + y_b + y_f)}{\beta^{1/\gamma} + A^{(\gamma-1)/\gamma}}, \tag{11}$$

and the equilibrium loan is $\ell = y_b + d$. *(iii) The equilibrium consumptions are given by*

$$c_0^* = \frac{A^{(\gamma-1)/\gamma}(y_h + y_b + y_f)}{\beta^{1/\gamma} + A^{(\gamma-1)/\gamma}} \tag{12}$$

$$c_1^* = \frac{A\beta^{1/\gamma}(y_h + y_b + y_f)}{\beta^{1/\gamma} + A^{(\gamma-1)/\gamma}} = (A\beta)^{1/\gamma} c_0^* \tag{13}$$

(iv) The equilibrium household utility is

$$u^* = \frac{1 + A^{(1-\gamma)/\gamma}\beta^{1/\gamma}}{1-\gamma}(c_0^*)^{1-\gamma}.$$

This proposition, which is proved in Appendix 8, provides the benchmark for what follows. Because the bank has a linear objective, there is no equilibrium with a positive spread between the deposit rate and the loan rate. If such a spread existed, the bank would want an infinite amount of deposits.

In order to have an interior equilibrium it is necessary that the household wants to save/invest. Otherwise the household would simply consume everything in period zero. There would be no bank deposits or equity investments. A key requirement for this is that the household endowment (adjusted for discounting) must be large enough relative to the endowments of the bank and the firm (adjusted for productivity). This is the requirement given by condition (10).

Provided that the household wishes to invest, the question is whether to invest in bank deposits or in new equity issued by the firm. Because we are following Christiano and Ikeda (2011) in assuming no uncertainty, any investment by the household must have an expected return of at least the risk-free rate. Since both investments offer the same return, the household is indifferent between them. The sum $d + e$ is determined, but the individual values for d and e are not pinned down. Due to the fact that household deposits at the bank are not fully pinned down, neither is the amount that the bank will lend to the firm. The bank lends its endowment plus any deposits received from the household.

Household investments are increasing in the household endowment and decreasing in bank and firm endowments. Household consumption in both periods is increasing in household endowment, bank endowment and firm endowment. Household consumption in the two periods is tightly linked through the discount parameter and the productivity parameter. Household consumption in period 0 is decreasing in the productivity of capital (A). Household consumption in period 1 is decreasing in the productivity if the productivity level is low (specifically, if $A \leqslant \left(\frac{1}{\beta\gamma^{\gamma}}\right)^{1/(1-\gamma)}$), and is increasing in the productivity otherwise.

The equilibrium interest rate is pinned down by the productivity of capital. Understanding the impact of a change in the interest rate on household saving is thus equivalent to understanding the impact of the productivity of capital. If capital becomes more productive (an increase in A) then for interior parameters, period zero consumption declines while savings and period one consumption increases. Utility also increases when productivity increases.

Suppose that $d \geqslant 0$ and $e \geqslant 0$, but the equation (10) is not satisfied. Then the equilibrium will be at a corner with $d = e = 0$, $c_0 = y_h$, $c_1 = \pi_b + \pi_f = A(y_b + y_f)$, $\pi_b = Ay_b$ and $\pi_f = Ay_f$. The interest rates are not fully pinned down. The interest rates must satisfy $A \leqslant 1 + r_d \leqslant \frac{1}{\beta}(\frac{A(y_b+y_f)}{y_h})^{\gamma}$, $A \leqslant 1 + r_e \leqslant \frac{1}{\beta}(\frac{A(y_b+y_f)}{y_h})^{\gamma}$, and $A - 1 \leqslant r_\ell \leqslant r_d$.

In corporate finance theory it is often assumed that investors are risk-neutral with infinitely deep pockets. Such an assumption – which we do not make – would extend the indifference between debt and equity investments, to settings with risky production. In corporate finance, theory is also frequently simplified by assuming that there is no discounting for time. In our model that means setting $\beta = 1$. That assumption would simplify some expressions, but it would not fundamentally alter the character of the friction-free equilibrium.

4 The Impact of Frictions

A great deal of attention has been paid to 'financial frictions' as motivation for policy activism. The idea is that this may help to account for the importance of having banks adequately capitalized. Following papers such as Kehoe and Levine (1993), Rampini and Viswanathan (2010), Christiano and Ikeda (2011) and Gertler and Kiyotaki (2011) we therefore introduce a particularly simple limited enforcement problem, or 'friction' at the bank. If the bank is inadequately capitalized it can steal a fraction of the cash flow without any further consequences for the bank, but leaving the depositors unpaid. Knowing this, in equilibrium the households will not make deposits at the bank such that the bank would be induced to cheat. This limits the range of things that can happen in an equilibrium.

In corporate finance much of the focus is on incentive problems at the corporate level. We therefore assume that if the firm is inadequately capitalized it too will cheat. Knowing this households have an upper bound on their willingness to make new equity investments in the firm. Banks can monitor firms and enforce repayment. Thus in our model bank loans to firms have the conventional enforcement advantage when compared to equity.

4.1 Bank Problem with Friction

The bank problem with an enforcement friction is

$$\max_{d_b \geqslant 0} \quad \pi_b = (1+r_\ell)y_b + (r_\ell - r_d)d_b \tag{14}$$

$$\text{s.t.} \quad \theta_b(1+r_\ell)(y_b + d_b) \leqslant (1+r_\ell)y_b + (r_\ell - r_d)d_b \tag{15}$$

The objective function is the same as in the model with no friction.

The difference from the friction-free problem is given by inequality (15). The idea is that it is not possible to fully control the bank's managers. They might decide to steal some of the firm's money and run away. If that were to happen the depositors would not get paid.[7] Bank depositors worry about this, and so will not make any level of deposit that would induce the bank to want to default and run away. The bank that cheats takes a fraction θ_b of the revenue, where $0 < \theta_b < 1$.

The constraint on bank deposits is given by inequality (15) and it can be rewritten as

$$0 \leqslant (1 - \theta_b)(1 + r_\ell)y_b - (1 + r_d - (1 - \theta_b)(1 + r_\ell))d_b \tag{16}$$

In order to avoid cheating by the bank manager, the returns on the bank's original endowment (y_b) must be large enough to be worth hanging onto relative to the impact of payments on bank deposits (d_b).

Consider what happens if $1 + r_d \leqslant (1 - \theta)(1 + r_\ell)$. The rate that the bank pays on deposits is low relative to the rate the bank receives on loans. In that case the coefficient on d_b is negative and so the inequality (16) holds for sure. The optimal solution is the same as the version of the model with no friction. The bank solution is simple. If the loan rate is below the deposit rate ($r_\ell < r_d$) then the bank has no demand for deposits ($d_b = 0$) and profits are ($\pi_b = (1 + r_\ell)y_b$). If the loan rate equals the deposit rate ($r_\ell = r_d$) then the bank is equally willing to take any level of deposits ($d_b \in [0, +\infty)$) and profits are ($\pi_b = (1 + r_\ell)y_b$). If the loan rate exceeds the deposit rate ($r_\ell > r_d$) the bank has an infinite willingness to supply loans ($d_b = +\infty$) which would generate infinite profits ($\pi_b = +\infty$).

Now consider what happens if $1 + r_d > (1 - \theta)(1 + r_\ell)$. Inequality (16) can be expressed as a restriction on the maximal amount of bank deposits that are consistent with the firm not cheating,

$$d_b \leqslant \frac{(1 - \theta)(1 + r_\ell)y_b}{1 + r_d - (1 - \theta)(1 + r_\ell)}.$$

The endowment must be large enough relative to the deposits or else the constraint would be violated. The bank decisions are summarized as follows.

Proposition 4 *Suppose that $1 + r_d > (1 - \theta_b)(1 + r_\ell)$. If the loan rate is below the deposit rate*

[7] The worries of bank depositors has a long history. It is a major motivation for deposit insurance provided by the government. Such insurance mitigates this concern. However it does not fully get around all of the associated costs. For analytic simplicity we do not model deposit insurance. As long as the insurance is imperfect, very similar results to that provided would apply.

($r_\ell < r_d$) then the bank has no demand for deposits ($d_b = 0$) and the profit is $\pi_b = (1 + r_\ell)y_b$. If the loan rate equals the deposit rate ($r_\ell = r_d$) then the bank is equally willing to take any level of deposits up to a bounded value ($d_b \in [0, \frac{1-\theta_b}{\theta_b} y_b)$) and the profit is $\pi_b = (1 + r_\ell)y_b$. If the loan rate exceeds the deposit rate ($r_\ell > r_d$), then the bank has a bounded demand for deposits ($d_b = \frac{(1-\theta_b)(1-r_\ell)y_b}{1+r_d-(1-\theta_b)(1+r_\ell)}$) a finite willingness to supply loans ($d_b + y_b$) which generates a profit of $\pi_b = \frac{(1+r_\ell)(1+r_d)\theta_b y_b}{1+r_d-(1-\theta_b)(1+r_\ell)}$.

When the bank deposit and lend rates are equal, the constraint places an upper bound on the amount of deposits that can be raised by the bank. When the lending rate exceeds the deposit rate, the friction free bank would raise an infinite amount of deposits and issue an infinite amount of loans. The constraint places an upper bound on both the deposit taking and the lending by the bank.

4.2 Firm Problem with Friction

The macro financing frictions literature has primarily focused on issues that afflict banks. Given the financial crisis, this focus is certainly understandable. The older corporate finance tradition focused on frictions at the firm level. Both have merit. So the next step is to consider the impact of potential cheating by the firm's manager. To do this it is necessary to take a stand on the ability of investors to enforce their claims. The previous section provided an analysis of the limited ability of investors to enforce their claims on the bank. To what extent can investors enforce their claims on the firm? To what extent can the bank enforce its claims on the firm?

Our approach to answering these two questions follows the corporate finance approach. It is well understood that the enforcement problem at the bank and in the equity market differ. Equity investors have very minimal ability to monitor and enforce claims. Following Grossman and Hart (1980) it is generally thought that equity investors suffer from a free-rider problem. Each equity investor would want other equity investors to bear the monitoring costs. In equilibrium no equity investor monitors the firm. Knowing that this will happen, each equity investor must take this into account when making the initial equity investment. Equity will only get its promised money if making such payment is in the interest of the firm.

