Auctions of Real Options*

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Nov 20, 2015

Abstract

Governments and corporations frequently sell assets with embedded real options to competing buyers using security bids. Examples include the sales of natural resources, real estate, patents and licenses, and start-up firms with growth options. This paper models these auctions of real options, incorporating both endogenous auction timing and post-auction option exercise. I characterize the ways common security bids distort investments and strategic auction timing affects auction initiation, security ranking, equilibrium bidding, and investment. Revenue-maximizing sellers inefficiently delay auctions, including optimal auctions which align investment incentives using a combination of down payment and royalty payment. When sellers do not restrict security design, bidding and allocation outcomes are equivalent to cash auctions. Finally, informed bidders always initiate the auctions when they could. The results are broadly consistent empirical observations and underscore that auction timing and sellers' commitment should be jointly considered with security design in selling real options.

JEL Classification: D44; D81; D82; G13; G31; G32; L24

Keywords: Real Options; Auctions; Investments; Acquisitions; Security Design; Agency Conflicts; Timing Games; Optimal Stopping; Oil and Gas Leases

*I am indebted to Steven R. Grenadier, Ilya Strebulaev, and Darrell Duffie for their guidance and support. I am grateful to John Beshears, Jeremy Bulow, Peter DeMarzo, Mike Harrison, Zhiguo He, Andrey Malenko, and Andy Skrzypacz for many fruitful discussions. I also thank Anat Admati, Shai Bernstein, Bruno Biais, Jules van Binsberg, Bradyn Breon-Drish, Douglas Diamond, Han Hong, Dirk Jenter, Arthur Korteweg, Ilan Kremer, Charles Lee, Richard Lowery, Paul Milgrom, Dmitry Orlov, Minyu Peng, Paul Pfleiderer, Ken Singleton, Pavel Zryumov, Jeff Zwiebel, and conference discussants and seminar participants at Stanford University, Northwestern Kellogg, University of Minnesota, University of Notre Dame, University of Illinois at Urbana-Champaign, NYU Stern, Boston University, Wharton, UT Austin, UCLA Anderson, University of British Columbia, Chicago Booth, Michigan State University, HKUST, INSEAD, Tokyo University, the Finance Theory Group, the Trans-Atlantic Doctoral Conference, the Australasian Banking and Finance Conference, and the Utah Winter Finance Conference for helpful comments and suggestions. I also would like to acknowledge Caryl Fagot from the Bureau of Ocean Energy Management and James Swaney from the Bureau of Safety and Environmental Enforcement for lending their expertise, and Stanford Graduate School of Business, Stanford Institute for Economic Policy Research, Stanford Vice Provost for Graduate Education, and Stanford Institute for Innovation in Developing Economies for their research support. All remaining errors are mine.

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1 Introduction

On March 30, 2013, the Bureau of Ocean Energy Management held an oil lease auction that netted the U.S. government \$1.2 billion. Exxon-Mobil emerged as the highest bidder and was entitled, but not obliged, to explore and drill on seven of the 320 auctioned tracts in the Central Gulf of Mexico for 5–10 years, and was to pay 18.75% royalty of future revenues from oil production to the Department of the Interior.[1] Less than two years later, Shire PLC acquired NPS Pharmaceutical, Inc. for approximately \$5.2 billion in cash to grow NPS's portfolio of licenses and products through its global footprint, market expertise in gastrointestinal disorders, and core capabilities in rare disease patient management.[2] These two deals involve classic examples of a large class of assets with embedded real options whose sale and exercise underlie some of the most crucial decisions for entrepreneurs, firm executives, and government officials. These transactions also routinely involve competing bids in combinations of cash and contingent securities, and can be effectively viewed through the lens of security-bid auctions.[3] Why did Shire make an all-cash offer? Why are nearly 72% of oil and gas tracts offshore, and 56% of those are on federal lands that are neither producing nor under active exploration?[4] More fundamentally, how should a seller trade off rent extraction and incentive provision in using security bids? How does one jointly decide the optimal security choice and auction timing? What is the role of the seller's commitment and who initiates a competitive negotiation in equilibrium?

