A Theory of Liquidity and Risk Management*

Incomplete and Do not Distribute

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Abstract

We formulate a theory of optimal corporate liquidity and risk management for a firm run by a risk-averse entrepreneur, who cannot irrevocably commit her human capital to the firm. The firm's balance sheet comprises illiquid capital and cash or marketable securities on the asset side, and on the liability side equity and a line of credit, with an endogenously derived limit. The firm's operations are subject to both idiosyncratic and aggregate shocks. The entrepreneur seeks to optimally smooth consumption and investment, and to manage the firm's risk by choosing the optimal loading on the idiosyncratic and market risk factors for the firm's savings, subject to the constraint that she will continue to employ her changing human capital at the firm. Besides this inalienability of human capital constraint we do not assume any other financial market imperfection. The unique state variable in the entrepreneur's dynamic optimization problem is the liquidity-to-capital ratio. As this state variable approaches the lower bound where the firm's credit limit is exhausted we show that the entrepreneur optimally responds by increasingly cutting investment, consumption and the risk exposure of the firm's liquid savings. The main general implication of our theory is that corporate liquidity and risk management adds value for firms subject to inalienability of risky human capital constraints, even in the absence of any financial market imperfection.

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1 Introduction

A corollary of the Modigliani and Miller irrelevance theorem is that firms cannot create any value through corporate liquidity and risk management. The basic logic is that any retained earnings or corporate hedging positions can be undone or replicated by the firm's investors, so that there is no value added in the firm doing this for them. Accordingly, first-generation theories of corporate liquidity and risk management have invoked financial market imperfections, such as tax distortions, or a wedge between internal and external funding costs, as a basic rationale for corporate risk management. While such financial imperfections are clearly important to explain why some firms do engage in liquidity and risk management, they do not account for all the benefits of corporate risk management.

In this paper we develop a theory of liquidity and risk management that emphasizes benefits over and above those that spring from tax or asymmetric information distortions. These benefits have to do with the firm's greater ability to offer optimal compensation and thereby improve the firm's chances to retain talent and valuable human capital. Our theory considers the problem faced by a risk-averse entrepreneur, who cannot irrevocably commit her human capital to the firm. The firm's operations are subject to both idiosyncratic and aggregate shocks. The entrepreneur has constant relative risk-averse preference and seeks to smooth consumption. To best retain the entrepreneur it is efficient for the firm to compensate her with current and future promised consumption. But to back up these promises the firm must engage in liquidity and risk management.

The firm's balance sheet is composed of illiquid capital and cash or marketable securities on the asset side. On the liability side the firm has equity and a line of credit with a limit that is endogenously determined. Illiquid capital can be augmented via investment and is subject to stochastic depreciation. The firm's operations are exposed to both idiosyncratic and aggregate risk. The firm's liquidity is augmented via retained earnings from operations and financial returns from its portfolio of marketable securities. The firm manages its risk by choosing the optimal loading of its securities holdings on the idiosyncratic and market risk factors. The unique state variable of the entrepreneur's dynamic optimization problem is the firm's liquidity-to-capital ratio. When this state variable approaches an endogenously determined lower bound where the firm's credit limit is exhausted the entrepreneur optimally responds by increasingly cutting investment, consumption and the risk exposure of the firm's liquid savings.

The model we develop generalizes the limited commitment frameworks of Hart and Moore (1994) and Rampini and Viswanathan (2010, 2013). Hart and Moore (1994) formulate a theory of debt and endogenous debt capacity arising from the inalienability of a risk-neutral entrepreneur's human capital. In a finite-horizon model with a single fixed project, deterministic cash flows and fixed human capital, they show that there is a finite debt capacity for the firm, which is given by the maximum repayment that the entrepreneur can credibly promise: any higher repayment and the entrepreneur would aband the firm.

We generalize Hart and Moore (1994) along several important dimensions: first, we introduce risky human capital and cash flows; second, we assume that the entrepreneur is risk averse; third, we consider an infinitely-lived firm with ongoing investment; and, fourth we also add a limited liability or commitment constraint for investors. In this more realistic model we are nevertheless able to derive the optimal investment, consumption, liquidity and risk management policy of the firm.

Rampini and Viswanathan (2010, 2013) also generalize Hart and Moore (1994) by introducing risky operations, capital accumulation and state-contingent limited commitment constraints. They develop a theory of liquidity and risk management focusing on the tradeoff between exploiting current versus future investment opportunities. If the firm invests today it may exhaust its debt capacity and thereby forego future investment opportunities. If instead the firm foregoes investment and hoards its cash it is in a position to be able to exploit potentially more profitable investment opportunities in the future. The main addition we bring to their model is the risk aversion of the entrepreneur and the modeling of limited commitment in the form of risky inalienable human capital. With this addition we focus on a different aspect of corporate liquidity and risk management, namely the management of risky human capital.

Our theory can thus explain the observed corporate policies of human capital intensive, high-tech, firms. These firms often hold substantial cash pools, which may be necessary to make credible future compensation promises and thereby retain highly valued employees with attractive alternative job opportunities. Indeed, employees in these firms are largely paid in the form of deferred stock compensation. When their stock options vest and are exercised the companies often engage in stock repurchases so as to avoid excessive stock dilution. But such repurchase programs require funding, which could explain why these companies retain so much cash.

The firm's optimal investment and consumption policies in our model can be characterized as straightforward generalizations of respectively the classical q-theory of investment and the permanent-income theory of consumption, which adjust the optimal investment and consumption policies further away from the fist-best policies as the firm's shadow value of liquidity rises. Similarly, the firm's optimal liquidity and risk management policies can be characterized as a generalization of Merton's classical intertemporal portfolio-choice rules that condition the portfolio weights on the firm's liquidity-to-capital ratio. We show that the firm's limited commitment constraint in the most reduced formulation of the entrepreneur's optimization problem takes the form of an endogenous lower bound for the firm's liquidity-to-capital ratio. The closer the firm is to this lower bound the tighter is its financial constraint and the more the firm's investment, consumption, and risk management policies deviate from first-best optimality.

We also show that the entrepreneur's optimal liquidity and risk management problem can be reformulated as a dual optimal contracting problem between an optimally diversified investor and a risk-averse entrepreneur subject to an inalienability-of-human-capital constraint. More concretely, the state variable in the optimal contracting problem between risk-neutral investors and the risk-averse entrepreneur is the promised wealth to the entrepreneur per unit of capital, w, and the value of the firm to investors per unit of capital is $p(w)$. Moreover, under the optimal contract the firm's investment and financing policies and the entrepreneur's consumption are all expressed as functions of w. As Table 1 below summarizes, we show that this dual contracting problem is equivalent to the entrepreneur's liquidity and risk management problem with corporate savings per unit of capital, $s = -p(w)$, as the state variable and with the objective function of the entrepreneur $m(s) = w$. The key observation in the formulation of this dual problem is that the firm's endogenously determined credit limit is the outcome of an optimal financial contracting problem. In other words, the firm's financial constraint is an optimal credit limit that reflects the entrepreneur's inability to irrevocably commit her human capital to the firm.

We extend the simplest formulation of the model in two directions. First, we also introduce a limited commitment (or limited liability) constraint for investors. Second we introduce productivity shocks in the form of a two-state Markov transition process from high and low productivity states. In the two-sided commitment problem, where a limited liability constraint for investors must also hold, we obtain further striking results. The firm

Table 1: Equivalence between primal contracting and dual implementation

	Primal Optimization	Dual Contracting
State Variable	s	w
Value Function	$m(s)$	$p(w)$

may now over-invest and the entrepreneur may over-consume (compared to the first-best benchmark). The intuition is as follows. To make sure that investors do not default on their promised future utility for the entrepreneur, w, this promise cannot exceed an upper bound $\overline{w}$ given by $p(\overline{w}) = 0$. In other words, the firm's liquidity, s, cannot be too high (exceed $\overline{s} = 0$), otherwise, investors would simply syphon off the excess liquidity. As a result, the entrepreneur needs to substantially increase investment and consumption in order to satisfy the investors' limited-liability constraint.

In the extension where the firm's productivity can switch from a high to a low productivity state we show that

Related literature. Our paper builds on the dynamic contracting methodology in continuous time following Holmstrom and Milgrom (1987), Schaettler and Sung (1993), DeMarzo and Sannikov (2006), and Sannikov (2008), among others. Our paper provides foundations for a dynamic theory of liquidity and risk management based on risky inalienable human capital. As such it is obviously related to the important, early contributions on corporate risk management by Stulz (1984), Smith and Stulz (1985) and Froot, Scharfstein, and Stein (1993). Unlike our setup, they consider static models with exogenously given financial frictions to show how corporate cash and risk management can create value by relaxing these financial constraints.

Our paper is also evidently related to the corporate security design literature, which seeks to provide foundations for the existence of corporate financial constraints, and for the optimal external financing by corporations through debt or credit lines. This literature can be divided into three separate strands. The first approach provides foundations for external debt financing in a static optimal contracting framework with either asymmetric information and costly monitoring (Townsend, 1979, and Gale and Hellwig, 1985) or moral hazard (Innes, 1990, and Holmstrom and Tirole, 1997).

The second more dynamic optimal contracting formulation derives external debt and credit lines as optimal financial contracts in environments where not all cash flows generated by the firm are observable or verifiable.[1]

The third approach which is closely related to the second provides foundations for debt financing based on the inalienability of human capital (Hart and Moore, 1994, 1998). Harris and Holmstrom (1982) is an early important paper that generates non-decreasing consumption profile in a model where workers are unable to commit to long-term contracts. Berk, Stanton, and Zechner (2010) incorporate capital structure and human capital bankruptcy costs into Harris and Holmstrom (1982). Rampini and Viswanathan (2010, 2013) develop a model of corporate risk management building on similar contracting frictions. A key result in their model is that hedging may not be an optimal policy for firms with limited capital available as collateral. For such firms, hedging demand in effect competes for limited collateral with investment demand. They show that for growth firms the return on investment may be so high that it crowds out hedging demand. Li, Whited, and Wu (2014) structurally estimate optimal contracting problems with limited commitment along the line of Rampini and Viswanathan (2013) providing empirical evidence in support of these class of models.

The latter two approaches are often grouped together because they yield closely related results and the formal frameworks are almost indistinguishable under the assumption of risk-neutral preferences for the entrepreneur and investors. However, as our analysis with risk-averse preferences for the entrepreneur makes clear, the two frameworks are different. The models based on non-contractible cash flows require dynamic incentive constraints that restrict the set of incentive compatible financial contracts, while the models based on inalienable human capital only impose (dynamic) limited-commitment constraints for the entrepreneur. With the exception of Gale and Hellwig (1985) the corporate security design literature makes the simplifying assumption that the contracting parties are risk neutral. By allowing for risk-averse entrepreneurs, we not only generalize the results of this literature on the optimality of debt and credit lines, but we are also able to account for the fundamental role of corporate savings and risk management.

[1] See Bolton and Scharfstein (1990), DeMarzo and Fishman (2007), Biais, Mariotti, Plantin, and Rochet (2007), DeMarzo and Sannikov (2006), Piskorski and Tchistyi (2010), Biais, Mariotti, Rochet, and Villeneuve (2010), and DeMarzo, Fishman, He and Wang (2012). See Sannikov (2012) and Biais, Mariotti, and Rochet (2013) for recent surveys of this literature.

In contemporaneous and independent work, Ai and Li (2013) analyze a closely related contracting problem. Their motivation is different from ours: where we emphasize the inalienability of risky human capital and the implementation of the optimal contract via dynamic liquidity and risk management, they study the dynamics of optimal managerial compensation and investment under limited commitment. In addition, we incorporate stochastic productivity shocks and establish the optimality of contingent capital and insurance contracts. Also closely related is Lambrecht and Myers (2012) who consider an intertemporal model of a firm run by a risk-averse entrepreneur with habit formation and derive the firm's optimal dynamic corporate policies. They show that the firm's optimal payout policy resembles the famous Lintner (1956) payout rule of thumb.

Our financial implementation of the optimal financial contract is also related to the portfolio choice literature featuring illiquid productive assets and under-diversified investors in an incomplete-markets setting. Building on Merton's intertemporal portfolio choice framework, Wang, Wang, and Yang (2012) study a risk-averse entrepreneur's optimal consumption-savings decision, portfolio choice, and capital accumulation when facing uninsurable idiosyncratic capital and productivity risks. Unlike Wang, Wang, and Yang (2012), our model features optimal liquidity and risk management policies that arise endogenously from an underlying financial contracting problem.

Our framework also provides a micro-foundation for the dynamic corporate savings models that take external financing costs as exogenously given. Hennessy and Whited (2007), Riddick and Whited (2009), and Eisfeldt and Muir (2014) study corporate investment and savings with financial constraints. Bolton, Chen, and Wang (2011, 2013) study the optimal investment, asset sales, corporate savings, and risk management policies for a firm that faces external financing costs. It is remarkable that although these models are substantially simpler and more stylized the general results on the importance of corporate liquidity and risk management are broadly similar to those derived in our paper based on more primitive assumptions. Conceptually, our paper shows that to determine the dynamics of optimal corporate investment, in addition to the marginal value of capital (marginal q), a critical variable is the firm's marginal value of liquidity. Indeed, we establish that optimal investment is determined by the ratio of marginal q and the marginal value of liquidity, which reflects the tightness of external financing constraints.[2] Our model thus shares a similar

[2] Faulkender and Wang (2006), Pinkowitz, Stulz, and Williamson (2006), Dittmar and Mahrt-Smith (2007), and Bolton, Schaller, and Wang (2014) empirically measure the marginal value of cash.

focus on the marginal value of liquidity as Bolton, Chen, and Wang (2011, 2013) and Wang, Wang, and Yang (2012).

Our paper also relates to the macroeconomics literature that studies the implications of dynamic agency problems for firms' investment and financing decisions. Green (1987), Thomas and Worrall (1990), Marcet and Marimon (1992), Kehoe and Levine (1993) and Kocherlakota (1996) are important early contributions on optimal contracting under limited commitment. Alvarez and Jermann (2000, 2001) study welfare and asset pricing implications of endogenously incomplete markets due to limited contract enforcement constraints. Albuquerque and Hopenhayn (2004), Quadrini (2004), and Clementi and Hopenhayn (2006) characterize firms' financing and investment decisions under limited commitment and/or asymmetric information. Lorenzoni and Walentin (2007) study q theory of investment under limited commitment. Grochulski and Zhang (2011) consider a risk sharing problem under limited commitment. Miao and Zhang (2014) develop a duality-based solution method for limited commitment problems.[3] Finally, our paper is clearly related to the voluminous economics literature on human capital that builds on Ben-Porath (1967) and Becker (1975).