Bank loans are different from equity. Following Diamond (1984) it is generally thought that banks are good at monitoring and enforcing the terms of the loan. For simplicity we suppose that the bank's enforcement ability is perfect. This gives the bank an advantage relative to equity finance. But of course, in the background, there is the serious question about the willingness of the bank to treat the bank depositors properly.

The firm problem with a friction is,

$$\begin{aligned} \max_{\ell_f \geqslant 0, e_f \geqslant 0} \quad & \pi_f = A(y_f + e_f + \ell_f) - (1 + r_\ell)\ell_f - (1 + r_e)e_f \\ \text{s.t.} \quad & \theta_f[A(y_f + e_f + \ell_f) - (1 + r_\ell)\ell_f] \leqslant A(y_f + e_f + \ell_f) - (1 + r_\ell)\ell_f - (1 + r_e)e_f \end{aligned}$$

The firm's objective function is the same as in the no friction case. The firm's endowment plus any new funds are invested in physical capital for production ($k = y_f + e_f + \ell_f$).

The difference from the friction-free case is given by the new constraint. A firm that cheats takes a fraction θ_f of the revenue after debt repayment, where $0 < \theta_f < 1$. Notice that whether the firm cheats or not, the bank is still getting paid $(1 + r_\ell)\ell_f$. This is due to the assumption that the bank can perfectly enforce its claim.

The incentive constraint on the firm is otherwise very similar to the incentive constraint on the bank modelled in the previous section. The payoff to the firm that does not cheat must be at least as big as the payoff to the firm that does cheat. If that condition were violated, new equity would not be willing to invest in the firm.

It is convenient to group terms in the constraint,

$$0 \leqslant (1 - \theta_f)Ay_f + [(1 - \theta_f)A - (1 + r_e)]e_f + (1 - \theta_f)[A - (1 + r_\ell)]\ell_f. \tag{17}$$

High values of y_f make it more likely that the constraint is satisfied. Due to the impact of θ_f a firm that cheats loses part of the returns to the endowment. The larger the endowment the more costly that loss will be, and so the firm is less likely to cheat. The impact of high values of e_f balances the impact of productivity (A) against the cost of equity (r_e). Similarly, the impact of high values of ℓ_f balance impact of productivity (A) against the cost of the bank loan (r_ℓ).

To start with consider the firm's demand for bank loans (ℓ_f). Provided the money is sufficiently productive ($1+r_\ell < A$) the objective function is increasing in ℓ_f. The bank can always enforce repayment and so the solution will be $\ell_f = +\infty$. If $1 + r_\ell = A$, then any nonnegative value of bank loans is equally good and so $\ell_f \in [0, +\infty)$. If the money is not sufficiently productive ($1 + r_\ell > A$) then the firm has no demand for a bank loan ($\ell_f = 0$).

Next consider the firm's demand for new equity finance. This may be affected by what is happening in the loan market since that will affect whether the constraint is satisfied or not. If $1 + r_\ell < A$, then the constraint for e_f is not binding due to the fact that $\ell_f = +\infty$. Accordingly in an equilibrium it must be the case that $1 + r_e \geqslant A$.

Next suppose that $1 + r_\ell > A$, so that bank loans are not appealing to the firm. It now matters to equity investors whether their claims will be honored. Since $\ell_f = 0$, the constraint is now $0 \leqslant (1-\theta_f)Ay_f + ((1-\theta_f)A-(1+r_e))e_f$, or $\frac{(1-\theta_f)Ay_f}{(1+r_e)-(1-\theta_f)A} \leqslant e_f$. Suppose that $1 + r_e \leqslant (1 - \theta_f)A$, then any nonnegative value for e_f will satisfy the constraint. In the objective function this is sufficient to ensure that the coefficient on e_f is positive and so the solution will be $e_f = +\infty$. Clearly this cannot happen in an equilibrium.

Suppose instead that $(1-\theta_f)A < 1+r_e < A$. Again the coefficient on e_f in the objective function is positive. So the firm will choose to pick the largest value of e_f that is consistent with the constraint just holding. Accordingly $e_f = \frac{(1-\theta_f)Ay_f}{(1+r_e)-(1-\theta_f)A}$. If instead $1 + r_e = A$, then the obvious substitution gives that $e_f \in [0, \frac{1-\theta_f}{\theta_f}y_f]$. Finally suppose that $A < 1 + r_e$. This time the objective function is decreasing in e_f, and so the firm picks a value of zero for e_f.

We summarize the above as follows.

Proposition 5 ***Loan demand.*** *Suppose that* $(1 - \theta_f)A < 1 + r_e < A$*. If* $1 + r_\ell < A$ *then the firm would have an infinite demand for bank loans* ℓ_f*. If* $1 + r_\ell = A$*, then the firm just breaks even on bank loans and so any nonnegative value is equally good,* $\ell_f \in [0, +\infty)$*. If* $1 + r_\ell > A$ *then the firm loses money on loans and so the demand for bank loans is zero,* $\ell_f = 0$*.* ***Equity demand.*** *If* $1 + r_e \leqslant (1 - \theta_f)A$*, then* $e_f = +\infty$*. If* $(1 - \theta_f)A < 1 + r_e < A$*, then* $e_f = \frac{(1-\theta_f)Ay_f}{(1+r_e)-(1-\theta_f)A}$*. If* $1 + r_e = A$ *then the firm breaks even and any finite value of* e *that is not too high will work,* $e_f \in [0, \frac{1-\theta_f}{\theta_f}y_f]$*. If* $A < 1 + r_e$ *then new equity is undesirable and so* $e_f = 0$*.*

The firm friction has effects that are analogous to the bank friction. The constraint require that the new equity not have too large a claim to the firm's revenues. If the new equity claim were too high, the firm would rather just default. Knowing this investors are unwilling to invest too much in the firm in the form of new equity.

Corollary 6 *The optimal firm profit depend on the required rates of return on bank loans and equity. If the required rates of returns are very low,* $1 + r_\ell < A$ *or* ($1 + r_\ell \geqslant A$ *and* $1 + r_e \leqslant (1 - \theta_f)A$)*, then profits are infinite,* $\pi_f = +\infty$*. If the required rates of return are very high then the firm does not raise any outside funding and so profits are* $\pi_f = Ay_f$*. If the bank loan rate is very high, but the equity rates is moderate,* $1 + r_\ell \geqslant A$ *and* $(1 - \theta_f)A < 1 + r_e < A$*, then firm profits are* $\pi_f = \frac{\theta_f(1+r_e)Ay_f}{1+r_e-(1-\theta_f)A}$.

4.3 Equilibrium with Both Frictions

Given the possible impacts of the bank and firm frictions, what kind of an equilibrium might emerge? If there is only a friction at the bank, the investors will avoid using the bank. If there is a friction only in the equity market then investors will avoid using the equity market. In Christiano and Ikeda (2011) the policy relevant case happens when there is an interior equilibrium with a positive interest rate spread between what the bank pays on deposits and what it charges on loans. For such an equilibrium to arise in our model it is necessary that both the bank and the firm be subject to binding incentive constraints. Just one such constraint is not enough. A useful result is the following proposition which is proved in Appendix 8.

Proposition 7 *A necessary condition for the existence of an equilibrium with* $0 < d < +\infty$, $0 < e < +\infty$, $0 < \ell < +\infty$, *and* $r_d < r_\ell$*, is that* $r_\ell = A - 1$*, and* $r \equiv r_d = r_e$ *satisfying* $(1 - \theta_b)A - 1 < r$, $(1 - \theta_f)A - 1 < r$ *and* $r \leqslant A$.

To determine the interest rate $r = r_d = r_e$ in the proposition above, we use the savings relation $d + e = S$ from the household problem, and note S given in (5), $d = d_b$ given in Proposition 4 and $e = e_f$ given in Proposition 4. This gives,

$$\frac{(1-\theta_f)Ay_f}{1+r-(1-\theta_f)A} + \frac{(1-\theta_b)Ay_b}{1+r-(1-\theta_b)A} = \frac{y_h\beta^{1/\gamma} - (\pi_b + \pi_f)(1+r)^{-1/\gamma}}{\beta^{1/\gamma} + (1+r)^{(\gamma-1)/\gamma}}, \tag{18}$$

where we note that the bank profit π_b and the firm profit π_f depend on the interest rate r. The above equation would determine the interest rate r. Define

$$\varphi(x) \equiv \frac{(1-\theta_f)Ay_f}{x-(1-\theta_f)A} + \frac{(1-\theta_b)Ay_b}{x-(1-\theta_b)A} - \frac{y_h\beta^{1/\gamma} - (\pi_b(x)+\pi_f(x))x^{-1/\gamma}}{\beta^{1/\gamma}+x^{(\gamma-1)/\gamma}}, \tag{19}$$

where we use the functions,

$$\pi_b(x) = \frac{\theta_b x A y_b}{x-(1-\theta_b)A} \quad \text{and} \quad \pi_f(x) = \frac{\theta_f x A y_f}{x-(1-\theta_f)A},$$

to make explicit the dependence of π_b and π_f on r and hence x.

We want to show that there is a solution $1+r$ to $\varphi(x)=0$ on the interval $((1-\theta_f)A, A)$, under the following condition.

$$y_h - \frac{A^{(\gamma-1)/\gamma}(y_h+y_b+y_f)}{\beta^{1/\gamma}+A^{(\gamma-1)/\gamma}} > \frac{1-\theta_b}{\theta_b}y_b + \frac{1-\theta_f}{\theta_f}y_f. \tag{20}$$

Proposition 8 *A necessary and sufficient condition for the existence of a strictly interior equilibrium that has a positive interest rate spread is that condition (20) holds. In that case, the corresponding equilibrium quantities are as follow:*

1. *Equilibrium interest rates are given by* $r_\ell = A-1$ *and* $r \equiv r_d = r_e = x^*-1$, *where* x^* *is the unique solution in the interval* $((1-\theta_f)A, A)$ *to the equation* $\varphi(x)=0$.