Without a model of auctions of real options, simultaneously addressing these important

[1] ENERCOM Consulting, 360 articles: Central Gulf of Mexico Lease Sale - Come on in, the Water if Fine! (March 21, 2013), http:oilandgas360.comcentral-gulf-of-mexico-lease-sale-come-on-in-the-water-is-fine.

[2] Shire News: Shire to Acquire NPS Pharma as Further Step in Building a Leading Biotech. http://www.shire.com/shireplc/en/investors/irshirenews?id=1052

[3] Oil leases have been auctioned using cash, bonus-bid, royalty and profit-share contracts. In technology transfers, such as the licensing in pharmaceuticals, rivals bid contingent contracts (Vishwasrao (2007) and Bessy and Brousseau (1998)). In sales of large assets, such as the wireless spectrum auction for FCC bandwidth, aggressive bidders can declare bankruptcy and the bids are essentially debts (Board (2007a) and Zheng (2001)). Equities, preferred convertibles, and call options are frequently used in M&A and venture capital financing (Martin (1996), Kaplan and Stromberg (2003), and Hellmann (2006)). Other examples include advance and royalty payments in publishing contracts (Dessauer (1981) and Caves (2003)), motion picture deals (Chisholm (1997)), business licenses such as electronic gambling machines with pre-specified profit tax, and military procurement contracts (McAfee and McMillan (1987b)).

[4] "Oil and Gas Lease Utilization,". Report to the President" by Department of the Interior dated May 2012. This revelation has triggered a huge public outcry and heated debate in Congress on the reason for the purported sluggish development of natural resources despite the imbalance in supply and demand, and has policy implications in the backdrop of Obama's proposal to increase onshore royalties by 50%.

questions is difficult. This paper does so by endogenizing auction timing and tying together optimal stopping with selling mechanisms in a tractable framework. I derive the following main results under the unifying intuition that economic agents enjoy different optionalities in the sale and operation of an asset: First, common security bids cause inefficient and often suboptimal investments, and unlike results in previous studies, security ranking depends on auction timing and the number of bidders. Second, strategic auction timing should be considered jointly with security design and option exercise. In particular, all security-bid auctions are inefficiently delayed, including cash auctions and auctions with optimal security design that combines cash and royalties. Third, when a seller lacks commitment to the auction design, bidding equilibria are equivalent to those in cash auctions, and bidders always initiate the auction when feasible. These findings imply that many conclusions from traditional auction and real-options models need to be modified in dynamic settings with learning and strategic interactions.

Auctions of real options are prevalent in licensing and patent acquisitions, leasing of natural resources, real estate development, M&A deals, venture capital and private equity markets, and privatization of large national enterprises.[5] They entail tremendous financial resources and mainly come in two categories in practice.[6] Whereas in formal settings such as oil lease auctions, wireless spectrum auctions, or privatization auctions, the seller specifies explicitly and commits to allocation rules and an ordered set of security bids, many other sales to competitive buyers can be thought of as *informal auctions* in which the seller lacks such commitment; that is, bidders bid anything they want and can potentially revise their offers.[7] Prominent examples of informal auctions include corporate takeovers and project finance, where bidders decide what to offer and often can initiate the contact or negotiation. Still others, such as licensing agreements and contracts in the entertainment industry, appear in both categories. This paper addresses both formal and informal auctions.

Prior studies on security-bid auctions typically take auction initiation as exogenous and do not consider how security design influences post-auction investments. Similarly, studies

[5] Bolton, Roland, Vickers, and Burda (1992) describe the privatization policies in Central and Eastern Europe. Pakes (1986) and Schwartz (2004) discuss patents as real options.

[6] In the Gulf of Mexico alone, the oil and gas leases auctioned by the U.S. federal government in 1954-2007 have exceeded \$300 billion, and annual licensing deals by pharmaceutical giants exceed \$20 billion even in the aftermath of the financial crisis. M&A volume worldwide is also in the trillions of dollars annually.