2 Model

We consider the intertemporal optimization problem faced by a risk-averse entrepreneur, who optimally chooses her consumption, savings, capital investment, and exposures to both systematic and idiosyncratic risks of the firm, subject to the limited commitment constraint that she cannot promise to operate the firm indefinitely under any circumstances. This limited-commitment problem for the entrepreneur will induce an *endogenous* financial constraint for the firm. To best highlight the central economic mechanism arising from this limited commitment constraint, we remove all other financial frictions from the model by assuming that financial markets are fully competitive and dynamically complete (we show how dynamic completeness is constructed through spanning in Section 2.2). The detailed model description begins below with the entrepreneur's production technology and preferences.

[3]See Ljungqvist and Sargent (2004) Part V for a textbook treatment on the limited-commitment-based macro models.

2.1 Production Technology and Preferences

Production Technology and Capital Accumulation. We adopt the capital accumulation specification of Cox, Ingersoll, and Ross (1985) and Jones and Manuelli (2005). The firm's capital stock K evolves as follows:

$$dK_t = (I_t - \delta_K K_t)dt + \sigma_K K_t \left(\sqrt{1-\rho^2} d\mathcal{Z}_{1,t} + \rho d\mathcal{Z}_{2,t} \right), \tag{1}$$

where I is the firm's rate of gross investment, $\delta_K \geq 0$ is the expected rate of depreciation, and σ_K is the volatility of the capital depreciation shock. Without loss of generality, we decompose risk into two orthogonal components: an idiosyncratic shock represented by the standard Brownian motion $\mathcal{Z}_1$ and a systematic shock represented by the standard Brownian motion $\mathcal{Z}_2$. The parameter ρ measures the correlation between the firm's capital risk and systematic risk, so that the firm's systematic volatility is equal to $\rho\sigma_K$ and its idiosyncratic volatility is given by

$$\epsilon_K = \sigma_K \sqrt{1-\rho^2}\,. \tag{2}$$

Production requires combining the entrepreneur's inalienable human capital with the firm's capital stock K_t, which together yield revenue AK_t. Without the entrepreneur's human capital the capital stock K_t does not generate any cash flows. Investment involves both a direct purchase and an adjustment cost, so that the firm's *free cash flow* (after capital expenditures) is given by:

$$Y_t = AK_t - I_t - G(I_t, K_t), \tag{3}$$

where the price of the investment good is normalized to one and $G(I, K)$ is the standard adjustment cost function in the q-theory of investment. Note that Y_t can take negative values, which simply means that investors are then financing part of the investment costs and the entrepreneur's consumption from sources other than contemporaneous revenue AK_t. We simplify the model by assuming that the firm's adjustment cost $G(I, K)$ is homogeneous of degree one in I and K (a common assumption in the q-theory of investment), so that $G(I, K)$ takes the following separable form:

$$G(I, K) = g(i)K \tag{4}$$

where $i = I/K$ denotes the firm's investment-capital ratio and $g(i)$ is increasing and convex in i. As Hayashi (1982) has shown, given this homogeneity property Tobin's average and marginal q are equal under perfect capital markets.[4] However, under limited commitment an endogenous wedge between Tobin's average and marginal q will emerge in our model.[5]

Preferences. The infinitely-lived entrepreneur has a standard concave utility function over positive consumption flows $\{C_t; t \geq 0\}$ given by:

$$J_t = \mathbb{E}_t \left[\int_t^{\infty} \zeta e^{-\zeta(v-t)} U(C_v) dv \right] , \tag{5}$$

where $\zeta > 0$ is the entrepreneur's subjective discount rate, $\mathbb{E}_t [\cdot]$ is the time-t conditional expectation, and $U(C)$ takes the standard constant-relative-risk-averse utility (CRRA) form:

$$U(C) = \frac{C^{1-\gamma}}{1-\gamma} , \tag{6}$$

with $\gamma > 0$ denoting the coefficient of relative risk aversion. As is standard we normalize the flow payoff with ζ in (5), so that the utility flow is given by $\zeta U(C)$.[6]

2.2 Complete Financial Markets

We assume that financial markets are perfectly competitive and complete. Market completeness is obtained through dynamic spanning with three long-lived assets as in the Black-Merton-Scholes framework (Duffie and Huang, 1985): Given that the firm's production is subject to two shocks, $\mathcal{Z}_1$ and $\mathcal{Z}_2$, financial markets are dynamically complete if the following three non-redundant financial assets can be dynamically and frictionlessly traded:

[4]Lucas and Prescott (1971) analyze dynamic investment decisions with convex adjustment costs, though they do not explicitly link their results to marginal or average q. Abel and Eberly (1994) extend Hayashi (1982) to a stochastic environment and a more general specification of adjustment costs.

[5]An endogenous wedge between Tobin's average and marginal q also arises in cash-based optimal financing and investment models such as Bolton, Chen, and Wang (2011) and optimal contracting models such as DeMarzo, Fishman, He, and Wang (2012).

[6]For example, see Sannikov (2008). In terms of preferences, we can generalize our model to allow for a coefficient of relative risk aversion that is different from the inverse of the elasticity of intertemporal substitution, *à* la Epstein and Zin (1989). Indeed, as Epstein-Zin preferences are homothetic, allowing for such preferences in our model will not increase the dimensionality of the optimization problem. Details are available upon request.

a. a risk-free asset that pays interest at a constant risk-free rate r;

b. a risky asset that is perfectly correlated with the idiosyncratic shock $\mathcal{Z}_1$. The incremental return dR_t on this risky asset over the time interval dt is

$$dR_{1,t} = rdt + \epsilon_K d\mathcal{Z}_{1,t} \,. \tag{7}$$

Note that the expected return on this risky asset equals the risk-free rate r. As it is only subject to an idiosyncratic shock it earns no risk premium. We let the volatility of this risky asset to be ϵ_K without loss of generality;

c. a risky asset that is perfectly correlated with the systematic shock $\mathcal{Z}_2$. The incremental return dR_t of this asset over the time interval dt is

$$dR_{2,t} = \mu_R dt + \sigma_R d\mathcal{Z}_{2,t} \,, \tag{8}$$

where μ_R and σ_R are constant mean and volatility parameters. As this risky asset is only subject to the systematic shock, we refer to it as the *market portfolio*.

Dynamic and frictionless trading with these three securities implies that the following unique stochastic discount factor (SDF) exists:

$$\frac{d\mathbb{M}_t}{\mathbb{M}_t} = -rdt - \eta d\mathcal{Z}_{2,t} \,, \tag{9}$$

where $\mathbb{M}_0 = 1$ and η is the Sharpe ratio of the market portfolio given by:

$$\eta = \frac{\mu_R - r}{\sigma_R} \,.$$

Note that the SDF $\mathbb{M}$ follows a geometric Brownian motion with the drift equal to the negative risk-free rate, as required under no-arbitrage. By definition the SDF is only exposed to the systematic shock $\mathcal{Z}_2$. Fully diversified investors will only demand a risk premium for their exposures to systematic risk. The entrepreneur, however, is not fully diversified given her exposure to the risky venture.

2.3 Limited Commitment and Endogenous Borrowing Capacity

The entrepreneur has at all time an outside option, which is to abscond with a fraction $\alpha \in (0,1)$ of the firm's capital stock and start afresh with zero liabilities. Another interpretation of this outside option is that the entrepreneur's human capital can be deployed elsewhere albeit less efficiently. The loss in efficiency is then captured by the parameter α, which measures the maximum relative size of the next best alternative open to the entrepreneur. Under this interpretation there is no misappropriation involved and the entrepreneur's outside option simply reflects the market value of her accumulated human capital.[7]

Limited commitment and inalienability-of-human-capital constraints have been widely invoked to explain endogenous corporate financial constraints and debt capacity (e.g., Hart and Moore, 1994, 1998, Albuquerque and Hopenhyan, 2004, Kyiotaki and Moore, 1997), but they have not been linked to corporate liquidity and risk management as we do here. The reason is that all corporate finance models with limited commitment assume that the investor and entrepreneur have risk neutral preferences and that the inalienability-of-human-capital constraint is constant. Rampini and Viswanathan (2010) introduce a risk management motive into a model with risk neutral preferences by inserting stochastic collateral constraints into their model. We generalize Rampini and Viswanathan (2010) by introducing risk aversion and consumption smoothing.

Before characterizing the solution under limited commitment, we derive the first-best optimum under full commitment.

3 First Best

Under dynamically complete markets the entrepreneur's savings, portfolio allocation, and consumption problem to maximize her utility can be separated from the corporate investment problem to maximize firm value (see Duffie, 2001). There are two ways of formulating the first-best optimization model: either as a static maximization problem with a single intertemporal budget constraint, or as a dynamic programming problem with continuous, dynamic, portfolio rebalancing. The latter construction provides a more direct link to the optimization problem under limited commitment, since it is the limit formulation when the

[7] There are other, perhaps more realistic, ways of modeling the outside option, but this is a particularly simple and parsimonious formulation.

entrepreneur's commitment friction vanishes. Accordingly, we shall rely on the dynamic programming method to characterize the first-best solution, which can be framed without loss of generality as a dynamic liquidity and risk management problem for the firm.

3.1 Liquidity and Risk Management

The entrepreneur's wealth is comprised of her *liquid* financial holdings and her ownership of the *illiquid* productive capital K. Let $\{S_t : t \geq 0\}$ denote the entrepreneur's liquid wealth process. The entrepreneur continuously allocates $\{S_t : t \geq 0\}$ to any admissible positions $\{\Phi_{1,t}, \Phi_{2,t} : t \geq 0\}$ in the two risky financial assets, whose returns are given by (7) and (8) respectively, and the residual amount $(S_t - \Phi_{1,t} - \Phi_{2,t})$ to the risk-free asset. Her liquid wealth then stochastically evolves as follows:

$$dS_t = (rS_t + Y_t - C_t)dt + \Phi_{1,t}\epsilon_K d\mathcal{Z}_{1,t} + \Phi_{2,t}[(\mu_R - r)dt + \sigma_R d\mathcal{Z}_{2,t}] \,. \tag{10}$$

The first term in (10), $rS_t+Y_t-C_t$, is simply the sum of the firm's interest income rS_t and net operating cash flows, Y_t-C_t, the second term, $\Phi_{1,t}\epsilon_K d\mathcal{Z}_{1,t}$, is the exposure to the idiosyncratic shock $\mathcal{Z}_1$, which earns no risk premium, and the third term, $\Phi_{2,t}[(\mu_R - r)dt + \sigma_R d\mathcal{Z}_{2,t}]$, is the excess return from the investment in the market portfolio.

In the absence of any risk exposure $rS_t + Y_t - C_t$ is simply the rate at which the entrepreneur saves when $S_t \geq 0$ or dissaves (by drawing on a line of credit (LOC) at the risk-free rate r, when $S_t < 0$). In general, saving all liquid wealth S at the risk-free rate is sub-optimal. By dynamically engaging in risk taking and risk management, through the risk exposures Φ_1 and Φ_2, the entrepreneur will do better, as we show next.

The Entrepreneur's Optimization Problem. The entrepreneur dynamically chooses consumption C, real investment I, idiosyncratic risk hedging demand Φ_1, and the market portfolio exposure Φ_2 to maximize her utility given in (5)-(6) subject to the liquidity accumulation dynamics (10) and the transversality condition $\lim_{s\to\infty} \mathbb{E}_t \left[e^{-\zeta(s-t)}|J_s|\right] = 0$, where J_s is the entrepreneur's time-s value function.

The entrepreneur's value function $J(K, S)$ depends on the firm's capital stock K and her liquid savings S. By the standard dynamic programming argument, $J(K, S)$ satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{aligned}\zeta J(K,S) = \max_{C,I,\Phi_1,\Phi_2} \quad & \zeta U(C) + (rS + \Phi_2(\mu_R - r) + AK - I - G(I,K) - C)J_S \\ & + (I - \delta_K K)J_K + \frac{\sigma_K^2 K^2}{2} J_{KK} + \left(\epsilon_K^2 \Phi_1 + \rho\sigma_K\sigma_R\Phi_2\right) K J_{KS} \\ & + \frac{(\epsilon_K\Phi_1)^2 + (\sigma_R\Phi_2)^2}{2} J_{SS}\,. \end{aligned} \tag{11}$$

The first term on the right side of (11) represents the entrepreneur's normalized flow utility of consumption; the second term (in J_S) represents the marginal value of incremental liquid savings S; the third term (in J_K) represents the marginal value of net investment $(I - \delta_K K)$; and the last three terms (in respectively J_{KK}, J_{KS}, and J_{SS}) capture the valuation of risk.

Given the concavity of the utility function and the convexity of the investment adjustment cost function, the optimal consumption $C(K,S)$ and investment $I(K,S)$ rules are characterized by the following first-order conditions (FOCs):

$$\zeta U'(C) = J_S, \tag{12}$$

and

$$1 + G(I,K) = \frac{J_K(K,S)}{J_S(K,S)}\,. \tag{13}$$

Equation (12) is the standard FOC for consumption, equating the marginal utility of consumption with the marginal value of savings J_S, and equation (13) states that the marginal cost of investing is equal to the entrepreneur's marginal value of investing, measured as the ratio of the entrepreneur's marginal value of illiquid capital J_K and the marginal value of liquid savings J_S.[8]

Differentiating the HJB equation (11) with respect to Φ_1 and Φ_2 we similarly obtain the following FOCs characterizing the firm's optimal risk-management policy:

$$\Phi_1 = -\frac{J_{KS}}{K J_{SS}}\,, \tag{14}$$

[8]Equation (13) generalizes the optimality condition for a risk-neutral firm that the marginal cost of investing is equal to the ratio of marginal q and the marginal value of cash in Bolton, Chen, and Wang (2011).

and

$$\Phi_2 = -\frac{\mu_R - r}{\sigma_R^2}\frac{J_S}{KJ_{SS}} - \frac{\rho\sigma_K}{\sigma_R}\frac{J_{KS}}{KJ_{SS}}. \tag{15}$$

We refer to Φ_1 as the *idiosyncratic-risk hedging demand*: the only reason for holding this risky asset is for hedging purposes against the firm's idiosyncratic risks. The entrepreneur's market portfolio holding Φ_2 is given by the classical exposure to the market excess return (the first term) and a hedge against the firm's systematic-risk exposure (the second term). Equations (11), (12), (13), (14) and (15) jointly characterize the solution to the entrepreneur's optimization problem.