2. *Equilibrium deposit* d, *equity investment* e *and bank loan* ℓ *are given by*

$$\begin{aligned} d &= \frac{(1-\theta_b)Ay_b}{1+r-(1-\theta_b)A}. && (21)\\ e &= \frac{(1-\theta_f)Ay_f}{1+r-(1-\theta_f)A}, && (22) \end{aligned}$$

   and $\ell = y_b + d$.

3. *Equilibrium consumptions are given by*

$$c_0 = \frac{(1+r)^{-1/\gamma}[(1+r)y_h + \pi_b(1+r) + \pi_f(1+r)]}{\beta^{1/\gamma}+(1+r)^{(\gamma-1)}} \quad \textit{and} \quad c_1 = (\beta(1+r))^{1/\gamma}c_0.$$

Proposition 8, which is key for policy, is proved in the Appendix 8. Table 1 provides a summary of the comparisons between the no friction interior equilibrium and the interior equilibrium with both frictions. The household is better off in the equilibrium with no frictions. In that case there is less consumption in period zero, but that it more than offset by the impact of greater investment which leads to greater period 1 consumption.

The rate that the bank charges to firms on loans is the same in the friction and the no friction equilibria. This is a reflection of the fact that the bank can perfectly monitor. In the equilibrium with both frictions the interest paid on bank loans and the rate paid on equity investments are both lower than the rate charged on bank loans to firms. In the friction free equilibria both the bank and the firm get compensated for their endowments. But neither earns anything beyond that compensation. In the interior equilibrium with both frictions, both the bank and the firm earn higher profits than they get in the friction-free equilibrium.

Corollary 9 *When the condition (20) does not hold, at least one of the equilibria for the no-friction problem is an equilibrium for the friction problem.*

In comparison to the non-friction case, the main difference for the friction equilibrium is the added constraint (16) for the bank problem and the added constraint (17) for the firm problem. If the no-friction equilibrium satisfy these constraints, then the no-friction equilibrium is also the friction equilibrium. In this case the friction equilibrium is socially efficient and no special policy is needed. So it is important to determine when this happens and when it does not.

Recall the no-friction equilibrium solution in Proposition 3. In that solution $r_d = r_e = r_\ell = A - 1$, $d \geqslant 0$ and $e \geqslant 0$ satisfying

$$d + e = y_h - \frac{A^{(\gamma-1)/\gamma}(y_h + y_b + y_f)}{\beta^{1/\gamma} + A^{(\gamma-1)/\gamma}}, \tag{23}$$

and $\ell = y_b + d$. This equilibrium will satisfy (16) and (17) if and only if

$$d \leqslant \frac{1-\theta_b}{\theta_b} y_b \quad \text{and} \quad e \leqslant \frac{1-\theta_f}{\theta_f} y_f.$$

Note that the no-friction equilibrium d and e is not unique and can be any nonnegative d and e satisfying (23). On the other hand, we only look for one no-friction equilibrium that is a friction equilibrium. Combining the above inequalities and the equality (23), we find a necessary and sufficient condition for the existence of one no-friction equilibrium that is also the friction equilibrium.

$$y_h - \frac{A^{(\gamma-1)/\gamma}(y_h + y_b + y_f)}{\beta^{1/\gamma} + A^{(\gamma-1)/\gamma}} \leqslant \frac{1-\theta_b}{\theta_b} y_b + \frac{1-\theta_f}{\theta_f} y_f,$$

This condition is exactly the complementary condition for (20).

5 Social Welfare

5.1 No Friction Benchmark

The social welfare problem is written,

$$\begin{aligned} \max \quad & U(c_0, c_1) = u(c_0) + \beta u(c_1) \\ \text{s.t.} \quad & c_0 + k = y_h + y_b + y_f \\ & c_1 \leqslant Ak \\ & c_0 \geqslant 0, c_1 \geqslant 0. \end{aligned} \tag{24}$$

The government (or "social planner") cares about the household welfare. The resources available to the government in period 0 is the sum of the endowments of the household, the bank, and the firm. These resources can be used for period 0 consumption or for investing in production. In period 1 consumption is limited by the output generated from using k.

It is clear that $c_1 \leqslant Ak$ must be binding. So the problem can be rewritten as

$$\max_{0 \leqslant k \leqslant y_h + y_b + y_f} u(y_h + y_b + y_f - k) + \beta u(Ak).$$

The first order condition is

$$\beta u'(Ak) = u'(y_h + y_b + y_f - k).$$

Assuming power utility, this implies $\beta A(Ak)^{-\gamma} = (y_h + y_b + y_f - k)^{-\gamma}$. The socially optimal level of production is

$$k^* = \frac{\beta^{1/\gamma}(y_h + y_b + y_f)}{\beta^{1/\gamma} + A^{(\gamma-1)/\gamma}}.$$

The equilibrium level of production is

$$k^e = \ell_f + y_f + e_f = y_b + y_f + d + e.$$

Does $k^* = k^e$? Consider the interior equilibrium when the condition (10) holds. From Proposition 3,

$$k^e = y_b + y_f + \left(y_h - \frac{A^{(\gamma-1)/\gamma}(y_h + y_b + y_f)}{\beta^{1/\gamma} + A^{(\gamma-1)/\gamma}}\right) = k^*,$$

and the equilibrium consumptions are also the same as the social welfare optimal consumptions. So the interior equilibrium does maximize social welfare under the condition (10).

When the condition (10) fails, the equilibrium is a corner solution equilibrium in which $d = e = 0$. In this case, $k^e = y_b + y_f > k^*$. Such equilibria are not socially efficient. To see how this can happen, let $y_h = 0$. Then clearly $d_h = e_h = 0$. To have $d_b = d_h = 0$, we need $r_\ell \leqslant r_d$. To have $e_f = e_h = 0$, we need $1 + r_e \leqslant A$. To have $\ell_f = \ell = y_b$, we need $1 + r_\ell \leqslant A$, and $y_b(1 + r_\ell - A) = 0$. It is clear that there is a set $\{r_d, r_e, r_\ell\}$, satisfying these conditions. Thus there exists a $d = e = 0$, $\ell = y_b$ that is an equilibrium, and so

$$k^e = y_b + y_f > \frac{\beta^{1/\gamma}(y_h + y_b + y_f)}{\beta^{1/\gamma} + A^{(\gamma-1)/\gamma}} = k^*.$$

Such an equilibrium is not socially efficient.

What is the intuition for the social inefficiency? If the household initial endowment is not sufficiently large, the period 0 consumption is not big enough, assuming that $d \geqslant 0$, $e \geqslant 0$ are enforced. The government would wish to transfer resources to the household for consumption purposes. But there is no mechanism to do so. The only way to get resources from the firm and the bank back to the household is to invest them in firm production. In this case a policy that provided for extra consumption in period 0 at the expense of subsequent consumption would be welfare improving. Thus the importance of this kind of social inefficiency depends on whether the inequality constraints are regarded as realistic.

When the household has no initial endowment it cannot consume anything in period 0, again assuming that $d \geqslant 0$, $e \geqslant 0$ are enforced. But social optimality would not have zero consumption in that period. The socially optimal period 0 consumption is

$$c_0^* = y_h + y_b + y_f - k^* = \frac{A^{(\gamma-1)/\gamma}(y_h + y_b + y_f)}{\beta^{1/\gamma} + A^{(\gamma-1)/\gamma}}.$$

Thus a necessary condition for the equilibrium to be socially efficient with $d_h = e_h = 0$, is

$$y_h \geqslant c_0^*.$$

The reason that the equilibrium with too low y_h is not socially efficient, can be viewed as a missing market problem. There is no means to transfer the period 0 resources to the household. An approach to improve social welfare would be to introduce such a mechanism, such as permitting negative d or negative e.

5.2 Social Welfare with Frictions

Suppose that there is only a bank level friction. Assuming that y_h is big enough, the no friction equilibrium was just shown to be efficient. The same consumptions and social welfare are also obtainable as an equilibrium with just a bank friction. To see that this claim is true, note that the only additional constraint on the equilibrium is

$$0 \leqslant (1-\theta_b)(1+r_\ell)y_b - [1+r_d-(1-\theta_b)(1+r_\ell)]d. \tag{25}$$

In the friction free equilibrium d and e are substitutable. Only the sum $d+e$ matters. Hence choosing $d=0$ would provide a no-friction equilibrium that also satisfies condition (25). Hence this equilibrium is also an equilibrium with the bank friction and this equilibrium is socially efficient.

Suppose that there is only a firm level friction. This is essentially similar to the bank friction only case. The bank is able to monitor the loan and ensure that it is repaid. So bank loan financing can be used to avoid the cheating friction that afflicts the equity finance. Again there will always be a socially efficient equilibrium.

In this case the extra constraint on the equilibrium is

$$0 \leqslant (1-\theta_f)Ay_f - [1+r_e-(1-\theta_f)A]e + (1-\theta_f)(1+r_\ell-A)\ell. \tag{26}$$

In the no friction case we have $1+r_\ell = A$. So if we pick $e=0$, then condition (26) clearly holds. So by picking $e=0$, we find an equilibrium for the no friction case that is also feasible as an equilibrium in the firm friction only case and clearly it is then socially optimal.

Suppose that the frictions affect both the bank and the firm. It is no longer possible to avoid one friction by using the other channel. So we notice how the bank only friction and the firm only friction results work. In each case an alternative financing channel is used to completely avoid the friction. As we will show next, allowing for both frictions at the same time is more complex. If we were to set $d=e=0$, then we would succeed in avoiding both frictions. But this comes at the cost of not getting household savings to the firm. So that is not a way around the problem.

In view of Proposition 8, the key question is whether inequality (20) holds or not. In one case the frictions are unimportant to the equilibrium. But in the other case they lead to a strictly interior equilibrium with a positive interest rate spread. In that case the equilibrium is not socially efficient and so there is potential for policy to improve welfare.

This is stated as a proposition.