[7] DeMarzo, Kremer, and Skrzypacz (2005) introduce a similar concept but rule out offer adjustments.

on real options analyze the sales and exercise of real options in isolation. This paper differs by considering how auction timing, security design, and post-auction investment interact. It therefore attempts to bridge the gap between auction theory and corporate finance, and adds to the emerging literature both on agency conflicts in real options and on auction initiation and security bids. Specifically, this paper models the sale and exercise of a typical investment option with endogenous participation. The baseline model involves a seller and multiple potential bidders who are risk neutral and maximize their expected payoffs. Time is continuous and in three sequential stages. In the first stage, the seller strategically times the auction. In the second stage, participants bid cash and contingent securities and the seller allocates the asset. The key difference between formal and informal auctions in this stage is whether the seller can commit to not entertaining offers outside a pre-specified ordered set. In the final stage, the winning bidder rationally times the exercise of the investment option and delivers the contingent payment to the seller.

The model contains two key frictions. The first is the non-contractibility of the bidders' private information. Contingent payment does not account for the bidders' private costs, and thus distorts investment incentives in the third stage. This misalignment leads to a tradeoff for the seller between the post-auction moral hazard in investments and the benefits of contingent bids, such as enhanced rent extraction.[8] As contingent bids become increasingly prevalent, this tradeoff could have a first-order impact on projects with high option values, such as the development of real estate and natural resources, as well as the transfer and licensing of technologies. Moreover, no "one-size-fits-all" exists in security ranking, because any comparison has to be made in conjunction with considerations of auction timing and the market environment.

The second friction is the cost associated with the ownership transfer that is well-recognized in the literature. Examples include the initial opportunity cost to the winning bidder, the seller's discontinued benefit from the asset's alternative use, or the irreversible loss of the option for more efficient allocation of the asset in the future.[9] Delaying the auc-

[8]Prima facie, the type of bids should not matter as a cash equivalent always exists. One advantage to contingent bids is that they enhance the seller's revenue by effectively linking payoff to a variable affiliated with bidders' private information—the "linkage" principle in Milgrom (1985). Contingent bids also mitigate liquidity or legal constraints and reduce valuations gaps among various parties.

[9]One illustration of the benefit from the asset's alternative use is Ecuador's estimation of \$3.6 billion in environmental benefits for not selling the Ishpingo-Tiputini-Tambococha oil block in the Yasuni National Park in its pristine rainforest.

tion saves the time value of money on these costs and encourages greater participation, but risks missing the opportune exercise of the investment option. These tradeoffs endogenize auction timing in the first stage. The seller times the auction to maximize the option value less the information rent, and a bidder times the auction to maximize the information rent, neither of which maximizes social welfare. I argue that strategic auction timing is a salient feature in real-life business practice and is integral to auction outcome, especially when an entrepreneur considers when to sell a startup, or when target and acquirer firms decide when to solicit or make an offer.

With standard security bids and endogenous auction timing, I show the seller faces a real option with an added exercise cost, namely, the information rent. In an optimal security design, the seller can pass this cost to the winning bidder by combining cash and royalty payments, which is consistent with the popular use of negotiated royalty payments and down payments in sales of marketing rights, licensing agreements, publishing and movie contracts, and many other franchise business practices. More generally for security-bid auctions, the seller's real option becomes in the money at a higher threshold due to the information rent, prompting her to inefficiently delay the auctions to delay incurring the cost of ownership transfer.

Furthermore, the seller's commitment to security design significantly influences the bidding and investment outcomes. Absent such commitment, bidding equilibria in both first-price and ascending informal auctions are equivalent to those in cash auctions. The intuition is that cash-like bids allow a bidder to generate the maximum social surplus, and at the same time outbid competitors in the cheapest way. For example, a bidder with higher valuation can more easily outbid others using cash versus equity shares, because the same shares cost him more than they cost someone with a lower valuation. When bidders can initiate, Bayesian updates of beliefs on the types that are present absent initiation further complicates the auction timing game.. In equilibrium, bidders always initiate, and consistent with empirical findings, invest more efficiently conditional on initiation.