Guided by the observation that the value function for the standard Merton portfolio-choice problem (without illiquid assets) inherits the CRRA form of the agent's utility function $U(\,\cdot\,)$, we conjecture and verify that the entrepreneur's value function in the first-best problem, denoted by $J^{FB}(K,S)$, takes the same form as in Merton's problem:

$$J^{FB}(K,S) = \frac{(bM^{FB}(K,S))^{1-\gamma}}{1-\gamma}\,, \tag{16}$$

where $M^{FB}(K,S)$ is the market value of the entrepreneur's wealth (to be derived) and b is the following constant:[9]

$$b = \zeta\left[\frac{1}{\gamma} - \frac{1}{\zeta}\left(\frac{1-\gamma}{\gamma}\right)\left(r + \frac{\eta^2}{2\gamma}\right)\right]^{\frac{\gamma}{\gamma-1}}\,. \tag{17}$$

3.2 The First-Best Solution

Given the model's homogeneity property in K, we can reduce the two-dimensional value function $J(K,S)$ in (11) to one dimension by dividing both sides of the HJB equation (11) by K. Our notation reflects this operation by turning all upper-case variables (investment I, liquidity S, capital adjustment cost G, consumption C, asset allocations Φ_1 and Φ_2) into lower-case variables representing each of the upper-case variable as a fraction of the capital stock K (for example, $i_t = I_t/K_t$ and $s_t = S_t/K_t$). The firm's optimization problem can then be expressed as the maximization of the entrepreneur's value per unit of capital. In other words, the homogeneity property implies that a unit of capital is worth the perpetuity

[9] In the special case when $\gamma = 1$ we have $b = \zeta \exp\left[\frac{1}{\zeta}\left(r + \frac{\eta^2}{2\gamma} - \zeta\right)\right]$.

value of its expected free cash flow given the optimal net expected first-best growth rate of capital $\left(i^{FB}-\delta\right)$. The value of capital, $Q_t^{FB}=q^{FB}K_t$, then follows a GBM process given by:

$$dQ_t^{FB}=Q_t^{FB}\left[\left(i^{FB}-\delta_K\right)dt+\left(\epsilon_K d\mathcal{Z}_{1,t}+\rho\sigma_K d\mathcal{Z}_{2,t}\right)\right] ,$$

with the drift $\left(i^{FB}-\delta_K\right)$, idiosyncratic volatility ϵ_K, and systematic volatility $\rho\sigma_K$, identical to those for the dynamics for $\{K_t : t\geq 0\}$.

Corporate investment, the value of capital Q^{FB}, and asset pricing. The following proposition characterizes the first-best solutions for corporate investment, the value of capital, and asset prices.

Proposition 1 *The value of capital, $Q^{FB}(K)$, is proportional to K, $Q^{FB}(K)=q^{FB}K$, where q^{FB} is Tobin's q solving:*

$$q^{FB}=\max_i \ \frac{A-i-g(i)}{r+\delta-i} , \tag{18}$$

and the maximand for (18), denoted by i^{FB}, is the first-best investment-capital ratio. The risk-adjusted capital depreciation rate, δ, is given by

$$\delta=\delta_K+\rho\eta\sigma_K . \tag{19}$$

The expected return μ^{FB} for the value of capital satisfies the CAPM equation:

$$\mu^{FB}=r+\rho\eta\sigma_K=r+\beta^{FB}\left(\mu_R-r\right) , \tag{20}$$

where

$$\beta^{FB}=\frac{\rho\sigma_K}{\sigma_R} . \tag{21}$$

This proposition generalizes the well known Hayashi conditions linking investment to Tobin's average (and marginal) q, by extending his framework to situations where the firm's operations are subject to both idiosyncratic and systematic risk, where systematic risk commands a risk premium. As in the q-theory of investment, capital adjustment costs create a wedge between the value of installed capital and newly purchased capital, so that $q^{FB}\neq 1$ in general. Optimal (scaled) investment i is given by the solution to the FOC for investment:

$$q^{FB} = 1 + g'(i^{FB}), \tag{22}$$

which equates marginal q to the marginal cost of investing, $1 + g'(i)$ at the optimum investment level i^{FB}. Jointly solving (18) and (22) yields the values for q^{FB} and i^{FB}.

Let μ^{FB} denote the expected return for the value of capital, Q_t^{FB}. Using Ito's formula, we may then express the expected return μ^{FB} as:

$$\mu^{FB} = \frac{A - i^{FB} - g(i^{FB})}{q^{FB}} + \left(i^{FB} - \delta_K\right) = r + \delta - i^{FB} + \left(i^{FB} - \delta_K\right) = r + \beta\left(\mu_R - r\right), \tag{23}$$

where the first equality gives the sum of the dividend yield and expected capital gains, the second equality uses (18), and the third uses (19) and $\delta - \delta_K = \rho\eta\sigma_K = \beta\left(\mu_R - r\right)$. Note that the CAPM holds for the value of capital Q_t^{FB}, with the risk-adjusted capital depreciation rate δ equalling the expected depreciation rate δ_K, augmented by the risk premium $\rho\eta\sigma_K$ as given in (19).

Scaled consumption c, scaled idiosyncratic risk hedging demand $\phi_1 = \Phi_1/K$ and scaled market portfolio demand $\phi_2 = \Phi_2/K$. The next proposition characterizes the first-best solutions for consumption and asset allocations.

Proposition 2 *The entrepreneur's optimal consumption policy is given by*

$$c_t = \chi\left(s_t + q^{FB}\right), \tag{24}$$

where χ is the marginal propensity to consume (MPC) given by

$$\chi = r + \frac{\eta^2}{2\gamma} + \gamma^{-1}\left(\zeta - r - \frac{\eta^2}{2\gamma}\right). \tag{25}$$

The first-best idiosyncratic (scaled) risk hedge ϕ_1 and the (scaled) market portfolio allocation

ϕ_2 *are respectively given by:*

$$\phi_1^{FB}(s) = -q^{FB}, \tag{26}$$
$$\phi_2^{FB}(s) = -\frac{\rho\sigma_K}{\sigma_R}q^{FB} + \frac{\mu_R - r}{\gamma\sigma_R^2}\left(s + q^{FB}\right). \tag{27}$$

In words, under complete markets and full commitment, the entrepreneur's total net worth, denoted by M_t^{FB}, is given by the sum of her liquid wealth S_t and the market value of capital:

$$M_t^{FB} = Q^{FB}(K_t) + S_t = q^{FB}K_t + S_t. \tag{28}$$

Again using Ito's formula, we can express the dynamics of $\{M_t : t \geq 0\}$ as:

$$dM_t^{FB} = M_t^{FB}\left[\left(r - \chi + \frac{\eta^2}{\gamma}\right)dt + \frac{\eta}{\gamma}d\mathcal{Z}_{2,t}\right]. \tag{29}$$

That is, total net worth M is a GBM process with drift $(r-\chi)+\eta^2/\gamma$ and volatility η/γ for the systematic shock $\mathcal{Z}_2$.

Two important observations follow from this proposition: First, note that M has zero net exposure to the idiosyncratic shock $\mathcal{Z}_1$. This is simply due to the fact that the entrepreneur is averse to any net exposure to risk which does not generate any risk premium. How does the entrepreneur achieve this? One way is for her to take an offsetting short idiosyncratic risk exposure in the financial markets by setting $\phi_1 = -q^{FB}$, so that her exposure to the idiosyncratic risk $\mathcal{Z}_1$ through her long position in the business venture is exactly offset by an equivalent short position in the financial asset that is exposed to the idiosyncratic risk $\mathcal{Z}_1$.

Second, under perfect and complete financial markets, the entrepreneur essentially capitalizes the entire present value of her capital stock K at a unit price of q^{FB}. She then constructs a Merton-type consumption and portfolio allocation that results in total wealth M_t^{FB}. That is why the marginal propensity to consume (MPC) and the dynamics for total wealth M are the same as respectively the MPC and the dynamics for liquid wealth in Merton (1971).

In summary, our first-best benchmark has the following important characteristics: 1) An optimal consumption rule that is linear in *total* wealth M; 2) An optimal liquidity and risk management policy such that the entrepreneur's net exposure to idiosyncratic risk is entirely eliminated as seen from (29), and her net exposure to systematic risk is η/γ as in Merton (1971); 3) A constant investment-capital ratio and a constant Tobin's q as in Hayashi (1982), but in a more general setting with a systematic risk premium; 4) An endogenous value for the capital process Q^{FB} that follows a GBM process as in the Black-Scholes economy.

4 Solution under Limited Commitment

We can also characterize the entrepreneur's optimization problem as a liquidity and risk management problem under limited commitment. However, the entrepreneur's inability to fully commit will constrain her ability to dynamically manage liquidity and risk over time and across states of nature, in particular by limiting her credit capacity. The entrepreneur responds to the constraints on her ability to obtain insurance through financial markets by engaging in self-insurance through liquidity management, which is critical to her due to her desire to smooth consumption.

Limited Commitment and Endogenous Credit Capacity. The entrepreneur's limited ability to commit to a given long-term contract (even if it is in her interest *ex ante*) arises from the fact that at any moment in time she has an option to abscond with a fraction $\alpha \in (0, 1)$ of the firm's capital stock and start afresh with zero liabilities.[10]

We denote by $\underline{S_t}$ the time-t endogenous lower boundary for the firm's liability at which the entrepreneur is indifferent between continuing with the firm and starting afresh with a smaller but liability-free firm. Given that it is never efficient for the entrepreneur to quit on the equilibrium path, we expect that the entrepreneur's value function satisfies the following condition:

$$J(K_t, S_t) \geq J(K_t, \underline{S_t}) = J(\alpha K_t, 0), \tag{30}$$

where the equality defines the endogenous lower boundary $\underline{S_t}$ implied by the indifference between *a*) the *status quo* value $J(K, \underline{S})$ of remaining in the contractual long-term relation and *b*) her outside option $J(\alpha K, 0)$.

[10]In practice entrepreneurs can sometimes partially commit themselves and lower their outside options by signing non-compete clauses. This possibility can be captured in our model by lowering the parameter α, which relaxes the entrepreneur's inalienability-of-human-capital constraints.

We show in Appendix XXX that (30) translates into the following intuitive constraint:

$$S_t \geq \underline{S_t} = \underline{S}(K_t)\,, \tag{31}$$

where $\underline{S}(K_t)$ is a function that defines the firm's credit capacity for any given capital stock K_t. When $S_t < 0$, the entrepreneur is in debt and draws down on a line of credit (LOC) granted by a bank to the entrepreneur. The entrepreneur can borrow on this LOC at the risk-free rate r up to the endogenously determined *credit capacity* $\underline{S}(K_t)$. This borrowing limit ensures that the entrepreneur does not walk away from the firm in an attempt to evade her debt obligations. As long as the entrepreneur works at the firm, the firm's credit line is risk free and hence can be financed at the risk-free rate.

The entrepreneur's optimization problem under limited commitment. Other than facing the additional endogenous credit constraint (31) induced by the limited commitment constraint (30), the entrepreneur faces essentially the same tradeoffs in the interior region as in the first-best problem of Section 3.1. In particular, the FOCs for C, I, Φ_1, and Φ_2 are given by (12), (13), (14), and (15), respectively.

Certainty-equivalent wealth. Again guided by the observation that the entrepreneur's value function under the first-best case inherits the CRRA form of her utility function $U(\,\cdot\,)$, we conjecture and verify that the entrepreneur's value function under limited commitment $J(K,S)$ also takes the same form:

$$J(K,S) = \frac{(bM(K,S))^{1-\gamma}}{1-\gamma}\,, \tag{32}$$

where $M(K,S)$ is the certainty-equivalent wealth that the entrepreneur would demand in exchange of permanently giving up her liquid wealth S and her risky venture of size K with her human capital.

Reducing the problem's dimensionality Because our model has the homogeneity property in capital stock K and liquidity S, we can express the entrepreneur's certainty equivalent wealth function $M(K,S)$, the consumption rule $C(K,S)$, the investment policy $I(K,S)$ as follows:

$$M(K,S) = m(s) \cdot K \ \ , \ C(K,S) = c(s) \cdot K, \quad I(K,S) = i(s) \cdot K \, . \tag{33}$$

The entrepreneur's two-dimensional optimization problem with state variables (K, S) can then be simplified to a one-dimensional problem, where the effective state variable is the firm's liquidity-capital ratio, $s = S/K$. The endogenous debt capacity constraint (31) then reduces to

$$s \geq \underline{s} \, , \tag{34}$$

where $|\underline{s}|$ is the maximal amount that the entrepreneur can borrow per unit of capital.