Proposition 10 *Under the condition (20), the equilibrium for the friction problem is socially inefficient. Conversely, when the condition (20) does not hold, there exists at least one equilibrium for the friction problem that is socially efficient provided the condition (10) holds.*

This proposition, which is proved in the Appendix 8, opens the door to justifiable policy intervention. Much of that intervention concerns getting the equilibrium with a friction to turn into an efficient equilibrium in which one or both of the frictions are not binding.

Corollary 11 *Suppose that the condition (20) holds. Under the equilibrium, the household consumes more in period zero and consumes less in the second period in comparison to the socially efficient solution.*

6 Policy to Improve Social Welfare

The equilibrium in which both frictions are binding is socially inefficient. There are variety of policies that can be considered to improve the equilibrium. Policy can target the bank, or the firm. Policy can take the form of an equity injection, a subsidized loan, or an interest rate subsidy. Any such policy must take into account the source of any funding required and the use of any fund generated. Some papers assume that the government has an enforcement advantage relative to private investors. Other papers assume that the government has no enforcement advantage, so that it would actually need to be in the interest of the policy target to make any planned payments.

Welfare is determined by the consumptions c_0 and c_1. Any policy that turns the friction consumption values (c_0^f, c_1^f) into the no friction values (c_0^n, c_1^n) will maximize social welfare. We know that $c_0^f > c_0^n$, and $c_1^f < c_1^n$. So an optimal policy will reduce period 0 consumption, transfer more resources into production, and increase period 1 consumption.

With two policy targets (bank, firm), three types of policy (equity, debt, interest rate), and two enforcement assumptions (fully enforceable, only self-interested), it is clear that there are many cases that can be considered. In each case we assume that any costs are funded by lump sum taxation on the household so that the policy is adequately funded.

Policies that achieve efficiency share a common focus. In each instance there is an inefficient equilibrium in which condition (20) from Proposition 8 holds. The goal is to operate on condition (20) so that the sign reverses. Once that happens, the equilibrium inherits the social efficiency from the friction-free equilibrium. In other words, once the policy pushes the system to the correct side of the inequality, provided no other constraints get violated in the process, private actions take over to ensure the specific consumption and savings values required for optimality.

Optimal policy is not unique. There are many values that are on the correct side of the inequality without disrupting any other equilibrium requirement. To go further in selecting a policy requires some further criterion. Within each class of policy, we focus on the optimal policy that minimizes the tax on the household. This secondary criterion makes sense if there are some minor costs associated with each dollar of tax.

The policies under consideration are: 1) equity injections to the bank and/or the firm, 2) loans to the bank and/or the firm, 3) interest rate subsidies on bank deposits or on equity investments in the firm. To fund equity injections requires imposing a tax on the household in period 0. In period 1 the household may benefit from enhanced profits received from the bank or the firm. To fund loans the household is again taxed in period 0. In this case the household will receive repayment of the loan (with interest) from the government in period 1. There may also be an effect on the value of profits received from the bank or the firm in period 1. Interest rate subsidies take place when the interest is due. This is in period 1. Thus this policy does not require any taxation of the household in period 0. There is a tax in period 1 and potentially some effect on the profits received from the bank and the firm in period 1.

With all six policies there are parameter values such that policy can achieve social optimum. Thus there is an important sense in which any of the policies are equally effective. If keeping the taxation of the household low is desirable, then they are not equivalent. Equity injections can achieve social optimality with a lower level of tax than is required for an optimal government loan.

6.1 Equity Injections

The first policy to be considered are direct equity injections into the bank and the firm. Equity injections during the crisis were primarily considered by the US government under the TARP (Troubled Asset Relief Plan). While this was coordinated between the Treasury Department and the Federal Reserve, it was primarily undertaken by the Treasury Department. Central banks outside of the USA have purchased shares in crises at times.[8]

The impact of equity injections acts much like an increase in an endowment. Both the bank and the firm can still default if they wish to do so. Let T_b and T_f denote the equity injections into the bank and the firm respectively. In order to carry out such injections the government will impose a lump sum tax on the household in period 0.

The household problem with equity injections is,

$$\begin{aligned} \max_{d_h \geqslant 0, e_h \geqslant 0} \quad & U(c_0, c_1) = u(c_0) + \beta u(c_1) \\ \text{s.t.} \quad & c_0 = y_h - (d_h + e_h) - T_b - T_f \\ & c_1 = (1 + r_d) d_h + (1 + r_e) e_h + \pi_b + \pi_f \\ & c_0 \geqslant 0, c_1 \geqslant 0. \end{aligned} \tag{27}$$

The government takes the necessary money away from the household using lump sum taxation. To ensure that such taxation is feasible assume that $y_h - T_b - T_f \geqslant 0$. These funds are injected into the bank and the firm. There are two ways to interpret the equity injection. It can be a pure gift to the existing owner. Or else it could involve taking ownership of a fraction of the firm. In the second interpretation, that return is paid back to the household. To minimize notation, we simply interpret π_b and π_f as inclusive of any government returns to the household. The household regards these as lump sum.

[8] A good example is the purchase of shares by the Hong Kong Monetary Authority during the crisis of 1998. See: http://www.trahk.com.hk/eng/homepage.asp and http://en.wikipedia.org/wiki/Hong_Kong_Monetary_Authority. There does seem to be evidence that this controversial action helped restore confidence in Hong Kong firms.

The frictional bank problem with equity injections is,

$$\max_{d_b \geqslant 0} \quad \pi_b = (1+r_\ell)(y_b + T_b) + (r_\ell - r_d)d_b \tag{28}$$

$$\text{s.t.} \quad \theta_b(1+r_\ell)(y_b + T_b + d_b) \leqslant (1+r_\ell)(y_b + T_b) + (r_\ell - r_d)d_b. \tag{29}$$

The subsidy means that y_b is replaced by $y_b + T_b$. On the left hand side of the constraint (29) is what the bank gets if it cheats. The bank fully collects on the bank loan to the firm, and is then able to abscond with a fraction θ_b of each dollar. On the right hand side of inequality (29) are the payments that are made if all contracts are fulfilled. The no cheating constraint thus says that households will not make any deposit at the bank which would induce the bank to cheat.

The frictional firm problem with equity injection is,

$$\max_{\ell_f \geqslant 0, e_f \geqslant 0} \quad \pi_f = A(y_f + T_f + e_f + \ell_f) - (1+r_\ell)\ell_f - (1+r_e)e_f \tag{30}$$

$$\begin{aligned} \text{s.t.} \quad & \theta_f[A(y_f + T_f + e_f + \ell_f) - (1+r_\ell)\ell_f] \\ & \leqslant A(y_f + T_f + e_f + \ell_f) - (1+r_\ell)\ell_f - (1+r_e)e_f. \end{aligned} \tag{31}$$

The subsidy means that y_f is replaced by $y_f + T_f$. Again, the government is entitled to a fraction of the equity which is passed back to the household. The constraint is similar in form to the constraint on the bank. The left hand side of inequality (31) is the payoff if the firm cheats, and the right hand side is the payoff if the firm does not cheat. Notice that the constraint assumes that the bank is able to enforce repayment of the bank loan, but neither the government nor the household have similar enforcement power. The firm level non-cheating constraint says that the household will not make any equity investment that would induce the firm to cheat.

To maximize social welfare the government needs values of T_b and T_f large enough so that the incentive constraints are not binding. The larger T_b the more likely the bank constraint will not be binding. The larger T_f the more likely the firm constraint will not be binding. How do we know that this will work? In a no-friction equilibrium given by Proposition 3, $r_d = r_e = r_\ell = A - 1$. Accordingly it is sufficient to find T_b and T_f such that

$r_d = r_e = r_\ell = A - 1$, there exists a nonnegative pair, $d_h = d_b \equiv d$, and $e_h = e_f \equiv e$, and $c_0 = c_0^*$ and $c_1 = c_1^*$. These must simultaneously solve i) the household problem (27), ii) the friction bank problem (28) and iii) the friction firm problem (30).

Assuming that $r_d = r_e = r_\ell = A - 1$, the constraints (29) and (31) lead to

$$d \leqslant \frac{1-\theta_b}{\theta_b}(y_b + T_b) \quad \text{and} \quad e \leqslant \frac{1-\theta_f}{\theta_f}(y_f + T_f). \tag{32}$$

If the bank cheats it has a gain of $\theta_b d$ and if it does not cheat it has a gain of $(1-\theta_b)(y_b + T_b)$, so to ensure that it is better not to cheat can be viewed as a restriction that d not be too large. The same argument applies to e. Any d and e satisfying these will solve the frictional bank problem (28) with $\pi_b = A(y_b + T_b)$ and the frictional firm problem (30) with $\pi_f = A(y_f + T_f)$. A large equity injection to the bank (T_b) helps satisfy the first inequality. A large equity injection to the firm (T_f) helps satisfies the second inequality.

From the household problem,

$$d + e = y_h - T_b - T_f - c_0^*,$$

where $c_0^* = \frac{A^{(\gamma-1)/\gamma}(y_h+y_b+y_f)}{\beta^{1/\gamma}+A^{(\gamma-1)/\gamma}}$, $\pi_b = A(y_b + T_b)$, and $\pi_f = A(y_f + T_f)$. Combining the household problem with inequality (32) gives a necessary and sufficient condition for the existence of a no-friction equilibrium that is also a friction equilibrium. This is,

$$y_h - T_b - T_f - c_0^* \leqslant \frac{1-\theta_b}{\theta_b}(y_b + T_b) + \frac{1-\theta_f}{\theta_f}(y_f + T_f). \tag{33}$$

It is readily seen that $c_0 = c_0^*$ and $c_1 = c_1^*$ with the above equilibrium quantities. Thus any combination of nonnegative T_b and T_f that satisfies inequality (33) provides an equity injection policy that is socially efficient.