These results offer insights to understand several puzzling empirical observations, and add to conventional theory. In particular, a high royalty rate in oil and gas lease auctions causes the winning bidder to delay exploration beyond efficient rational waiting due to optionality, especially in highly uncertain environments, which potentially explains the large number of idle tracts reported. Although having more bidders is often revenue-enhancing (Bulow and Klemperer (1996)), sellers in some corporate auctions still restrict the number of bidders (Hansen (2001) and French and McCormick (1984)). I show that even absent the entry-cost channel in Samuelson (1985), more bidders could decrease revenue and social welfare by exacerbating moral hazard associated with security bids. Another widely held belief is that security bids generate higher revenue than cash, but this paper argues cash dominates common securities as the bidders' market becomes very competitive. This paper also highlights the important role of endogenous auction timing and the seller's commitment. While many studies have focused on security design or security ranking, their conclusions are sensitive to auction timing. Post-auction investment distortion is only a concern in formal auctions, but we should pay attention to inefficient auction timing that is present even in auctions traditionally deemed efficient (including cash auctions).

This paper builds on studies of security-bid auctions and their applications in corporate finance.[10] DeMarzo, Kremer, and Skrzypacz (2005) give an extensive exposition of security-bid auctions, showing "steeper" securities lead to higher expected value to the seller. Samuelson (1987) suggests adverse selection and moral hazard complicate the effect. Che and Kim (2010) and Rhodes-Kropf and Viswanathan (2000) demonstrate, respectively, that adverse selection could reverse the ranking of securities and lead to inefficiencies in bankruptcy reorganizations and privatizations. This paper examines post-auction moral hazard—the second issue Samuelson (1987) emphasized. Kogan and Morgan (2010) compare equity and debt auctions under moral hazard in an experimental study. Laffont and Tirole (1987), Eső and Szentes (2007), and Riordan and Sappington (1987) also study how post-auction decisions affect auctions. McAfee and McMillan (1987a) is another earlier study that derives an optimal linear incentive contract under competition, information asymmetry, and moral hazard. Povel and Singh (2010) and Liu (2012) relate security-bid auctions to external financing. This paper is unique in considering post-auction moral hazard in a dynamic setting with persistent private information, emphasizing endogenous auction timing.

[10] Hansen (1985), Crémer (1987) and Riley (1988) are among the early contributions. Also related are the theories of incentive contracting, typically applied to defense procurement (Engelbrecht-Wiggans (1987), McAfee and McMillan (1986), and Laffont and Tirole (1987)). Hansen (2001) reviews the corporate auction process. Dasgupta and Hansen (2007) and Boone and Mulherin (2007) justify viewing corporate takeovers as auctions. Eckbo, Giammarino, and Heinkel (1990) consider the role of non-cash bids. Skrzypacz (2013) gives an overview.

This paper also complements the emerging literature on agency issues and auction initiation in a real-options framework. Maeland (2002), Grenadier and Wang (2005), and Cong (2012) study distortion of investment incentives due to adverse selection and moral hazard. Board (2007b) derives optimal selling mechanisms of options. This paper differs primarily in considering auction timing and linking the agency conflicts to a broader class of security bids. Gorbenko and Malenko (2014) examine bidder-initiated takeover attempts in cash and stocks with heterogeneous cash constraints. Gorbenko and Malenko (2015) offer another detailed study on auction initiation, focusing on time-varying types and signaling through initiation in cash auctions. This paper complements theirs by examining initiations driven by aggregate market conditions and Bayesian learning. In addition, I highlight the role of the seller's commitment, and link auction initiation to post-auction investment.

The remainder of the paper proceeds as follows. Section 2 describes the economic environment, sets up the model, and analyzes optimal investment strategies. Section 3 derives bidding equilibria, security ranking, and auction timing of formal auctions. Section 4 characterizes informal auctions as games of signaling, timing, and Bayesian learning. Section 5 concludes. The appendix contains all the proofs.