The dynamics of scaled liquidity $s = S/K$. Using Ito's formula, and given $c(s)$, $i(s)$, $\phi_1(s)$, and $\phi_2(s)$, we show that the scaled liquidity ratio s follows the process:

$$ds_t = \mu^s(s_t)dt + \sigma_1^s(s_t)d\mathcal{Z}_{1,t} + \sigma_2^s(s_t)d\mathcal{Z}_{2,t} \, , \ \ s_t \geq \underline{s} \ ,$$

where the drift function $\mu^s(\,\cdot\,)$ is given by:

$$\begin{aligned} \mu^s(s) \ \ &= \ \ (A - i(s) - g(i(s)) + \phi_2(s)(\mu_R - r) - c(s)) \\ &\quad\ \ + (r + \delta_K - i(s))s - (\epsilon_K \sigma_1^s(s) + \rho \sigma_K \sigma_2^s(s)), \end{aligned} \tag{35}$$

and the idiosyncratic volatility $\sigma_1^s(\,\cdot\,)$ and systematic volatility $\sigma_2^s(\,\cdot\,)$ are given by

$$\sigma_1^s(s) \ \ = \ \ (\phi_1(s) - s)\epsilon_K \, , \tag{36}$$

$$\sigma_2^s(s) \ \ = \ \ \phi_2(s)\sigma_R - s\rho\sigma_K \, . \tag{37}$$

Endogenous credit limit and credit capacity. Having described the dynamics for s in the interior region $s_t \geq \underline{s}$, we now turn to the *endogenous* boundary conditions that determine $\underline{s}$. An important first observation is that the endogenous credit constraint (34) generally does not bind. The reason is, as in the buffer-stock savings models of Deaton (1991) and Carroll (1992) for household finance, that the *risk-averse* entrepreneur manages her liquid holdings s with the objective of smoothing her consumption and thereby maximizing her value function. Setting $s_t = \underline{s}$ is costly in terms of consumption smoothing for a risk-averse agent. This is why risk aversion plays an important role in liquidity management.

Second, while the credit constraint given in (34) rarely binds, it has to be satisfied with probability one. Only then can we ensure that the credit limit is never exceeded and that the entrepreneur does not default. Given that s follows a diffusion process (and therefore that the path for s is continuous), we must require that the following conditions on the drift and volatility functions hold:

$$\begin{aligned} \lim_{s \to \underline{s}} \mu^s(s) &\geq 0, \\ \lim_{s \to \underline{s}} \sigma_1^s(s) &= 0, \text{ or } \lim_{s \to \underline{s}} \phi_1(s) = \underline{s}, \text{ and} \\ \lim_{s \to \underline{s}} \sigma_2^s(s) &= 0, \text{ or } \lim_{s \to \underline{s}} \phi_2(s) = \underline{s}\frac{\rho\sigma_K}{\sigma_R}. \end{aligned}$$

That is, at the lower boundary $\underline{s}$, where $m(\underline{s}) = \alpha m(0)$, the firm's liquidity should be weakly increasing and the entrepreneur should face vanishing risk with respect to changes in s.

The following proposition summarizes the limited commitment solution for $m(s)$. The derivations and proof are provided in the appendix.

Proposition 3 *In the interior region $s > \underline{s}$, the scaled certainty-equivalent wealth $m(s)$ satisfies the following ODE:*

$$\begin{aligned} 0 \;=\;& \frac{m(s)}{1-\gamma}\left[\gamma\chi m'(s)^{\frac{\gamma-1}{\gamma}} - \zeta\right] + \left[rs + A - i(s) - g(i(s))\right] m'(s) + (i(s)-\delta)(m(s) - sm'(s)) \\ & - \left(\frac{\gamma\sigma_K^2}{2} - \rho\eta\sigma_K\right)\frac{m(s)^2 m''(s)}{m(s)m''(s) - \gamma m'(s)^2} + \frac{\eta^2 m'(s)^2 m(s)}{2(\gamma m'(s)^2 - m(s)m''(s))}\,, \end{aligned} \tag{38}$$

subject to the following boundary conditions:

$$\lim_{s\to\infty} m(s) = q^{FB} + s\,, \tag{39}$$

$$m(\underline{s}) = \alpha m(0)\,, \tag{40}$$

$$\lim_{s \to \underline{s}} \sigma_1^s(s) = 0\,, \qquad \lim_{s \to \underline{s}} \sigma_2^s(s) = 0\,, \quad \textit{and} \quad \lim_{s \to \underline{s}} \mu^s(s) \geq 0\,. \tag{41}$$

The consumption and investment rules are:

$$\zeta U'(c) = b^{1-\gamma} m(s)^{-\gamma} m'(s), \tag{42}$$

$$1 + g'(i) = \frac{m(s)}{m'(s)} - s, \tag{43}$$

and the hedging strategies are given by:

$$\phi_1(s) = \frac{sm''(s)m(s) + \gamma m'(s)(m(s) - sm'(s))}{m(s)m''(s) - \gamma m'(s)^2}, \tag{44}$$

$$\phi_2(s) = \frac{\rho\sigma_K}{\sigma_R}\frac{sm''(s)m(s) + \gamma m'(s)(m(s) - sm'(s))}{m(s)m''(s) - \gamma m'(s)^2} - \frac{\mu_R - r}{\sigma_R^2}\frac{m'(s)m(s)}{m(s)m''(s) - \gamma m'(s)^2} \tag{45}$$

5 The Equivalent Optimal Contracting Problem

Before continuing with the numerical solution of the entrepreneur's problem under limited commitment it is helpful to underline how the entrepreneur's optimal dynamic liquidity and risk-management problem can be formulated equivalently as an optimal contracting problem between an optimally diversified investor and a risk-averse entrepreneur subject to an inalienability-of-human-capital constraint.

Thus, consider the optimal long-term contracting problem between an infinitely-lived fully diversified investor (*the principal*) and a financially constrained, infinitely-lived, risk-averse entrepreneur (*the agent*). The investor is a deep-pocketed individual who provides both the initial productive capital K_0 and working capital over time as needed. Suppose that the output process Y_t is publicly observable and verifiable. In addition, suppose that the entrepreneur cannot privately save.[11] The contracting game begins at time 0 with the investor making a take-it-or-leave-it long-term contract offer to the entrepreneur. The contract specifies an investment process $\{I_t; t \geq 0\}$ and a consumption allocation process $\{C_t; t \geq 0\}$ to the entrepreneur, both of which depend on the entire history of capital stock $\{K_t; t \geq 0\}$.

At the moment of contracting at time 0 the entrepreneur has a reservation utility V_0^*, so

[11] This is a standard assumption in the literature on dynamic moral hazard (see Bolton and Dewatripont, 2005 chapter 10).

that the optimal contract must satisfy the participation constraint:

$$V_0 \geq V_0^*, \tag{46}$$

where V_0 denotes the value of the contract to the entrepreneur at time 0. In addition, the entrepreneur's human capital is inalienable and she can at any time leave the firm. Let the outside payoff the entrepreneur obtains be denoted by $\widehat{V}(K_t)$. Then the entrepreneur's inalienability-of-human-capital constraint at each point in time t is given by:

$$V_t \geq \widehat{V}(K_t)\,, \quad t \geq 0. \tag{47}$$

The investor's problem at time 0 is to choose a dynamic investment I_t and consumption C_t policy to maximize the investor's time-0 discounted value of cash flows,

$$F_0 = \max_{I,C} \quad \mathbb{E}_0\left[\int_0^\infty \frac{\mathbb{M}_t}{\mathbb{M}_0}(Y_t - C_t)dt\right], \tag{48}$$

subject to the capital accumulation process (1), the production function (3), the entrepreneur's inalienability-of-human-capital constraint (47) at all t, and the entrepreneur's time-0 participation constraint (46).[12]

The participation constraint (46) is always binding under the optimal contract. Otherwise, the investor can always increase his payoff by lowering the agent's consumption and still satisfy all other constraints. However, the entrepreneur's inalienability-of-human-capital constraints (47) will often not bind as the investor dynamically trades off the benefits of providing the entrepreneur with risk-sharing/consumption smoothing and the benefits of extracting higher contingent payments from the firm.

As is well known (see e.g. DeMarzo and Sannikov, 2006), an important simplification of the contracting problem is to summarize the entire history of the contract in the entrepreneur's promised utility V_t conditional on the history up to time t. Under the optimal contract the dynamics of the agent's promised utility can then be written in the recursive form below. The sum of the agent's utility flow $\zeta U(C_t)dt$ and change in promised utility dV_t

[12] Additionally, we require that the investor's value at time 0, F_0, is (weakly) greater than the investors' second-best option F_0^*.

has the expected value $\zeta V_t dt$, or:

$$\mathbb{E}_t\left[\zeta U(C_t)dt + dV_t\right] = \zeta V_t dt\,. \tag{49}$$

We can write the stochastic differential equation (SDE) for dV implied by (49) as the sum of: *i*) the expected change (i.e., drift) term $\mathbb{E}_t\left[dV_t\right]$; *ii*) a martingale term driven by the Brownian motion $\mathcal{Z}_1$; and *iii*) a martingale term driven by the Brownian motion $\mathcal{Z}_2$. Accordingly, we may write the dynamics of the entrepreneur's promised utility process V as follows:

$$dV_t = \zeta(V_t - U(C_t))dt + x_{1,t}V_t d\mathcal{Z}_{1,t} + x_{2,t}V_t d\mathcal{Z}_{2,t}\,, \tag{50}$$

where $\{x_{1,t}; t \geq 0\}$ and $\{x_{2,t}; t \geq 0\}$ controls the diffusion idiosyncratic and systematic volatility of the entrepreneur's promised utility V, respectively.

Finally, we can write the investors' objective as a value function $F(K,V)$ with two state variables: *i*) the entrepreneur's promised utility V; *ii*) the venture's capital stock K. The optimal contract then specifies investment I, consumption C, idiosyncratic risk exposure x_1 and systematic risk exposure x_2 to solve the following optimization problem:

$$F(K_t, V_t) = \max_{C,\,I,\,x_1,\,x_2} \quad \mathbb{E}_t\left[\int_t^\infty \frac{\mathbb{M}_v}{\mathbb{M}_t}(Y_v - C_v)dv\right]\,, \tag{51}$$

subject to the entrepreneurs' inalienability-of-human-capital constraints (47) for all time t, and the entrepreneur's initial participation constraint (46). Applying Ito's Lemma to $F(K_t, V_t)$ it is straightforward to derive the following HJB equation for the investor's value function $F(K,V)$:

$$\begin{aligned} rF(K,V) = \max_{C,\,I,\,x_1,\,x_2} \quad & \Big\{(Y - C) + (I - \delta K)F_K + \sigma_K^2 K^2 F_{KK}/2 \\ & + [\zeta(V - U(C)) - x_2\eta V]F_V + \frac{(x_1^2 + x_2^2)V^2 F_{VV}}{2} + (x_1\epsilon_K + x_2\rho\sigma_K)KVF_{VK}\Big\} \end{aligned} \tag{52}$$

Again mapping the promised utility V into a promised certainty-equivalent wealth W and reducing the investor's problem into a one-dimensional problem with state variable $w =$

W/K, the investor's value function $F(K,V)$ can be rewritten as:

$$F(K,V) \equiv F(K,U(bW)) = P(K,W) = p(w) \cdot K, \tag{53}$$

where $p(w)$ is the solution to the ODE given below in (54).

We summarize the solution to the optimal contracting problem between the investor and entrepreneur in the proposition below. A detailed derivation and proof of the proposition can be found in the appendix.

Proposition 4 *In the region $w > \underline{w}$ the investors' scaled value $p(w)$ solves:*

$$\begin{aligned} rp(w) &= A - i(w) - g(i(w)) + \frac{\chi\gamma}{1-\gamma}\left(-p'(w)\right)^{1/\gamma} w + (i(w)-\delta)(p(w) - wp'(w)) \\ &\quad + \frac{\zeta}{1-\gamma} wp'(w) + \left(\frac{\gamma\sigma_K^2}{2} - \rho\eta\sigma_K\right)\frac{w^2 p'(w)p''(w)}{wp''(w)+\gamma p'(w)} - \frac{\eta^2}{2}\frac{wp'^2}{wp''(w)+\gamma p'(w)}, \end{aligned} \tag{54}$$

subject to the following boundary conditions:

$$\lim_{w\to\infty} p(w) = q^{FB} - w\,, \tag{55}$$

$$p(\overline{w}) = 0\,, \tag{56}$$

$$\lim_{w\to\underline{w}} \sigma_1^w(w) = 0\,, \quad \lim_{w\to\underline{w}} \sigma_2^w(w) = 0 \quad \text{and} \quad \lim_{w\to\underline{w}} \mu^w(w) \geq 0\,. \tag{57}$$

The optimal investment-capital ratio $i = I/K$, the entrepreneur's consumption-capital ratio $c = C/K$ and risk exposure x_1 and x_2 respectively satisfy

$$g'(i(w)) = p(w) - wp'(w) - 1\,, \tag{58}$$

$$c(w) = \chi\left(-p'(w)\right)^{1/\gamma} w\,, \tag{59}$$

$$x_1(w) = \frac{(1-\gamma)\epsilon_K w p''(w)}{wp''(w)+\gamma p'(w)}\,, \tag{60}$$

$$x_2(w) = \frac{(1-\gamma)(\rho\sigma_K w p''(w) + \eta p'(w))}{wp''(w)+\gamma p'(w)}\,. \tag{61}$$

Propositions (4) and (3) reveal how the two problems are linked. The optimization problem for entrepreneur is equivalent to the optimal contracting problem for the investor in (51) if and only if the borrowing limits, $\underline{S}(K)$, are such that for all K:

$$\underline{S}(K) = -P(K, \underline{W}), \tag{62}$$

where $P(K, \underline{W})$ is the investors' value when the entrepreneur's inalienability-of-human-capital constraint binds, that is, when $W = \underline{W}$. We provide a proof of the equivalence between the two problems in the Appendix.

6 Quantitative Analysis

6.1 Parameter Choices and Calibration

While our model is equally tractable for any homogeneous adjustment cost function $g(i)$, for numerical and illustrational simplicity purposes, we choose the following widely-used quadratic adjustment cost function:

$$g\left(i\right) = \frac{\theta i^2}{2}, \tag{63}$$

which gives explicit formulas for Tobin's q and optimal i in the first-best MM benchmark:

$$q^{FB} = 1 + \theta i^{FB}, \text{ and } i^{FB} = r + \delta - \sqrt{(r+\delta)^2 - 2\frac{A-(r+\delta)}{\theta}}. \tag{64}$$

Our model with no productivity shocks is parsimonious with only eight parameters. Three parameters essential for the contracting tradeoff between risk sharing and limited commitment are the entrepreneur's coefficient of relative risk aversion γ, the volatility of the capital shocks σ_K, and the parameter measuring the degree of human capital inalienability α. The other five parameters (the risk-free rate r, the entrepreneur's discount rate ζ, the risk-adjusted depreciation rate δ, the adjustment cost θ, and the productivity parameter A) are basic to any dynamic model with investment. We choose plausible parameter values to highlight the model's mechanism and main insights.