To select a policy from the set of policies that satisfy inequality (33) requires a criterion that goes beyond the model. We examine policy choices that minimizes the tax imposed on the household. This replaces inequality (33) with an equality

$$y_h - T_b - T_f - c_0^* = \frac{1-\theta_b}{\theta_b}(y_b + T_b) + \frac{1-\theta_f}{\theta_f}(y_f + T_f), \tag{34}$$

The relationship between the two levels of equity injections can also be written,

$$\frac{1}{\theta_b} T_b + \frac{1}{\theta_f} T_f = a, \tag{35}$$

where

$$a = y_h - c_0^* - \frac{1-\theta_b}{\theta_b} y_b - \frac{1-\theta_f}{\theta_f} y_f. \tag{36}$$

Note that a is the difference between the right hand side and the left hand side of inequality (20). One might call this the social efficiency protection wedge, or social protection wedge. A nonnegative value of the wedge is needed to ensure that the equilibrium is socially efficient. If both kinds of equity injections are in use, then the greater the injection to the firm, the lower the equity injection to the bank that is necessary to achieve social efficiency. The minimization problem for the equity injection is very simple: $\min_{T_b \geqslant 0, T_f \geqslant 0} T_b + T_f$, s.t., equation (35).

Proposition 12 *An equity injection policy that achieves social efficiency and also minimizes household tax works as follows. The lump sum tax on the household is* $T_b + T_f$. *The optimal equity injection policy sets* $T_b = \theta_b a$ *and* $T_f = 0$ *if* $\theta_b < \theta_f$. *It sets* $T_b = 0$ *and* $T_f = \theta_f a$ *otherwise. Under this policy regime, household consumption is socially efficient, and household investment is,*

$$d = \frac{1-\theta_b}{\theta_b}(y_b + T_b) \quad \textit{and} \quad e = \frac{1-\theta_f}{\theta_f}(y_f + T_f).$$

The government takes the funds that are needed from the household. These funds are injected into the bank or into the firm depending on which of them will be more responsive to the injection. Only in a knife edge case is some money injected into both. The more equity is injected to the bank, the higher the deposit into the bank. The more equity is injected into the firm, the higher the equity invested in the firm. The fact that $\theta_b > \theta_f$ means that the moral hazard at the firm is more severe than at the bank. In this case, it is cheaper then to incentivize the bank, and the cost effective optimal policy focuses entirely on the banks and not at the firms. In the reverse case, it is cheaper to entirely incentivize the firm and not the bank.

It is often suggested that the government has an enforcement advantage relative to private sector investors. If this is true then there is greater scope for successful policy. To see this suppose that neither the bank nor the firm can default on an equity injection. Consider injecting an amount of T_b into the bank and T_f into the firm. In this case, any combination of the injections that satisfies

$$y_h - T_b - T_f - c_0^* \leqslant \frac{1-\theta_b}{\theta_b} y_b + \frac{1}{\theta_b} T_b + \frac{1-\theta_f}{\theta_f} y_f + \frac{1}{\theta_f} T_f,$$

will work. The equality version of the above can be written as,

$$\frac{1+\theta_b}{\theta_b} T_b + \frac{1+\theta_f}{\theta_f} T_f = a.$$

Recall that a is defined in equation (36).

If there is secondary objective to tax the minimum amount from the household, then we find that $T_b = \left(\frac{\theta_b}{1+\theta_b}\right) a$ and $T_f = 0$ is optimal if $\theta_b < \theta_f$, while $T_b = 0$ and $T_f = \left(\frac{\theta_f}{1+\theta_f}\right) a$ are optimal otherwise. Direct comparison to Proposition 12 shows that with an enforcement advantage, a somewhat lower equity injection will suffice to achieve social efficiency.

6.2 Loans

Lending by the Federal Reserve is normally targeted at the banks. But, as pointed out by Ben Bernanke, it can also target individual firms, at least during an emergency. A loan consists of an amount of money ($L_i, i = b, f$) and an interest rate that is due ($\rho_i, i = b, f$) at the same time that the original loan is repaid.

The household problem with lending is,

$$\begin{aligned}
\max_{d_h \geqslant 0, e_h \geqslant 0} \quad & U(c_0, c_1) = u(c_0) + \beta u(c_1) \\
\text{s.t.} \quad & c_0 = y_h - (d_h + e_h) - L_b - L_f \\
& c_1 = (1 + r_d) d_h + (1 + r_e) e_h + \pi_b + \pi_f + (1 + \rho_b) L_b + (1 + \rho_f) L_f \\
& c_0 \geqslant 0, c_1 \geqslant 0.
\end{aligned}$$

In order for the government to extend loans to the bank and to the firm, the government must get money. To get that money it must tax the household in period zero. When the government gets the loan repayment, the repayment is transferred back to the household in period one.

The frictional bank problem with lending is,

$$\begin{aligned} \max_{d_b \geqslant 0} \quad & \pi_b = (1+r_\ell)(y_b + L_b) + (r_\ell - r_d)d_b - (1+\rho_b)L_b \\ \text{s.t.} \quad & \theta_b(1+r_\ell)(y_b + d_b + L_b) \leqslant (1+r_\ell)(y_b + L_b) + (r_\ell - r_d)d_b - (1+\rho_b)L_b. \end{aligned}$$

The constraint on the frictional bank shows that the bank can enforce payment on the bank loan to the firm. However, neither the household, nor the government can force the bank to repay if the bank would prefer to cheat. Since neither the government nor the household get anything if the bank cheats, the no cheating constraint is a limitation on the willingness of the household to extend funds. The bank gets the government loan in period zero and repays that loan with interest in period one.

The frictional firm problem with lending is,

$$\begin{aligned} \max_{\ell_f \geqslant 0, e_f \geqslant 0} \quad & \pi_f = A(y_f + L_f + e_f + \ell_f) - (1+r_\ell)\ell_f - (1+r_e)e_f - (1+\rho_f)L_f \\ \text{s.t.} \quad & \theta_f[A(y_f + L_f + e_f + \ell_f) - (1+r_\ell)\ell_f] \\ & \qquad \leqslant A(y_f + L_f + e_f + \ell_f) - (1+r_\ell)\ell_f - (1+r_e)e_f - (1+\rho_f)L_f. \end{aligned}$$

Much like the bank, the firm receives a government loan in period zero and repays it with interest in period one.

A government loan policy specifies a combination of loans (ρ_b, L_b) and (ρ_f, L_f). These have the property that under the no-friction equilibrium solution $r_d = r_e = r_\ell = A - 1$, there exists a nonnegative pair, $d_h = d_b \equiv d$ and $e_h = e_f \equiv e$, that simultaneously solves the household problem, the friction bank problem and the friction firm problem in the above.

The structure of the argument is similar to the equity injection policy analysis. The combination of loans (ρ_b, L_b) and (ρ_f, L_f) will achieve social efficiency if they satisfy $\rho_b <$

$(1-\theta_b)A-1$, $\rho_f < (1-\theta_f)A-1$, and

$$y_h - L_b - L_f - c_0^* \leqslant \frac{1-\theta_b}{\theta_b}y_b + \frac{1-\theta_f}{\theta_f}y_f + \frac{(1-\theta_b)A-(1+\rho_b)}{A\theta_b}L_b + \frac{(1-\theta_f)A-(1+\rho_f)}{A\theta_f}L_f.$$

These inequalities are not sufficient to fully pin down an optimal policy. To do that, we again look for an optimal policy that minimizes household tax. Accordingly the inequality is replaced by an equality. This equality is equivalent to

$$\frac{A-(1+\rho_b)}{A\theta_b}L_b + \frac{A-(1+\rho_f)}{A\theta_f}L_f = a, \tag{37}$$

with a defined in equation (36). Under this subsidy combination, a total subsidy of L_b+L_f is taxed from the household.

The minimization problem for the loan subsidy is: $\min_{L_b\geqslant 0, L_f\geqslant 0, \rho_b\geqslant 0, \rho_f\geqslant 0} L_b + L_f$, subject to equation (37). The optimal solution is to take $\rho_b = \rho_f = 0$; and the corresponding $L_b = \frac{A\theta_b}{A-1}$ and $L_f = 0$ if $\theta_b < \theta_f$, and $L_b = 0$ and $L_f = \frac{A\theta_f}{A-1}$ otherwise. If for any reason, we fix ρ_b and ρ_f at non-zero levels, then we would get correspondingly different optimal values of L_b and L_f. Note that we have to have a second objective function for this solution to be interesting.

Proposition 13 *For any given ρ_b and ρ_f, suppose that the objective is to take the minimum amount from the household while still maximizing social welfare. Then if $[A-(1+\rho_f)]\theta_b < [A-(1+\rho_b)]\theta_f$, it is optimal to subsidize the loan to bank only with $L_b = \left(\frac{A\theta_b}{A-(1+\rho_b)}\right)a$; otherwise, it is optimal to subsidize the firm only with $L_f = \left(\frac{A\theta_f}{A-(1+\rho_f)}\right)a$. Under this policy regime, household consumption is socially optimal, and household investment is,*

$$d = \frac{1-\theta_b}{\theta_b}y_b + \frac{(1-\theta_b)A-(1+\rho_b)}{A\theta_b}L_b \quad \textit{and} \quad e = \frac{1-\theta_f}{\theta_f}y_f + \frac{(1-\theta_f)A-(1+\rho_f)}{A\theta_f}L_f.$$

If ρ_b and ρ_f are chosen for the same objective, then it is optimal to set $\rho_b = \rho_f = 0$.

This optimal loan policy has much in common with the optimal equity injection policy. There is tax imposed on the household to fund the subsidy. The subsidy is given to either the bank or to the firm, depending on which will be more responsive. A subsidized loan involves an interest payment. But in terms of the incentives there is no benefit from having the interest payment be anything greater than zero. Higher interest repayments will require greater loan size and hence greater taxation of the household.

When the moral hazard at the firm is more severe than at the bank, it is cheaper to provide incentives to the bank. The cost effective optimal policy focuses on the bank and provides not special incentives to the firm. When the cost advantage is reversed the lose cost optimal policy is aimed at the firm and not at the bank.