2 Setup and Optimal Stopping

A risk neutral revenue-maximizing seller with discount rate $r > 0$ owns a project with an embedded option. Once developed, the project generates a verifiable lump sum cash flow whose value P_t is publicly observed and evolves stochastically according to a geometric Brownian motion (GBM)

$$dP_t = \mu P_t dt + \sigma P_t dB_t, \tag{1}$$

where B_t is a standard Brownian motion under the equivalent martingale measure, $\mu < r$ is the instantaneous conditional expected percentage change per unit time in P_t, and σ is the instantaneous conditional standard deviation per unit time.[11]

The seller does not have the expertise to exploit the option but can auction the project to N risk neutral potential bidders with the same discount rate r who have the expertise

[11] $r > \mu$ ensures a finite value of the option. See McDonald and Siegel (1986) or Dixit and Pindyck (1994). The lump sum could represent the present value of a stream of future cash flows.

to exploit the option.[12] When the seller holds the auction, a bidder i knows his private investment cost for the project θ_i.[13] He also learns that the distribution of types for other bidders is i.i.d. with positive support $[\underline{\theta}, \overline{\theta}]$. Denote the cumulative distribution and density function by $F(\theta)$ and $f(\theta)$, respectively. The project is worthless to him if it is never developed.

I assume that whereas the revenue from exercise P is observable and contractible, the cost θ and thus the profit $P-\theta$ are not. This assumption is realistic because contracts or security designs based on profit are rare due to high monitoring costs, limited comparability, and landowners' risk aversion (Robinson (1984)). The procurement literature has also established that profit reporting is subject to manipulations, and contracting on revenue is more feasible. Past experience in oil lease auctions has also shown considerable difficulties in reaching agreement on the proper profits (Opaluch, Grigalunas, Anderson, Trandafir, and Jin (2010)).

In addition, ownership transfer entails both benefits and costs. These "transaction costs" borne potentially by either the seller or the bidders are typically small for financial assets, but are often rather significant for real assets, especially when they are illiquid and complex. Similar to DKS, the winning bidder has to pay an up-front cost $X \geq 0$, which we can interpret as the initial resources the project requires, such as illiquid human capital, the social cost of underwriting securities, or simply his opportunity cost. The seller also loses a reservation value $Y \geq 0$ when the asset is sold. For example, she may lose a continuous stream of cash flow rY through alternative uses of the asset before the auction, or the option value of more efficient allocation of the asset when technology improves.[14] In the case of leasing natural resources, the winning bidder has to assemble a team and equipments to be ready for drilling any time, and the seller loses the environmental benefits or income from utilizing the land as a national park.[15] Basically, $X+Y$ represents in reduced form the

[12] For clarity, I refer to the seller as female and the bidders as male.

[13] The analysis also applies when bidders differ in other quantities, such as production capacities.

[14] Bleakley and Ferrie (2014) show that after an initial allocation of the frontier land in Georgia, land use took over a century to converge to post-allocation efficiency and land value was depressed by 20%. Another example is the FCC spectrum auction, where the government selling certain bandwidth to multiple firms has to consider the cost of losing the option to allocate it in the future to firms with better technology, because even for the federal government, repurchasing the bandwidth is hard because of the well-known hold-up problem involved in multilateral bargaining. Y could also simply be the intermediary fee, or legal and professional costs of holding the auction, which could be higher than 10% of the value for small firms. Other pre-contract costs are common too (French and McCormick (1984)). Hansen (2001) and Gorbenko and Malenko (2014) discuss costs associated with revealing proprietary information to rivals.

[15] $X+Y$ may be insignificant, especially when a winning bidder can contract with the seller to continue the

ownership-transfer frictions. In fact, either X or Y can be negative; to make the auction timing non-trivial, we need $X + Y > 0$. Even when $X + Y$ is small in magnitude, auction timing is still relevant in adverse market conditions, that is, when P is also small.