Thus, we take the widely used value for the coefficient of relative risk aversion, $\gamma = 2$; the annual risk-free interest rate $r = 5\%$; and the aggregate equity risk premium is $(\mu_R - r) =$

6%. The annual volatility of the market portfolio return is $\sigma_R = 20\%$ implying the Sharpe ratio for the aggregate stock market $\eta = (\mu_R - r)/\sigma_R = 30\%$. The subjective discount rate is set to equal to the risk-free rate, $\zeta = r = 5\%$. As for investment, we rely on the parameter findings suggested by Eberly, Rebelo, and Vincent (2009): we set the annual productivity A at 20% and the annual volatility of capital shocks at $\sigma_K = 20\%$. We set the correlation between the market portfolio return and the capital stock shock $\rho = 0.2$, which implies that the idiosyncratic volatility of the productivity shock $\epsilon_K = 19.6\%$.

Fitting the first-best values of q^{FB} and i^{FB} to the sample averages, we set the adjustment cost parameter at $\theta = 2$ and the (expected) annual capital depreciation rate at $\delta_K = 11\%$, which implies the risk-adjusted depreciation rate is $\delta = 0.122$. These parameters imply $q^{FB} = 1.264$ and an annual investment-capital ratio of $i^{FB} = 0.132$.

Finally, we choose the fraction of capital stock that the entrepreneur may start out with when she quits, α, to be 0.8, in line with some empirical estimates.[13]

The parameter values for our baseline case are summarized in Table 2. Note that all parameter values are annualized when applicable.

Table 2: Summary of Parameters

This table summarizes the parameter values used for numerical illustration.

A. Baseline model with no productivity shocks		
Parameters	Symbol	Value
Risk-free rate	r	5%
The entrepreneur's discount rate	ζ	5%
Correlation	ρ	20%
Excess return	$\mu_R - r$	6%
Volatility of market portfolio	σ_R	20%
The entrepreneur's relative risk Aversion	γ	2
Capital depreciation rate	δ_K	11%
Volatility of capital depreciation shock	σ_K	20%
Quadratic adjustment cost parameter	θ	2
Firm's productivity	A	20%
Inalienability of human capital parameter	α	80%

[13] See Li, Whited, and Wu (2014) for the empirical estimates of α. The averages are 1.2 for Tobin's q and 0.1 for the investment-capital ratio, respectively, for the sample used by Eberly, Rebelo, and Vincent (2009). The imputed value for the adjustment cost parameter θ is 2 broadly in the range of estimates used in the literature. See Hall (2004), Riddick and Whited (2009), and Eberly, Rebelo, and Vincent (2009).

6.2 Investors' Value is Entrepreneur's Liability: $P(K, W) = -S$

The primal contracting and dual implementation problems are linked as follows:

$$s = -p(w) \ \text{ and } \ w = m(s)\,, \tag{65}$$

where $p(w)$ is the scaled investors' value in the contracting problem, and $m(s)$ is the entrepreneur's scaled certainty equivalent wealth as a function of s in the implementation formulation. Thus, *liquidity* s for the entrepreneur is the payoff that the investor is giving up through the *promised wealth* w to the entrepreneur. Note that (65) implies that the composition of $-p$ and m, denoted by $-p \circ m$, yields the identity function: $-p(m(s)) = s$.

Scaled promised wealth w and scaled investors' value $p(w)$. Panel A and B of Figure 1 plots the investor's scaled value $p\,(w)$ and the sensitivity of the value to changes in promised wealth $p'\,(w) = P_W$ in Panels A and B respectively. In the first-best MM world, compensation to the entrepreneur is simply a one-to-one transfer away from investors, as we see from the dotted lines: $p(w) = q^{FB} - w = 1.264 - w$ and $p'(w) = -1$. With inalienability of human capital, investors' value $p(w)$ is decreasing and concave in w. That is, as w increases the entrepreneur is less constrained so that the marginal value $p'(w)$ decreases.

Additionally, $p(w)$ approaches $q^{FB} - w$, and $p'(w) \to -1$, as $w \to \infty$. That is, the first-best payoff obtains when the entrepreneur is unconstrained. However, the entrepreneur's inability to fully commit not to walk away *ex post* imposes a lower bound $\underline{w}$ on w. For our parameter values, $w \geq \underline{w} = 0.944$.

Finally, note that despite being risk neutral, the investor effectively behaves in a risk-averse manner due to the entrepreneur's inalienability-of-human-capital constraints. This is reflected in the concavity of the investors' scaled value function $p(w)$. This concavity property is an important difference of the limited commitment problem relative to the neoclassical problem, where volatility has no effect on firm value.

Scaled liquidity s and the entrepreneur's scaled certainty-equivalent wealth $m(s)$. Panel C and D of Figure 1 plot the entrepreneur's scaled savings $m(s)$ and the marginal value of liquidity $m'(s)$. As one might expect $m(s)$ is increasing and concave in s. The higher the liquidity s the less constrained the entrepreneur is. Additionally, as s increases

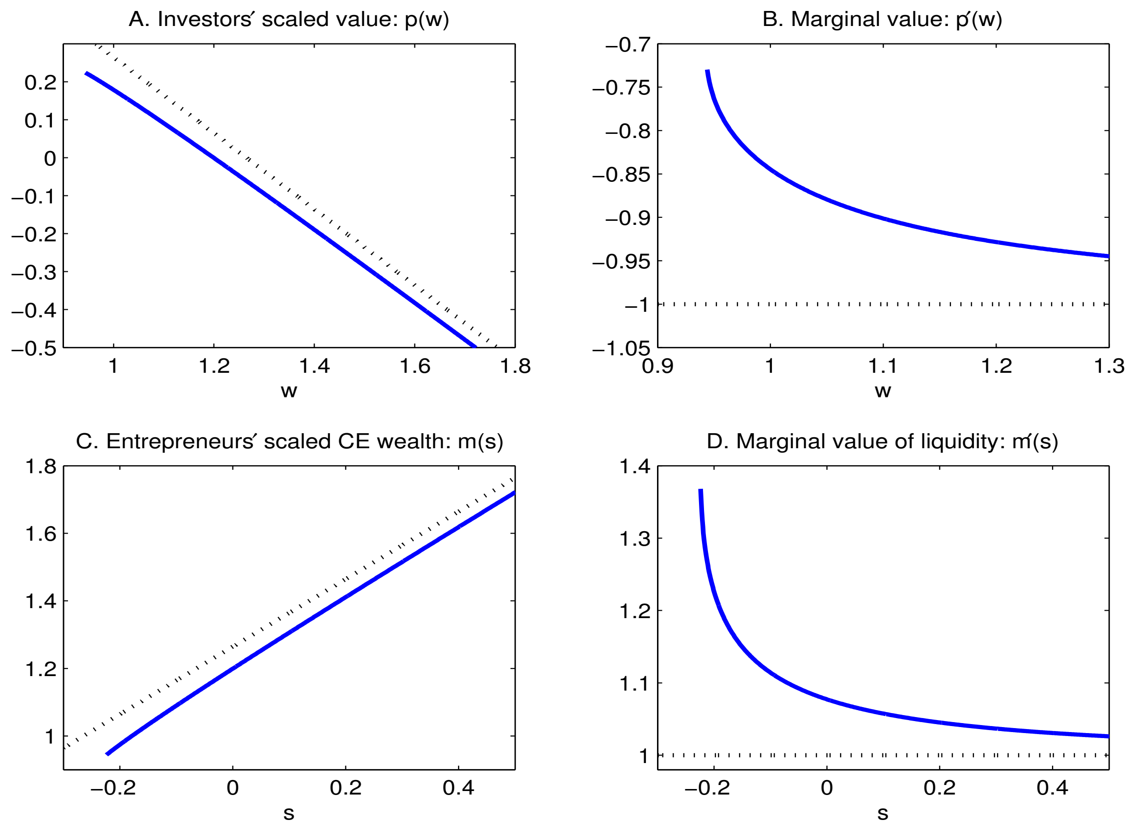

Figure 1: **Investors' scaled value $p(w)$, the marginal value of w, $p'(w)$, the entrepreneur's scaled certainty equivalent wealth $m(s)$ and marginal (certainty equivalent) value of liquidity, $m'(s)$.** For the limited-commitment case, $w \geq \underline{w} = 0.944$, and $p(w)$ is decreasing and concave in w. Similarly, for the limited-commitment case, $s \geq \underline{s} = -0.224$, and $m(s)$ is increasing and concave in s. The dotted line depicts the full-commitment MM results: $p(w) = q^{FB} - w$, $p'(w) = -1$, $m(s) = q^{FB} + s$ and $m'(s) = 1$.

the entrepreneur is less constrained so that the marginal value of savings $m'(s)$ decreases ($m''(s) < 0$). In the limited-commitment case the entrepreneur's scaled wealth $m(s)$ approaches $q^{FB} + s$ and $m'(s) \to 1$ as $s \to \infty$.[14] The entrepreneur's LOC limit, or in other words, her risk-free debt capacity $\underline{s} = -p(\underline{w})$ is given by -0.224.

We next discuss the optimal policy rules.

[14]See Wang, Wang, and Yang (2012) for similar conditions in a model with exogenously-specified incomplete-markets model of entrepreneurship.

6.3 Investment, Consumption, Liquidity and Risk Management

We first analyze the firm's investment decisions, then the entrepreneur's optimal consumption, and finally corporate liquidity and risk management.

6.3.1 Investment, marginal q, and the marginal value of liquidity $m'(s)$.

We can simplify the FOC for investment to:

$$1 + g'(i(s)) = \frac{J_K}{J_S} = \frac{M_K}{M_S} = \frac{m(s) - sm'(s)}{m'(s)}\,, \tag{66}$$

where the first equality is the investment FOC, the second equality follows from the definition of the value function in (32), and the last equality follows from the homogeneity property of $M(K,S)$ in K. Under perfect capital markets the entrepreneur's certainty equivalent wealth is given by $M(K,S) = m(s)\cdot K = (q^{FB}+s)\cdot K$ and the marginal value of liquidity is $M_S = 1$ at all times. Hence in this case, the FOC (66) specializes to the classical Hayashi condition for optimal investment, where the marginal cost of investing $1 + g'(i(s))$ equals marginal q.

Under limited commitment, $M_S > 1$ in general and the FOC (66) then states that the marginal cost of investing (on the left-hand side) equals the ratio between (a) marginal q, measured by M_K, and (b) the marginal value of liquidity measured by M_S. Unlike in the classical q theory of investment, here financing matters and M_S measures the (endogenous) marginal cost of financing generated by limited commitment constraints.

Figure 2 illustrates the effect of inalienability of human capital on marginal q and investment $i(s)$. The dotted lines in Panels A and B of Figure 2 give the first-best $q^{FB} = 1.264$ and $i^{FB} = 0.132$, respectively. With limited commitment, $i(s)$ is lower than the first-best benchmark $i^{FB} = 0.132$ for all s, and increases from -0.043 to $i^{FB} = 0.132$ as s increases from the left boundary $\underline{s} = -0.224$ towards ∞. This is to be expected: increasing financial slack mitigates the severity of under-investment for a financially constrained firm. Note however that, surprisingly, marginal q (that is, M_K) *decreases* with s from 1.25 to 1.20 in the credit region $s < 0$. What is the intuition? When the firm is financing its investment via credit at the margin (when $S < 0$), increasing K moves a negative-valued s closer to the origin thus mitigating financial constraints, which is an additional benefit of accumulating

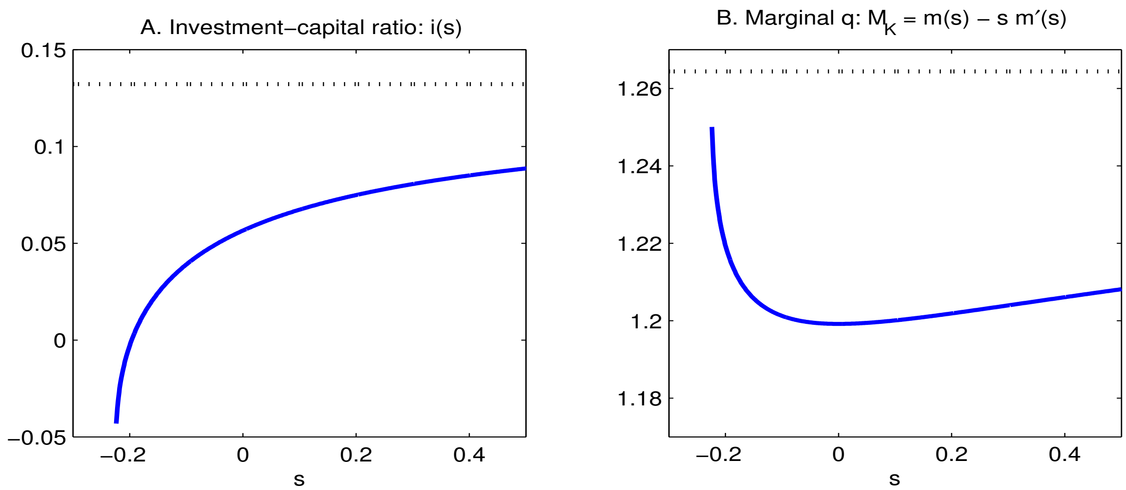

Figure 2: **Marginal q, $M_K = m(s) - sm'(s)$, and the investment-capital ratio $i(s)$.** For the limited-commitment case, the firm always under-invests and $i(s)$ increases with s. The dotted line depicts the full-commitment MM results where the marginal equals $q^{FB} = 1.264$ and the first-best investment-capital ratio $i(s) = i^{FB} = 0.132$.

capital.[15]

But why does a high marginal-q firm invest less in the credit region $s < 0$? And how do we reconcile an increasing investment function $i(s)$ with a decreasing marginal q function, $M_K = m(s) - sm'(s)$ in the credit region $s < 0$? The reason is simply that in the credit region ($s < 0$) a high marginal-q firm also faces a high financing cost. When $s < 0$ the marginal q and the marginal financing cost $m'(s)$ are perfectly correlated. And investment is determined by the ratio between the marginal q and $m'(s)$ as we have noted. At the left boundary $\underline{s} = -0.224$ marginal q is 1.25 and $m'(s)$ is 1.37 both of which are high. Together they imply that $i(-0.224) = -0.043$, which is low compared with the first-best $i^{FB} = 0.132$.