As with the equity injections, it is of interest to consider the simplifications that are possible when the government has an enforcement advantage. Suppose that neither bank nor firm can default on the loans. A combination of loans (ρ_b, L_b) and (ρ_f, L_f) achieves social efficiency if they satisfy $A > 1 + \rho_b$, $A > 1 + \rho_f$ and

$$y_h - L_b - L_f - c_0^* \leqslant \frac{1-\theta_b}{\theta_b} y_b + \frac{1-\theta_f}{\theta_f} y_f + \frac{A-(1+\rho_b)}{A\theta_b} L_b + \frac{A-(1+\rho_f)}{A\theta_f} L_f.$$

For the minimal tax (equality case), this turns into

$$\frac{A(1+\theta_b)-(1+\rho_b)}{A\theta_b} L_b + \frac{A(1+\theta_f)-(1+\rho_f)}{A\theta_f} L_f = a. \tag{38}$$

Recall that a is defined in equation (36). With this policy,

$$\begin{aligned} d &= \frac{1-\theta_b}{\theta_b} y_b + \frac{A-(1+\rho_b)}{A\theta_b} L_b, \\ e &= \frac{1-\theta_f}{\theta_f} y_f + \frac{A-(1+\rho_f)}{A\theta_f} L_f. \end{aligned}$$

An optimal subsidized loan policy that also minimizes household tax works as follows. The lump sum tax on the household is $L_b + L_f$ in period 0. This money is loaned to the bank and the firm respectively. Suppose that the interest rates are exogenously set as $\rho_b = \rho_f \equiv \rho$. If $\theta_b < \theta_f$, it is optimal to subsidize the loan to bank only with $L_b = \left(\frac{A\theta_b}{A(1+\theta_b)-(1+\rho)}\right) a$. Otherwise, it is optimal to subsidize the firm only with $L_f = \left(\frac{A\theta_f}{A(1+\theta_f)-(1+\rho)}\right) a$.

6.3 Interest Rate Subsidies

The Federal Reserve is actively engaged in setting the interest rates in the U.S. economy. With there being a large number of rates that apply to specific kinds of investments, it is natural to consider interest rate subsidies.

The household problem with interest rate subsidies is,

$$\begin{aligned}
\max_{d_h \geqslant 0, e_h \geqslant 0} \quad & U(c_0, c_1) = u(c_0) + \beta u(c_1) \\
\text{s.t.} \quad & c_0 = y_h - (d_h + e_h) \\
& c_1 = (1 + r_d)d_h + (1 + r_e)e_h + \pi_b + \pi_f - r_d\tau_d d_b - r_e\tau_e e_f \\
& c_0 \geqslant 0, c_1 \geqslant 0.
\end{aligned} \tag{39}$$

In this problem the household takes the quantity $(r_d\tau_d d_b + r_e\tau_e e_f)$ as given and beyond the control of the household.

The frictional bank problem with lending is,

$$\begin{aligned}
\max_{d_b \geqslant 0} \quad & \pi_b = (1 + r_\ell)y_b + (r_\ell - r_d(1 - \tau_d))d_b \\
\text{s.t.} \quad & \theta_b(1 + r_\ell)(y_b + d_b) \leqslant (1 + r_\ell)y_b + (r_\ell - r_d(1 - \tau_d))d_b.
\end{aligned}$$

Instead of paying r_d for each dollar of deposit, the bank now pays only $r_d(1 - \tau_d)$. The government pays the difference. In order to get the money to pay that difference, the government levies a tax on the household. The no cheating constraint, as usual, says that it must not pay for the bank to cheat. This places limits on the willingness of investors to provide money.

The frictional firm problem with lending is,

$$\begin{aligned}
\max_{\ell_f \geqslant 0, e_f \geqslant 0} \quad & \pi_f = A(y_f + e_f + \ell_f) - (1 + r_\ell)\ell_f - (1 + r_e(1 - \tau_e))e_f \\
\text{s.t.} \quad & \theta_f[A(y_f + e_f + \ell_f) - (1 + r_\ell)\ell_f] \leqslant A(y_f + e_f + \ell_f) - (1 + r_\ell)\ell_f - (1 + r_e(1 - \tau_e))e_f
\end{aligned}$$

As in the previous cases, the incentive constraint is the requirement that it not pay for the frictional firm to cheat. This in turn limits the willingness of investors to provide money to the firm.

Suppose there is a subsidy on the bank deposit at rate τ_d and a subsidy on new equity at rate τ_e at the same time. These rates must satisfy[9]

$$y_h - c_0^* \leqslant \frac{(1-\theta_b)Ay_b}{A\theta_b - (A-1)\tau_d} + \frac{(1-\theta_f)Ay_f}{A\theta_f - (A-1)\tau_e}.$$

Any pair of the nonnegative subsidy rates τ_d and τ_e that satisfies the above inequality will achieve social efficiency. The interest rate subsidy policy that achieves social efficiency and also minimizes household tax replaces the inequality with,

$$\frac{(A-1)(1-\theta_b)\tau_d y_b}{\theta_b[A\theta_b - (A-1)\tau_d]} + \frac{(A-1)(1-\theta_f)\tau_e y_f}{\theta_f[A\theta_f - (A-1)\tau_e]} = a$$

where a defined in equation (36).

Now consider the specific subsidy rate that would minimize the total household tax. The total amount taken from the household to subsidize the bank and the firm is given by

$$r_d\tau_d d_b + r_e\tau_e e_f = \frac{(A-1)\tau_d(1-\theta_b)Ay_b}{A\theta_b - (A-1)\tau_d} + \frac{(A-1)\tau_e(1-\theta_f)Ay_f}{A\theta_f - (A-1)\tau_e}.$$

The rates τ_d and τ_e that minimize the total amount taken from the household are given by

$$\begin{aligned} \min_{\tau_d \geqslant 0, \tau_e \geqslant 0} \quad & \frac{(A-1)\tau_d(1-\theta_b)Ay_b}{A\theta_b - (A-1)\tau_d} + \frac{(A-1)\tau_e(1-\theta_f)Ay_f}{A\theta_f - (A-1)\tau_e} \\ \text{s.t.} \quad & \frac{(A-1)(1-\theta_b)\tau_d y_b}{\theta_b[A\theta_b - (A-1)\tau_d]} + \frac{(A-1)(1-\theta_f)\tau_e y_f}{\theta_f[A\theta_f - (A-1)\tau_e]} = a. \end{aligned} \tag{40}$$

Proposition 14 *Suppose that the objective is to take the minimum amount from the household while still maximizing the social welfare. Then if $\theta_b > \theta_f$, it is optimal to set the subsidy interest rate*

$$\tau_d = \frac{A\theta_b^2 a}{(A-1)[\theta_b a + (1-\theta_b)y_b]}. \tag{41}$$

[9]This follows from solving the above household problem with $r_d = r_e = A-1$ and assuming $-r_d\tau_d d_b - r_e\tau_e e_f$ in (39) is a given constant.

and $\tau_e = 0$*; otherwise, it is optimal to set* $\tau_d = 0$ *and*

$$\tau_e = \frac{A\theta_f^2 a}{(A-1)[\theta_f a + (1-\theta_f)y_f]}. \tag{42}$$

Under this policy,

$$d = \frac{(1-\theta_b)Ay_b}{A\theta_b - (A-1)\tau_d} \quad \textit{and} \quad e = \frac{(1-\theta_f)Ay_f}{A\theta_f - (A-1)\tau_e},$$

and the total amount taken from the household is $\max\{\theta_b, \theta_f\}Aa$.

The policy has a similar spirit to the cases of equity injections and loan subsidies. When the moral hazard at the firm is more severe than at the bank, it is cheaper then to provide incentives to the bank. The cost effective optimal policy should focuses on the bank and provides no special incentives to the firm. In the parameters are reversed, the low cost optimal policy is aimed and the firm and ignores the bank.

6.4 Comparing Policies

In this subsection, we compare the household tax and the house savings under the different policies.

First, how does the household tax with an interest rate subsidy compare to the tax needed for equity injections or the bank loans? In general the minimum amount tax taken from the household with an interest rate subsidy is $\max\{\theta_b, \theta_f\}Aa$. This is clearly larger than the minimum amount tax taken from the household in the case of an equity injection, which is either $\min\{\theta_b, \theta_f\}a$ if the bank and the firm can cheat the government, or $\min\{\frac{\theta_b}{1+\theta_b}, \frac{\theta_f}{1+\theta_f}\}a$ if the bank and the firm cannot cheat the government.

We note that the minimum total amount taken from the household in the loan subsidy which is achieved by setting $\rho_b = \rho_f = 0$, is $\min\{\theta_b, \theta_f\}\frac{Aa}{A-1}$ if both the bank and the firm can cheat the government, or $\min\{\frac{A\theta_b}{(1+\theta_b)A-1}, \frac{A\theta_f}{(1+\theta_f)A-1}\}a$ if neither can cheat the government. The amount taken is clearly larger than the case with the equity injection, but may be larger than or smaller than the case with the rate subsidy, dependent on the specific ranges of the parameters.

Beyond the tax level there is also a timing difference to consider. With an interest rate subsidy the household is taxed in period 1, while for an equity injection of a loan the tax is in period 0. With an interest rate subsidy neither the bank nor the firm can cheat the government in the manner modeled in this paper. Because there will be an extra tax in period 1, the household prepares by increasing savings.

We summarize the above with the following proposition.

Proposition 15 *To achieve social efficiency, equity injection policy requires less tax on the household than loan subsidy policy that charge interest.*

Finally, we compare the household savings. We assume $\theta_b > \theta_f$ in our analysis (and the conclusion is the same when $\theta_b \leqslant \theta_f$. Under the minimum rate subsidy policy, it follows from Proposition 14 that

$$d = a + \frac{1-\theta_b}{\theta_b} y_b \quad \text{and} \quad e = \frac{1-\theta_f}{\theta_f} y_f,$$

or the total amount of the savings is given by

$$d + e = a + \frac{1-\theta_b}{\theta_b} y_b + \frac{1-\theta_f}{\theta_f} y_f = y_h - c_0^* = S^*.$$

(In the above expression, the first equality follows from the definition for a in equation (36) and the second equality follows from Proposition 3.) That is, this policy induces socially efficient savings.