More generally, we consider a measure of investment-cash sensitivity given by $i'(s)$. Taking the derivative of investment-capital ratio $i(s)$ in (66) with respect to s, we have

$$i'(s) = -\frac{1}{\theta}\frac{m(s)m''(s)}{m'(s)^2} > 0. \tag{67}$$

As $m(s)$ is concave in s regardless of whether $s \geq 0$ or $s < 0$, $i(s)$ is increasing in liquidity.[16]

[15]Formally, this result follows from $dM_K/ds = -sm''(s) < 0$ when $s < 0$ and from the concavity of $m(s)$.

[16]See Bolton, Chen, and Wang (2011) for related discussions on how cash and credit influence the behaviors of investment, marginal q, and marginal value of liquidity.

6.3.2 Consumption

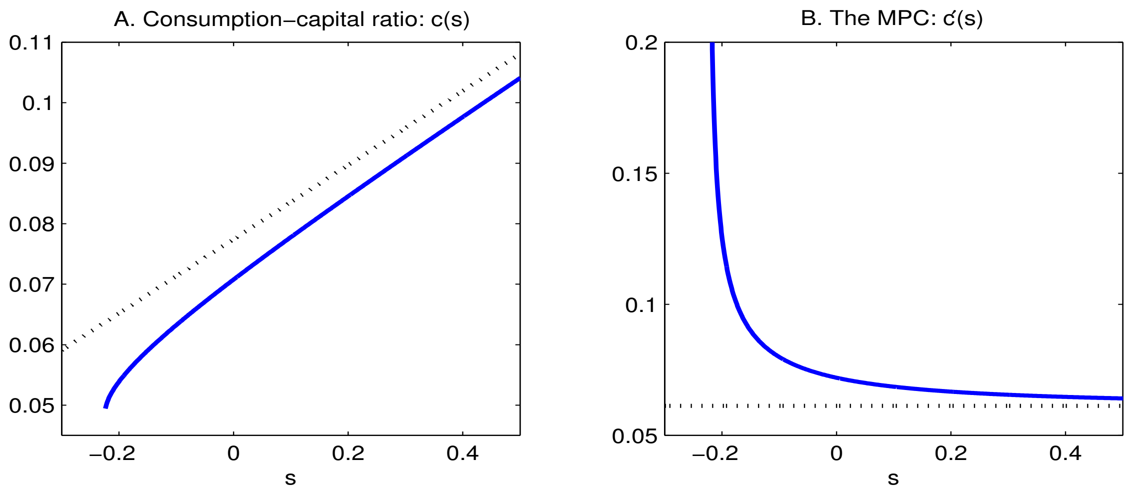

Figure 3: **Consumption-capital ratio** $c(s)$ **and the MPC** $c'(s)$**.** For the limited-commitment case, the entrepreneur always under-consumes compared with the full-commitment case and $c(s)$ increases with s. The dotted line depicts the full-commitment consumption-smoothing results: $c(s) = \chi(s + q^{FB})$ and the MPC $c'(s) = \chi = 5\%$.

The entrepreneur's optimal consumption rule $c(s)$ is given by:

$$c(s) = \chi m'(s)^{-1/\gamma} m(s)\,, \tag{68}$$

where χ is given in (25). Figure 3 plots the optimal consumption-capital ratio $c(s)$, and the MPC $\mathfrak{c}'(s)$ in Panels A and B respectively. The dotted lines in Panels A and B of Figure 3 give the first-best $c(s) = (s + q^{FB})$ and MPC $c'(s) = 6.13\%$, respectively. The solid line gives the entrepreneur's consumption, which is lower than the first-best benchmark. Additionally, the higher the financial slack s the higher is $c(s)$ as seen in the figure. Moreover, we have $m(s) \to q^{FB} + s$ and the marginal value of liquidity $m'(s) \to 1$ as $s \to \infty$, so that $c(s) \to \chi\left(q^{FB} + s\right)$, the permanent-income consumption benchmark. Panel B shows that the MPC $c'(s)$ decreases significantly with s and approaches the permanent-income benchmark $\chi = 6.13\%$ as $s \to \infty$. Thus, financially constrained entrepreneurs deep in debt (with s close to $\underline{s}$) have MPCs that are substantially higher than the permanent-income benchmark.

Next we turn to the firm's optimal hedging policy.

6.3.3 Hedging

Before delving into the analysis, we first review the entrepreneur's total wealth holdings in our implementation, which consist of three parts: (1) a 100% equity stake in the underlying business; (2) a mark-to-market futures position; and (3) a liquidity asset holding in the amount of s (negative when the firm is borrowing.)

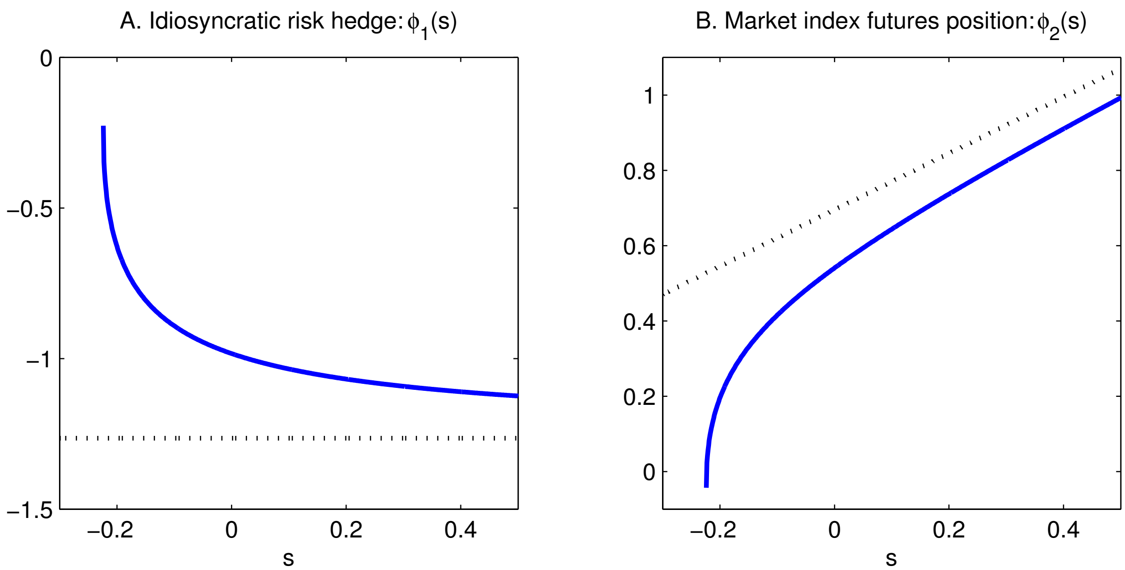

Figure 4: **Hedging position.** For the limited-commitment case, the entrepreneur partially hedges her firm's equity idiosyncratic risk exposure by shorting in the idiosyncratic risky asset, $\phi_1(s) < 0$. Note that idiosyncratic risky hedging $|\phi_1(s)|$, and market index futures position $\phi_2(s)$ increases with s. And the The dotted line depicts the entrepreneur's full-commitment hedging results with $\phi_1(s) = -q^{FB} = -1.264$ and $\phi_2(s) = -0.2 * q^{FB} + 0.75 * (s + qFB)$.

Panel A of Figure 4 plots the futures position against idiosyncratic shocks $\phi_1(s)$. First, under full commitment, the risk-averse entrepreneur is fully insured against the idiosyncratic business risk by taking a perfectly offsetting short futures position $\phi_1(s) = -q^{FB} = -1.264$. See the dotted line in Panel A of Figure 4.

With limited commitment the entrepreneur cannot fully hedge her equity exposure. How does $\phi_1(s)$ depend on s in this case? The solid line gives the futures position $\phi_1(s)$: As the firm becomes less constrained (s increases) the entrepreneur increases the futures hedging position $|\phi_1(s)|$. Thus, a less constrained firm has a larger hedging position (after controlling for firm size), and in the limit as $s \to \infty$ the entrepreneur can fully diversify the idiosyncratic business risk by taking a short futures position: $\phi_1(s) = -q^{FB} = -1.264$, attaining the full-commitment perfect insurance benchmark. Rampini, Sufi, and Viswanathan (2014) provide empirical evidence supporting this result. Note that here liquidity and hedging are complements.

Panel B plots the market index futures position $\phi_2(s)$. Notice that under full commitment, the risk-averse entrepreneur is fully insured against the systematic business risk by shorting the position $-\frac{\rho\sigma_K}{\sigma_R}q^{FB}$ and he also has the standard mean-variance demand given as $\frac{\eta}{\gamma\sigma_R}m^{FB}(s)$, so the total market index futures position $\phi_2(s) = -\frac{\rho\sigma_K}{\sigma_R}q^{FB} + \frac{\eta}{\gamma\sigma_R}m^{FB}(s) = -0.2 * q^{FB} + 0.75 * (s + q^{FB})$ as represented by the dotted line in Panel B of Figure 4. It's interesting that for low s, it's very important to manage the risk

7 Two-sided Limited Commitment

As we have shown so far, under the one-sided commitment solution investors must be able to commit to incurring losses. As Figure 1 illustrates $p(w)$ takes negative values when w exceeds 1.18. To be able to retain the entrepreneur, investors then promise such a high wealth w to the entrepreneur that they end up committing to making losses in these states of the world. But, what if they cannot commit to such loss-making wealth promises to the entrepreneur? What if investors are protected by limited liability and cannot commit to a long-term contract that yields a negative net present value at some point in the future? We explore this issue in this section and derive the optimal contract when neither the entrepreneur nor investors are able to fully commit. Specifically, we introduce the additional set of constraints for investors that guarantee at any time t that investors receive a non-negative payoff value under the contract:

$$F_t \equiv \mathbb{E}_t\left[\int_t^\infty e^{-r(v-t)}(Y_v - C_v)dv\right] \geq 0\,. \tag{69}$$

As it turns out, solving the two-sided limited commitment problem does not involve major additional complexities. The main change relative to the one-sided problem is that the upper boundary is now $s = 0$. Indeed, any promise of strictly positive savings $s > 0$ is not credible as this involves a negative continuation payoff for investors.

Again, we simply modify the upper boundary condition in Proposition 3. The upper boundary is now given by $s = 0$ rather than the natural limiting boundary $s \to \infty$ in the one-sided case. We thus replace condition (39) with the following conditions at the new upper boundary $s = 0$:

$$\lim_{s \to 0} \sigma_H^s(s) = 0 \quad \text{and} \quad \lim_{s \to 0} \mu_H^s(s) \leq 0\,. \tag{70}$$

As before, the volatility $\sigma_n^s(\,\cdot\,)$ must be zero at $s = 0$ and the drift needs to be weakly negative to pull s to the interior so as to ensure that s will not violate the constraint $s \leq 0$.

Similarly, we also consider the special case when the firm's productivity is constant, $A^L = A^H = A$, so that the only shock is the diffusion capital shock Z.

7.1 Investment and Risk Management

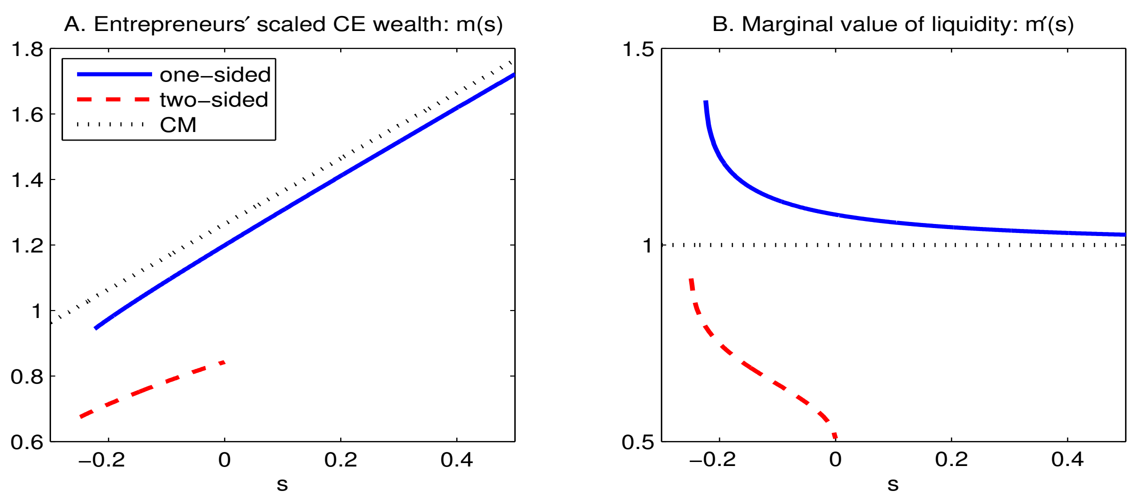

Figure 5: **The entrepreneur's scaled certainty equivalent wealth $m(s)$ and marginal (certainty equivalent) value of liquidity, $m'(s)$ under two-sided limited-commitment case.**

Figure 5 shows the entrepreneur's scaled certainty equivalent wealth $m(s)$ and marginal (certainty equivalent) value of liquidity, $m'(s)$. It shows that in the two-sided limited-commitment case, s lies between $\underline{s} = -0.249$ and $\overline{s} = 0$, so that the entrepreneur has a larger LOC limit of $|\underline{s}| = 0.249$. But this comes at the expense of lower promised utility, which translates into no corporate savings in this implementation ($\overline{s} = 0$). Indeed, if we had $\overline{s} > 0$ the investors' value would be strictly negative violating the investors' limited-liability condition. In sum, the additional investor limited-liability condition limits the entrepreneur's self savings capacity, which in turn increases the entrepreneur's demand for relying on an LOC. Remarkably, here a firm with a larger debt capacity is not necessarily less constrained and may have a lower value!

While $m'(s) \geq 1$ holds for the one-sided case, $m'(s)$ can be less than 1 in the two-sided limited-commitment case. This is again due to the fact that in the two-sided case the benefit of relaxing financial constraints for the entrepreneur with an increase in s may not be sufficient to offset the cost to the investor (due to a shorter distance investors' limited-liability constraint) implying that $m'(s) < 1$ in the region $\underline{s} < s \leq 0$.

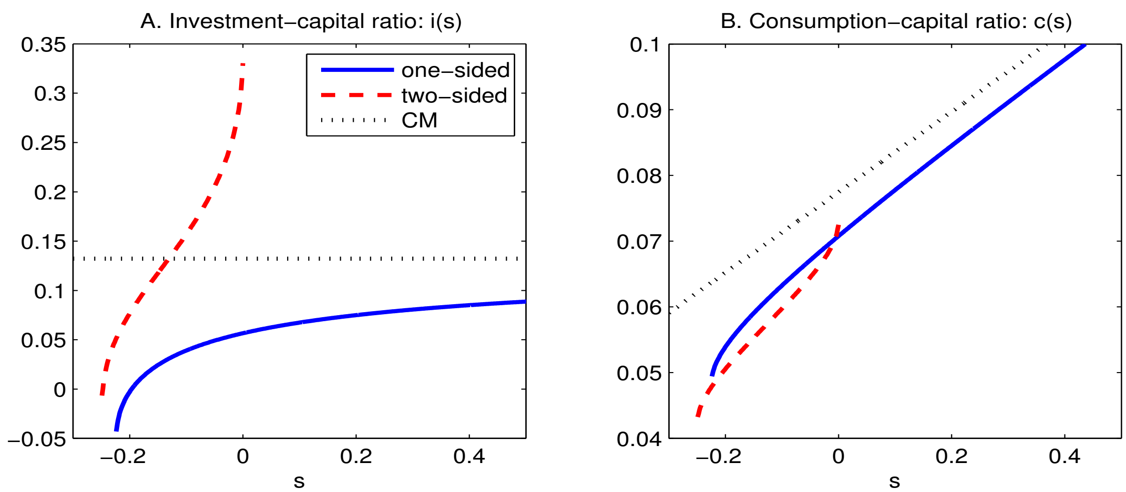

Figure 6: **Optimal investment-capital ratio $i(s)$ and consumption-capital ratio $c(s)$ under two-sided limited-commitment case.**

Figure 6 reports the two-sided limited-commitment solution for investment and consumption in Panels A and B, respectively. Comparing the two-sided and one-sided limited commitment solutions for investment in Panel A, we observe that the limited-liability constraint for investors prevents the entrepreneur from owning positive liquid wealth, so that there is only a credit region in the two-sided case: $s \leq 0$. This is necessary for the investors to have positively-valued stake in the firm. Remarkably, in this case the firm may either under-invest or over-invest compared with the first-best benchmark. The firm under-invests when $s < -0.13$ but over-invests when $-0.13 < s \leq 0$. Whether the firm under-invests or over-invests depends on the net effects of the entrepreneur's limited-commitment and the investors' limited-liability constraints. For sufficiently low values of s (when the entrepreneur is deep in debt) the entrepreneur's constraint matters more and hence the firm under-invests. For values of s sufficiently close to zero, the investors' limited-liability constraint has a stronger influence on investment as the investors' value is close to zero. To ensure that s will drift back into the credit region the entrepreneur needs to "save" in the form of the illiquid productive asset (by increasing K) by borrowing more. By over-investing, the firm optimally chooses to keep s between $\underline{s}$ and 0. In summary, given that the entrepreneur cares about the total compensation $W = w \cdot K$ and given that investors are constrained by their ability to promise the entrepreneur w beyond an upper bound, investors reward the entrepreneur along the extensive margin, firm size K, which induces over-investment but allows the entrepreneur to build more human capital.

Similarly, comparing the two-sided and one-sided limited commitment solutions for consumption in Panel B, we observe that in both cases consumption is increasing with financial slack s, and the entrepreneur always under-consume. Interestingly, the entrepreneur will significantly increasing the consumption under two-sided limited commitment, which may exceed the consumption under one-sided case, to ensure s drift back into the credit region when s is close to 0. And the economic intuition is similarly with the behavior of the firm's over-investment.

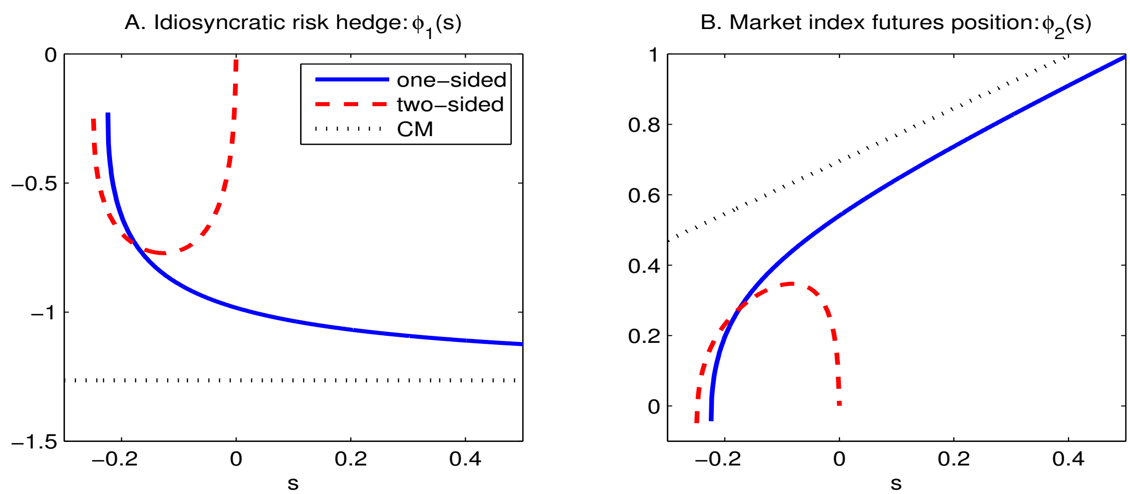

Figure 7: **Hedging strategy under two-sided limited-commitment case.**

Figure 7 plots the hedging position $\phi_1(s)$ and $\phi_2(s)$. It illustrates that both of $\phi_1(s)$ and $\phi_2(s)$ are non-monotonic in s for the two-sided case: Although the entrepreneur can afford to build larger hedging positions when s is larger, investors find these large hedging positions incompatible with their limited-liability constraints. To prevent investors from reneging on their promises volatility must be turned off at $s = 0$, which is achieved by setting $\phi_1(s) = s$ and $\phi_2(s) = s$ at $s = 0$, as implied by the volatility boundary condition (36) for $\sigma_1(s)$ and (37) for $\sigma_2(s)$. This nonlinear hedging result illustrates the complexity of firms' liquidity and risk management policies and point to the subtle interaction between a firm's risk management and its financial slack.

8 Persistent Productivity Shocks: Insurance

The shocks in our baseline model of Section 2 are capital depreciation shocks that we model via continuous diffusion processes. In this section, we extend the model to allow for persistent productivity shocks that also have first-order implications on corporate liquidity and risk management. Naturally, the risk-averse entrepreneur optimally insure against such shocks. We characterize the optimal insurance contract against such shocks and how investment, executive compensation/consumption, liquidity/risk management and the firm's credit limit vary with the firm's productivity. For simplicity, we only consider the one-sided limited commitment case where productivity shock is purely idiosyncratic. We have generalized this setting to allow for systematic productivity shocks and/or two-side limited-commitment, which are available upon request, but these features out due to space considerations.

To keep the analysis simple we model persistent productivity shocks $\{A_t;\, t \geq 0\}$ as a two-state Markov switching process, $A_t \in \left\{A^L, A^H\right\}$ with $0 < A^L < A^H$. Similarly, we denote by $\lambda_t \in \left\{\lambda^L, \lambda^H\right\}$ with λ^L being the transition intensity from state L to H and λ^H being the intensity from state H to L. The counting process $\{\mathcal{N}_t;\, t \geq 0\}$ (starting with $\mathcal{N}_0 = 0$ without loss of generality) increases by one whenever the state switches either from H to L or from L to H, and remains unchanged otherwise, in that $d\mathcal{N}_t = \mathcal{N}_t - \mathcal{N}_{t-} = 1$ if and only if $A_t \neq A_{t-}$. Otherwise, $d\mathcal{N}_t = 0$.

Suppose that the current time is $t-$ and the firm is considering the decision over the time interval $(t-, t)$. First, we introduce the contingent claim that the firm can buy or issue/sell to insure the change of productivity shock. One unit of this insurance contract over time interval $(t-, t)$ pays the holder one lump-sum unit at time t, if and only if $d\mathcal{N}_t = 1$. Under the assumptions that markets are perfectly competitive and productivity shocks are idiosyncratic, the actuarially fair premium on a unit of this insurance claim over the time interval $(t-, t)$ is then λ_{t-} per unit of time.[17] Therefore, if the firm buys Π_{t-} units of insurance over the time interval $(t-, t)$, the firm will be pays insurance premium $\lambda_{t-}\Pi_{t-}dt$.

We can write the dynamics of liquidity S_t as follows:

$$dS_t = (rS_t + Y_t - C_t + \Phi_{2,t}(\mu_R - r) - \lambda_{t-}\Pi_{t-})\,dt + \Phi_{1,t}\epsilon_K d\mathcal{Z}_{1,t} + dt + \Phi_{2,t}\sigma_R d\mathcal{Z}_{2,t} + \Pi_{t-}d\mathcal{N}_t\,. \tag{71}$$

The first term in (71) gives the firm's savings rate. In addition to the term $rS_t + Y_t - C_t + \Phi_{2,t}(\mu_R - r)$ as we have in the baseline case, we also have the insurance premium term $-\lambda_{t-}\Pi_{t-}$. If $\Pi_{t-} > 0$, in that the firm is buying an insurance with a lump-sum payment in the amount of Π_{t-}, the firm has to pay an insurance premium which lowers the firm's savings rate at $\lambda_{t-}\Pi_{t-}$ per unit of time. The insurance company delivers the payment $\Pi_{t-} > 0$ if and only if $d\mathcal{N}_t = 1$, i.e., when the productivity suddenly switches from time $t-$ to t. If $\Pi_{t-} < 0$, the firm sells insurance which increases the firm's savings rate locally, but exposes the firm to the risk of delivering a lump-sum payment $-\Pi_{t-}$ at time t when $d\mathcal{N}_t = 1$.

9 Conclusion

Our generalization of Hart and Moore (1994) to introduce risky human capital, risk aversion, and ongoing consumption reveals the optimality of corporate liquidity and risk management for financially constrained firms. Most of the existing corporate security design literature has confined itself to showing that debt financing and credit line commitments are optimal financial contracts. By adding risky human capital and risk aversion for the entrepreneur, two natural assumptions, we show that corporate liquidity and hedging policies are also an integral part of an optimal financial contract. When productivity shocks are persistent, we find that insurance contracts and/or equilibrium default by the entrepreneur on her debt obligations is part of an optimal contract. We have thus shown that the inalienability-of-human-capital constraint naturally gives rise to a role for corporate liquidity and risk management, dimensions that are typically absent from existing macroeconomic theories of investment under financial constraints following Kiyotaki and Moore (1997).

Although our framework is quite rich, we have imposed a number of strong assumptions,

[17] We can generalize this to allow for systematic risk premium via a change of measure by choosing different jump intensities under the physical measure and the risk-neutral measure. Notes are available upon request.

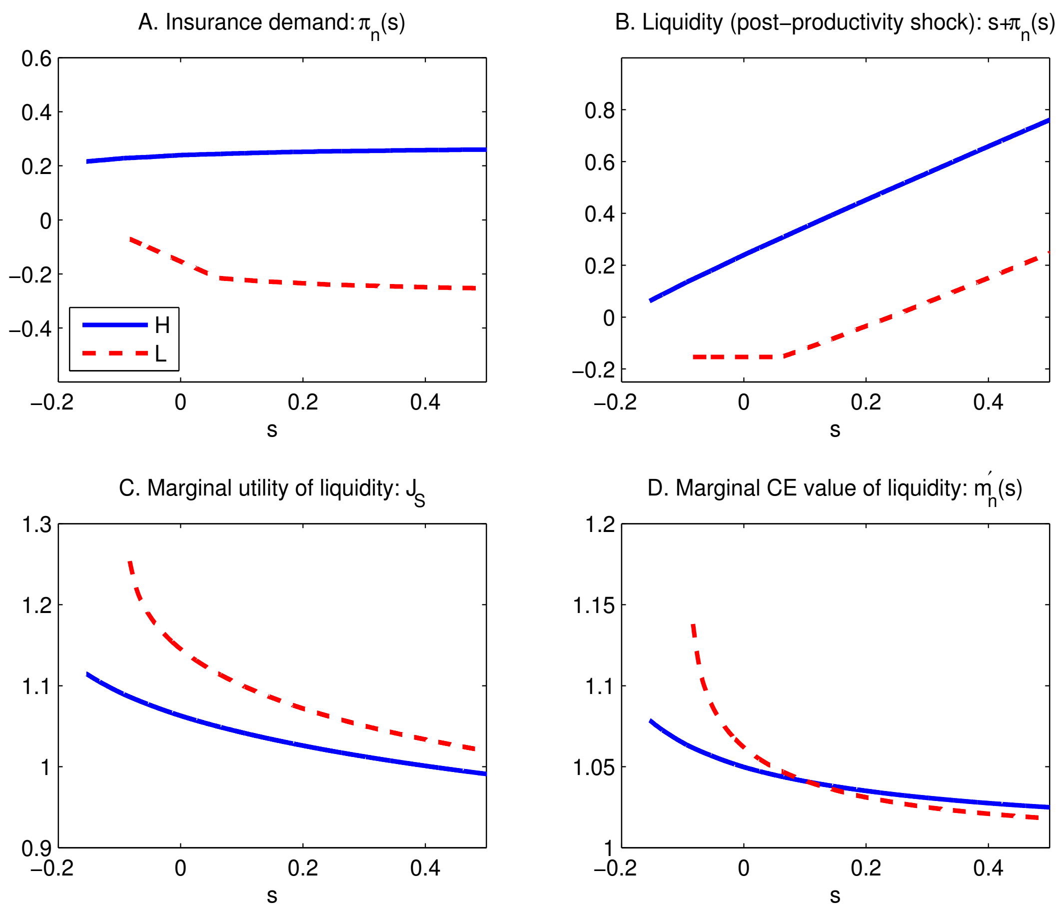

Figure 8: **Insurance demand.** Parameter values: $\gamma = 0.1$, $\zeta = 0.1$, $\sigma_K = 1$, $\eta = 0$, $\theta = 5$, $A^L = 0.05$, $A^H = 0.2$, $\lambda_H = \lambda_L = 0.05$. The constrained region of π_n under L state is $-0.09 < s < 0.06$.

which are worth relaxing in future work. For example, one interesting direction is to allow for equilibrium separation between the entrepreneur and the investors. This could arise, when after an adverse productivity shock the entrepreneur no longer offers the best use of the capital stock. Investors may then want to redeploy their capital to other more efficient uses. By allowing for equilibrium separation our model could be applied to study questions such as the expected and optimal life-span of entrepreneurial firms, the optimal turnover of managers, or the optimal investment in firm-specific or general human capital.