Under a policy of equity injection, the total amount savings is given by

$$y_h - c_0^* - T_b - T_f = S^* - T_b - T_f,$$

where S^* is the socially efficient equilibrium saving. Notice that this equality is independent of whether the bank and the firm can or cannot cheat the government. Of course, T_b and T_f would depend on whether the bank and the firm can cheat. In other words, the summation of the total tax and the total savings in this case achieves socially efficient savings as it should. As the total tax is positive, the total amount savings in this case is less than the socially optimal savings (under the no-friction case).

Under the loan injection, the total amount savings is given by

$$y_h - c_0^* - L_b - L_f = S^* - L_b - L_f.$$

Similarly, the summation of the total tax and the total savings in this case achieves the socially efficient savings. As $T_b + T_f < L_b + L_f$, the equity injection would induce more savings than the loan subsidy.

We summarize the above discussion with the following proposition.

Proposition 16 *Optimal policy that takes the form of an equity injection or a loan subsidy reduces household savings. Optimal interest rate subsidies increase household savings and achieve socially optimal savings.*

7 Conclusion

As described in Bernanke (2013) the unusual circumstances of the financial crisis led to unusual policy on the part of the Federal Reserve and the Treasury. The actual policy interventions motivated the construction of models to help account for what had taken place. In these models it is common to assume that there are efficient firms financed by banks, and banks are themselves financed by households. When the bank has an incentive to deviate from first best, the equilibrium is inefficient. The inefficiency motivates policy.

Our concern is that empirically banks are not the dominant channel by which household savings reach firms. Furthermore, banks are not the only source of potential deviations from optimal behavior. Firms may also have incentives to deviate from first best. This motivates our study of an economy in which banks and firms may both have incentive to cheat on promises to investors, and in which the bank coexists with a market for corporate equity.

Socially efficient equilibria exist even when the bank has an incentive to cheat, provided the equity market operates efficiently. Similarly, even if the equity market is distorted, the equilibrium will be efficient if the bank has an incentive to behave efficiently.

To attain social efficiency it is important that at least one channel operate properly. It is not necessary that they all operate perfectly.

Within our model it is possible to justify policy intervention. If both the bank and equity markets are distorted the equilibrium is not socially efficient. Suitable government policy can improve social welfare. The policy can be aimed at the bank or at the firm. The policy can take the form of an equity injection, a subsidized loan, or an interest rate subsidy. Suitably designed, any of these can achieve social optimality. Equity injections can achieve social optimality with a lower level of tax imposed on the household than is required if loans or interest rate subsidies are used.

Optimal policy that takes the form of a loan subsidy or an equity injection crowds out the private sector in the sense that the household reduces savings. The household consumption is still optimal since the policy is optimal. Consumption is sustained by greater profits from the bank and/or the firm that are making use of the money received from the government. Optimal policy that takes the form of an interest rate subsidy increases household savings.

Overall, we observe that financial markets may be more flexible than often depicted in the recent policy literature. Thus the case for any form of extraordinary policy must be more broadly conceived. However, when needed, there are pluses and minuses to interest rate subsidies and equity injections. Interest rate subsidies may require a higher level of taxation, but that taxation takes place later, and there is less of a concern about cheating. However, equity investments by the government in response to a crisis can reestablish social optimality while keeping down the tax imposed on the household.

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Table 1: Comparing No-friction and Friction Equilibrium Values

To facilitate comparisons, this table presents a summary of the equilibrium values described in Proposition (3) and Proposition (7).

Variable	No Frictions	Relation	Both Frictions
π_b	Ay_b	$<$	$\frac{\theta_b(1+r)Ay_b}{1+r-(1-\theta_b)A}$
π_f	Ay_f	$<$	$\frac{\theta_f(1+r)Ay_f}{1+r-(1-\theta_f)A}$
r_d	$A-1$	$>$	$r \in ((1-\theta_f)A-1, A-1)$
r_e	$A-1$	$>$	$r \in ((1-\theta_f)A-1, A-1)$
r_ℓ	$A-1$	$=$	$A-1$
$d+e$	$\frac{y_h\beta^{1/\gamma}-(y_b+y_f)A^{(\gamma-1)/\gamma}}{\beta^{1/\gamma}+A^{(\gamma-1)/\gamma}}$	$>$	$\frac{(1-\theta_b)Ay_b}{1+r-(1-\theta_b)A}+\frac{(1-\theta_f)Ay_f}{1+r-(1-\theta_f)A}$
c_0	$\frac{A^{(\gamma-1)/\gamma}(y_h+y_b+y_f)}{\beta^{1/\gamma}+A^{(\gamma-1)/\gamma}}$	$<$	$\frac{(1+r)^{-1/\gamma}(y_h(1+r)+\pi_b+\pi_f)}{\beta^{1/\gamma}+(1+r)^{(\gamma-1)/\gamma}}$
c_1	$(A\beta)^{1/\gamma}c_0$	$>$	$(\beta(1+r))^{1/\gamma}c_0$
U	$\frac{1+A^{(1-\gamma)/\gamma}\beta^{1/\gamma}}{1-\gamma}c_0^{1-\gamma}$	$>$	$\frac{1+(1+r)^{(1-\gamma)/\gamma}\beta^{1/\gamma}}{1-\gamma}c_0^{1-\gamma}$

8 Appendix: Some Elementary Results and Various Proofs

8.1 A Lemma for a Household Problem

Consider the household's problem is

$$\begin{aligned}
\max_{d_h \geqslant 0, e_h \geqslant 0} \quad & U(c_0, c_1) = u(c_0) + \beta u(c_1) \\
\text{s.t.} \quad & c_0 = Y - (d_h + e_h) \\
& c_1 = (1 + r_d)d_h + (1 + r_e)e_h + \Pi \\
& c_0 \geqslant 0, c_1 \geqslant 0,
\end{aligned} \tag{43}$$

where Y is the household net initial endowment and Π is the net profit/contribution to the household in period 1.

This formulation takes several household problem formulations as the special cases. Foe example, in the benchmark household model, $Y = y_h$ is the household initial endowment and $\Pi = \pi_b + \pi_f$. Under the equity injection policy, $Y = y_h - T_b - T_f$ is the household initial endowment minus the injection amount (which is taken away from the household). Under the loan subsidy policy, $\Pi = \pi_b + \pi_f + (1 + \rho_b)L_b + (1 + \rho_f)L_f$. Under the interest rate subsidy, $\Pi = \pi_b + \pi_f - r_d\tau_d d_b - r_e\tau_e e_f$; in this case, we note that Π is treated as a constant in the optimization.

Lemma 17 *Let $r_{\max} = \max\{r_d, r_e\}$. The optimal solution to the problem (43) falls into two cases dependent upon whether the inequality*

$$Y(\beta(1 + r_{\max})^{1/\gamma} \geqslant \Pi. \tag{44}$$

When the inequality (44) holds, then the optimal solution is to have $d_h = 0$ and $e_h = S$ if $r_d < r_e$, to have $d_h = S$ and $e_h = 0$ if $r_d > r_e$, and to have any nonnegative pair (d_h, e_h) with $d_h + e_h = S$, where

$$S = \frac{Y\beta^{1/\gamma} - \Pi(1 + r_{\max})^{-1/\gamma}}{\beta^{1/\gamma} + (1 + r_{\max})^{(\gamma-1)/\gamma}},$$

with corresponding optimal consumption levels

$$\begin{aligned}
c_0 &= \frac{Y(1 + r_{\max})^{(\gamma-1)/\gamma} + \Pi(1 + r_{\max})^{-1/\gamma}}{\beta^{1/\gamma} + (1 + r_{\max})^{(\gamma-1)/\gamma}}, \\
c_1 &= \frac{\beta^{1/\gamma}(Y(1 + r_{\max}) + \Pi)}{\beta^{1/\gamma} + (1 + r_{\max})^{(\gamma-1)/\gamma}} = (\beta(1 + r_{\max}))^{1/\gamma}c_0.
\end{aligned}$$

When the inequality (44) does not hold, the optimal solution is $d_h = e_h = 0$ with corresponding consumptions $c_0 = Y$ and $c_1 = \Pi$

The proof of this lemma is elementary, and we only sketch a proof. First by observing that if $r_d > r_e$, then the household optimal decision is to set $e_h = 0$, and if $r_d < r_e$, then the household optimal decision is to set $e_d = 0$. Then, let $S = d_h + e_h$ and $r_{\max} = \max\{r_e, r_d\}$.

Then, the household problem is equivalent to maximizing the total utility over S with the constraints

$$c_0 = Y - S \quad \text{and} \quad c_1 = (1 + r_{\max})S + \Pi.$$

The solution can be easily found through the first-order condition.

8.2 Proofs

Proof (of Proposition 3). It follows from the argument prior to the proposition statement that for an interior equilibrium, it must be that $r_d = r_e = r_\ell = A - 1$. Hence, there is no equilibrium that has a positive interest spread. With $r_d = r_e = r_\ell = A - 1$, we have $\pi_b = (1 + r_\ell)y_b = Ay_b$ from the bank optimization solution and $\pi_f = Ay_f$ from the firm optimization solution. Next, using these quantities in the household optimization solutions, we immediately establish the rest of the propositions; specifically, we obtain S^* from (5) and c_0^* and c_1^* respectively from (6) and (7). The condition (10) is the same as the condition $d + e = S^* > 0$. ∎

Proof (of Proposition 7). From the household problem, in order for both $d = d_h > 0$ and $e = e_h > 0$, it is necessary to have $r_d = r_e \equiv r$. From the firm problem with friction, in order for $0 < \ell_f = \ell < \infty$ and $0 < e_f = e < \infty$, it is necessary to have $(1-\theta_f)A < 1+r_e < A$ (or equivalently $(1 - \theta_f)A < 1 + r < A$ and $1 + r_\ell = A$ (in view of Proposition 5) and the argument before it). Next, from the bank problem with friction, in order for $r_d < r_\ell$ and $0 < d_b = d < \infty$, it is necessary to have $1 + r_d > (1 - \theta_b)(1 + r_\ell)$ (or equivalently $1 + r > (1 - \theta_b)A$) (in view of Proposition 4 and the argument before it). ∎

In preparing for the proof of Proposition 8, we first establish the following lemma.