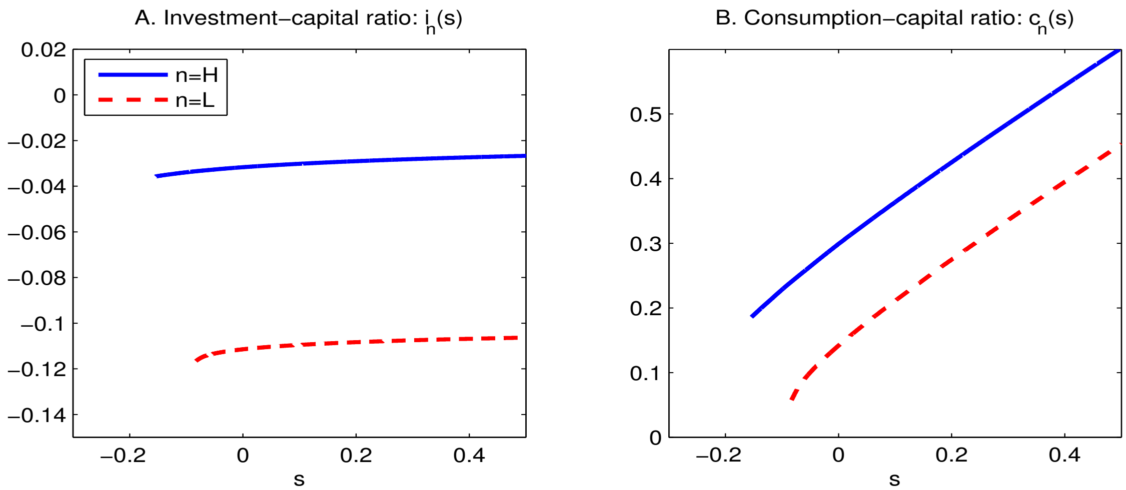

Figure 9: **Optimal investment-capital ratio $i_n(s)$ and consumption-capital ratio $c_n(s)$ under two states and one-sided limited-commitment case. Parameter values: $\gamma = 0.1$, $\zeta = 0.1$, $\sigma_K = 1$, $\eta = 0$, $\theta = 5$, $A^L = 0.05$, $A^H = 0.2$, $\lambda_H = \lambda_L = 0.05$.**

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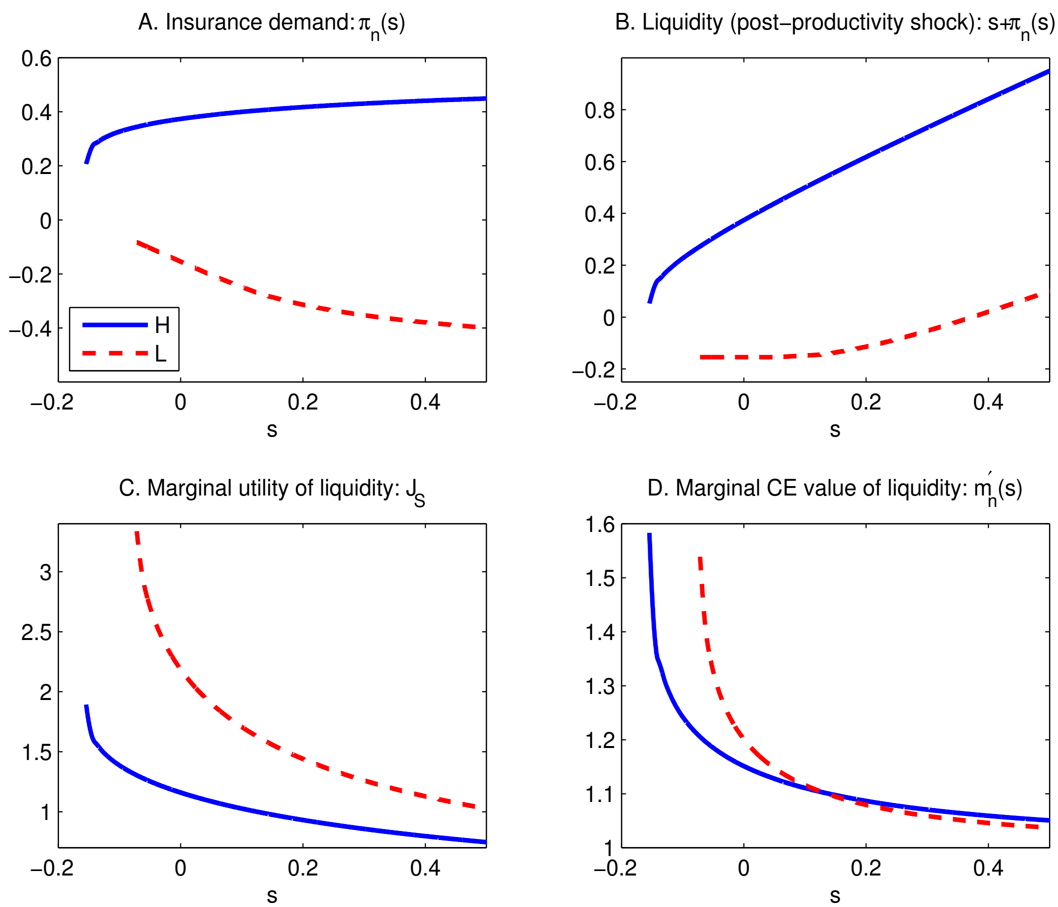

Figure 10: **Insurance demand.** Parameter values: $\gamma = 0.8$, $\zeta = 0.05$, $\sigma_K = 0.2$, $\eta = 0.5$, $\theta = 5$, $A^L = 0.02$, $A^H = 0.245$, $\lambda_H = \lambda_L = 0.1$. The constraint region of π_n under L state is $-0.072 < s < 0.051$.

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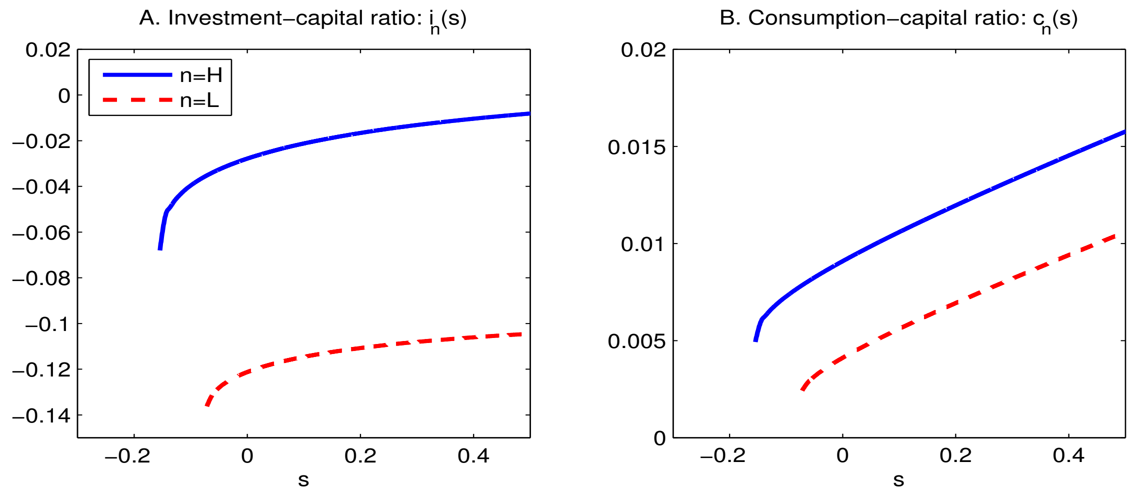

Figure 11: **Optimal investment-capital ratio $i_n(s)$ and consumption-capital ratio $c_n(s)$ under two states and one-sided limited-commitment case. Parameter values: $\gamma = 0.8$, $\zeta = 0.05$, $\sigma_K = 0.2$, $\eta = 0.5$, $\theta = 5$, $A^L = 0.02$, $A^H = 0.245$, $\lambda_H = \lambda_L = 0.1$.**

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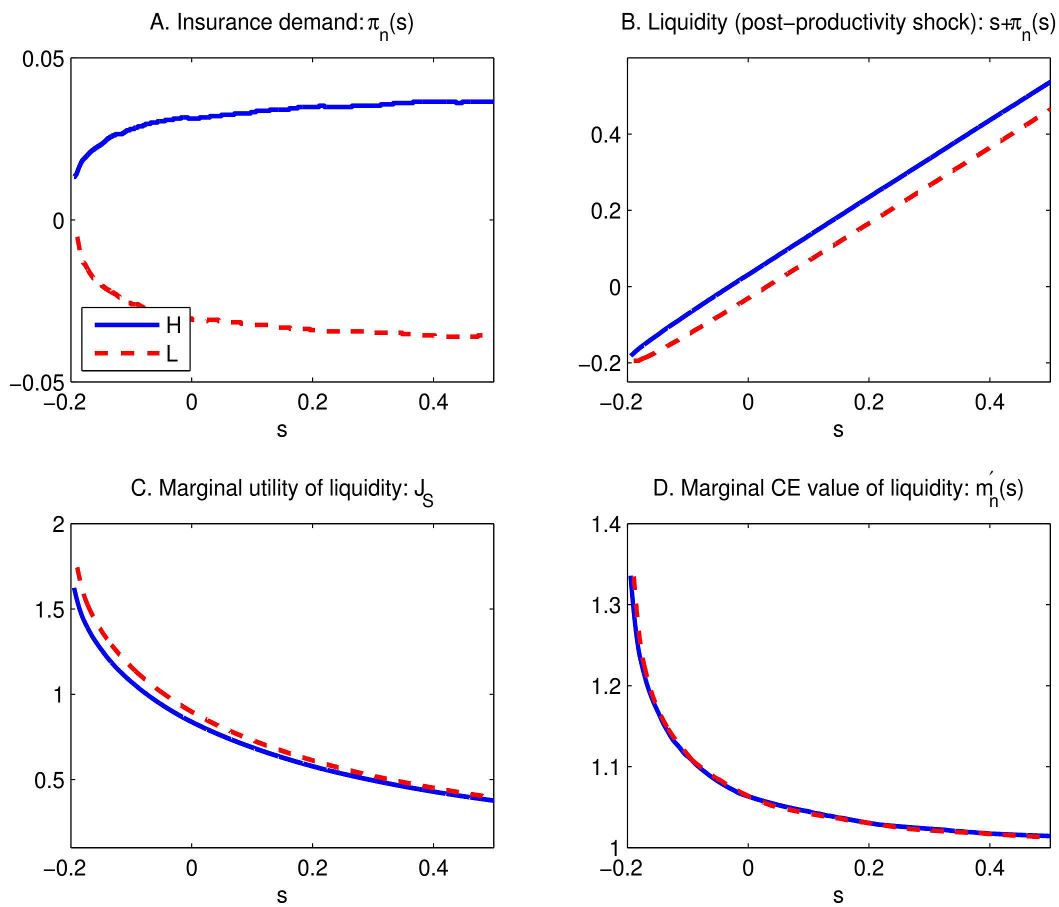

Figure 12: **Insurance demand.** Parameter values: $A^L = 0.18$, $A^H = 0.2$, $\lambda_H = \lambda_L = 0.2$ and the other parameters are the same with baseline model. The constraint region of π_n under L state is $-0.19 < s < -0.183$.

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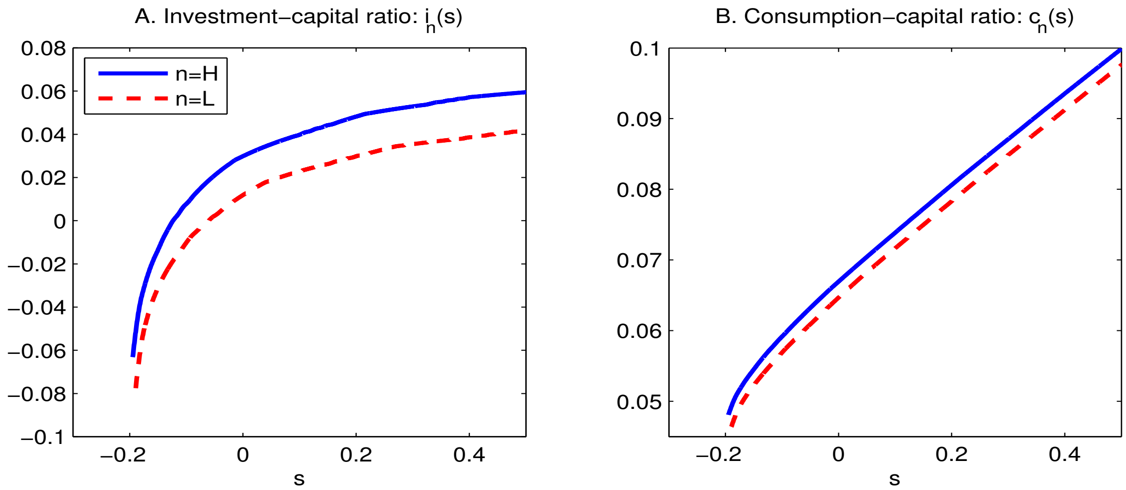

Figure 13: **Optimal investment-capital ratio $i_n(s)$ and consumption-capital ratio $c_n(s)$ under two states and one-sided limited-commitment case. Parameter values: $A^L = 0.18$, $A^H = 0.2$, $\lambda_H = \lambda_L = 0.2$ and the other parameters are the same with baseline model.**

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