Lemma 18 *The condition (20) is a necessary and sufficient condition for the equation* $\varphi(x) = 0$ *to have a unique solution in* $((1 - \theta_f)A, A)$.

Proof To establish the lemma, we only need to prove the following: (a) $\varphi(x)$ is decreasing in $((1 - \theta_f)A, A)$; (b) $\varphi(x)$ increases to $+\infty$ as x decreases to $(1 - \theta_f)A$; and (c) $\varphi(A) < 0$ if and only if the condition (20) holds. The verification for (a) and (b) is straightforward. To show (c), we note

$$\begin{aligned}
\varphi(A) &= \frac{(1-\theta_f)y_f}{\theta_f} + \frac{(1-\theta_b)y_b}{\theta_b} - \frac{y_h\beta^{1/\gamma} - (y_b + y_f)A^{(\gamma-1)/\gamma}}{\beta^{1/\gamma} + A^{(\gamma-1)/\gamma}} \\
&= \frac{1}{\beta^{1/\gamma} + A^{(\gamma-1)/\gamma}} \\
&\quad \times \left\{ \frac{1}{\theta_f}[A^{(\gamma-1)/\gamma} + \beta^{1/\gamma}(1-\theta_f)]y_f + \frac{1}{\theta_b}[A^{(\gamma-1)/\gamma} + \beta^{1/\gamma}(1-\theta_b)]y_b - \beta^{1/\gamma}y_h \right\},
\end{aligned}$$

which is negative if and only if the condition (20) holds. ∎

Proof (for Proposition 8). To prove sufficiency we only need to show that under the condition (20), the equilibrium quantities defined in the theorem indeed give the desired equilibrium. By Lemma 18 above, x^* in the theorem is well-defined, and hence, all the equilibrium quantities are well-defined. The strict interior condition and the positive interest rate spread condition would be obvious, if we show these quantities form an equilibrium. For the latter, we show that these quantities simultaneously solve the household problem, the bank problem and the firm problem.

Household Problem. First, we note that $d + e = S$ for S in (5) is the same as the condition $\varphi(1+t) = 0$; the latter holds since $x^* = 1 + r$ is the solution to $\varphi(x) = 0$. Next, it is immediate to check that c_0 and c_1 are the same as the ones given in (6) and (7) by noting $\pi_b = \pi_b(1+r)$ and $\pi_f = \pi_f(1+r)$ (with the functions $\pi_b(\cdot)$ and $\pi_f(\cdot)$ defined in the theorem).

Bank Problem. It is sufficient to verify Claim 4. It is clear that $1+r_d > (1-\theta_b)(1+r_\ell)$ and $r_\ell > r_d$ holds. Also noting

$$d_b = \frac{(1-\theta_b)(1-r_\ell)y_b}{1+r_d-(1-\theta_b)(1+r_\ell)} = \frac{(1-\theta_b)Ay_b}{1+r_d-(1-\theta_b)A} = d;$$

hence, d solves the bank problem.

Firm Problem. Note $r_e = r$ clearly satisfies $(1-\theta_f) < 1+r_e < A$; then $e = e_f$ from the frictional firm problem shows e solves the firm problem.

Now we return to prove the necessity of the condition (20). By Proposition 7, a necessary condition for a strictly interior equilibrium with a positive interest spread is that $r_\ell = A-1$, and $r \equiv r_d = r_e$ satisfying $(1-\theta_b)A - 1 < r$, $(1-\theta_f)A - 1 < r$ and $r \leqslant A$. In this case, we have from the friction bank problem, the equilibrium deposit must be

$$d \equiv d_b = \frac{(1-\theta_b)(1+r_\ell)y_b}{1+r_d-(1-\theta_b)(1+r_\ell)} = \frac{(1-\theta_b)Ay_b}{1+r_d-(1-\theta_b)A},$$

which is the same as (21), and we have from the friction firm problem, the equilibrium equity investment must be

$$e \equiv e_f = \frac{(1-\theta_f)(1+r_\ell)y_f}{1+r_e-(1-\theta_f)(1+r_\ell)} = \frac{(1-\theta_f)Ay_f}{1+r_e-(1-\theta_f)A},$$

which is the same as (22). Next in view of the optimal household saving equation (5), the equality $d + e \equiv d_h + e_h = S$ (again from the household problem) can be written as

$$\frac{(1-\theta_f)Ay_f}{1+r-(1-\theta_f)A} + \frac{(1-\theta_b)Ay_b}{1+r-(1-\theta_b)A} = \frac{y_h\beta^{1/\gamma} - (\pi_b+\pi_f)(1+r)^{-1/\gamma}}{\beta^{1/\gamma} + (1+r)^{(\gamma-1)/\gamma}}, \tag{45}$$

where π_b and π_f are respectively from the friction bank problem and the friction firm problem given by

$$\pi_b = \frac{\theta_b(1+r)Ay_b}{1+r-(1-\theta_b)A} \qquad \text{and} \qquad \pi_f = \frac{\theta_f(1+r)Ay_f}{1+r-(1-\theta_f)A}.$$

It is immediate to verify that the equation (45) is the same as the equation $\varphi(x) = 0$ with $x = 1 + r$. By Lemma (18), we conclude that the condition (20) is a necessary condition. ■

Proof (of Proposition 10). The converse clearly follows from Corollary 9 and the fact that under the condition (10), the no-friction equilibrium is socially efficient.

Now we prove the first part. We assume that the inequality (20) holds through the rest of the proof. To show that the equilibrium for the friction problem is socially inefficient, it is sufficient for us to show that the friction equilibrium solution (which is given in Proposition 8) (a) is a feasible solution to the social welfare problem (24); and (b) is not the optimal solution to social welfare problem (24).

Part (a). The proof is straightforward if we note that the equilibrium solution must simultaneously solve the household problem, the friction bank problem and the friction firm problem. First, it follows from the friction bank problem and the friction firm problem that we have $k = y_f + \ell + e = y_f + y_b + d + e$. Next, from the household problem, we know that $c_0 = y_h - (d + e)$. This establishes the first equality constraint in the optimization problem (24). It is elementary to verify the second inequality constraint holds with equality by noting $k = y_f + y_b + d + e$ and quantities d, e and c_1 in Theorem 8. The last nonnegative constraints, $c_0 \geqslant 0$ and $c_1 \geqslant 0$, are clear.

Part (b). First note that the optimal solution to the social welfare problem (24) is unique. Next note the optimal consumptions to the social welfare problem (24) are given by

$$\begin{aligned} c_0^* &= y_h + y_b + y_f - k^* = \frac{A^{(\gamma-1)/\gamma}(y_h + y_b + y_f)}{\beta^{1/\gamma} + A^{(\gamma-1)/\gamma}}, \\ c_1^* &= Ak^*. \end{aligned}$$

Then it is elementary to check that $c_0^* \neq c_0$, where c_0 is the period zero equilibrium consumption level given in Theorem 8. Hence, the friction equilibrium solution in Theorem 8 cannot be the optimal solution to social welfare problem (24). This completes the proof. ■

Proof (of Corollary 11). We show that the household consumes more in period zero and consumes less in the second period in comparison to the socially efficient solution, i.e., $c_0 > c_0^*$ and $c_1 < c_1^*$. Let

$$\psi_1(x) \equiv \frac{y_h\beta^{1/\gamma} - (\pi_b(x) + \pi_f(x))x^{-1/\gamma}}{\beta^{1/\gamma} + x^{(\gamma-1)/\gamma}}.$$

with $\pi_1(x)$ and $\pi_2(x)$ defined in Theorem 8. Observe that $\psi_1(x)$ is increasing in x; hence, $1 + r < A$ (where r is defined in Theorem 8) implies $\psi_1(1 + r) < \psi_1(A)$. Note that

$$c_0 = y_h - \psi_1(1 + r) \quad \text{and} \quad c_0^* = y_h - \psi_1(A);$$

this proves $c_0 > c_0^*$. Since $U(c_0, c_1) = u(c_0) + \beta u(c_1) < u(c_0^*) + \beta u(c_1^*) = U(c_0^*, u_1^*)$ (the inefficiency result in the first part of proposition) and $u(\cdot)$ is an increasing function, $c_0 > c_0^*$ implies that $c_1 < c_1^*$. This completes the proof. ■

Proof (of Proposition 14). First rewriting the above constraint in the optimization problem

(40),

$$\frac{(A-1)\tau_e(1-\theta_f)Ay_f}{A\theta_f-(A-1)\tau_d} = A\theta_f\left(a - \frac{(A-1)(1-\theta_b)\tau_d y_b}{\theta_b[A\theta_b-(A-1)\tau_d]}\right). \tag{46}$$

Substituting this into the objective function, the optimization problem can be written,

$$\min_{\tau_d\geqslant 0,\tau_e\geqslant 0} A\theta_f a + \left(1-\frac{\theta_f}{\theta_b}\right)\frac{(A-1)\tau_d(1-\theta_b)Ay_b}{A\theta_b-(A-1)\tau_d}.$$

Suppose that $\theta_b > \theta_f$. Then the above objective function is increasing in τ_d. The optimal solution is to choose the maximum τ_d such that the constraint (46) becomes zero because we must keep $\tau_e \geqslant 0$. This is equivalent to

$$\frac{(A-1)(1-\theta_b)\tau_d y_b}{\theta_b[A\theta_b-(A-1)\tau_d]} = a.$$

Solving the above for the expression for τ_d in (41). With $\theta_b > \theta_f$, under the pair τ_d as given in (41) and $\tau_e = 0$, the total tax taken from the household is

$$A\theta_f a + \left(1-\frac{\theta_f}{\theta_b}\right)\frac{(A-1)\tau_d(1-\theta_b)Ay_b}{A\theta_b-(A-1)\tau_d} = A\theta_b a.$$

The proof for the case $\theta_b < \theta_f$ is similar. ∎