When the auction is held at time t_a, bidders compete by offering security bids that are combinations of contingent payments from the cash flow of the project and non-contingent payments that, for simplicity, can be viewed as upfront cash at the time of the auction. **Unless stated otherwise, the remainder of the paper focuses on standard security bids as defined next.**

DEFINITION. *A **standard security bid*** is an upfront cash payment $C \in \mathbb{R}$ and a contingent payment at the time of investment τ given by continuous function $S(P_\tau) \in \mathbb{R}$.[16]

Standard security bids are simple and intuitive, and as discussed later, can implement the optimal auction design even in the augmented universe of security bids. They admit most securities and contracts used in practice. For example, with equity bids, the seller receives a fraction α of the payoff: $S(P) = \alpha P$; with call option bids, the seller can pay a strike price k for the project cash flow: $S(P) = [P - k]^+$; with bonus bids on fixed royalty rate ϕ, the seller receives bonus C and royalty payment $S(P) = \phi P$.

The agents interact in continuous time as shown in Figure 1. To analyze the dynamics, I work backward to first solve for the optimal investment strategy for the winning bidder, then derive the bidding equilibrium given the bidders' valuations based on their investment strategies, and then study the impact of strategically timing the auction.

Formal auctions and informal auctions mainly differ in the seller's commitment to the

original use before the option is exercised. Although such scenarios occur in professional sports, where teams sometimes buy a player and then immediately loan them back to the original team, in many business settings such efficient contracts are exceptions rather than norms, due to factors outside the model. For example, the federal government typically auctions areas of land or sea involving multiple leases in a shared ecosystem, and cannot contract with individual winners to keep certain areas intact to derive environmental benefits while allowing drilling in a neighboring tract. Due to political and ideological differences, the managers of national parks and environmental organizations are unlikely to collaborate with oil firms to maintain their operations before the oil firms start drilling. Moreover, as soon as a tract of land is sold, either the seller or the bidder has to pay preparation costs to relocate local habitats so that exploratory studies can be conducted.

[16] As known in practice and earlier studies, directly contracting on private cost θ faces problems of validating profits reported. Consequently, payments are usually contingent on top-line revenue in the development of natural resources, contracts on marketing and licensing rights, as well as franchise chain operations.

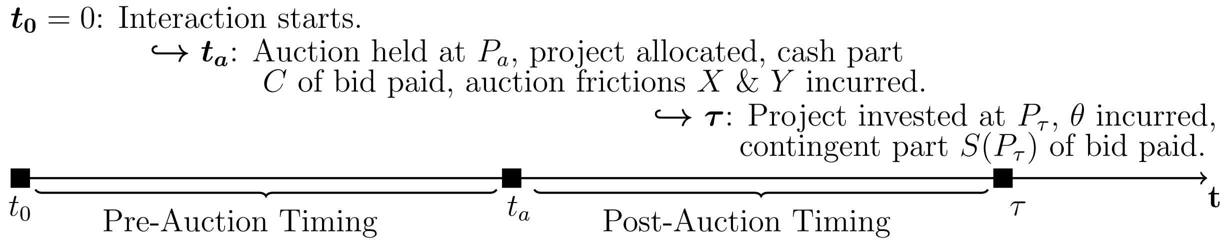

Figure 1: Timeline

auction timing and design. Throughout the paper, I focus on first-price auctions (FPAs) and second-price auctions (SPAs) in which the bidder with the highest bid wins and pays the highest bid or the second-highest bid, respectively. I assume the seller commits to no renegotiation post-auction, and to no contracting or resale to losing or non-participating bidders.

Welfare is defined as the total payoff to the seller and bidders, and efficiency in this paper means constrained efficiency from a global optimizer's perspective; that is, welfare maximizing under the same informational or institutional constraints as individual agents.

Cash Auctions as a Benchmark

In cash auctions, a bidder of type θ owns the project entirely upon winning, and optimally develops the project at time $t \geq t_a$ to maximize $\mathbb{E}[e^{-r(t-t_a)}(P_t - \theta)]$. The optimal strategy for this standard problem involves immediate investment upon reaching an upper threshold $P^*(\theta)$.[17] Let P_a denote the cash-flow level when the auction is held. The value of the investment option W and $P^*(\theta)$ are independent of X and t, and are given by

$$P^*(\theta) = \max\left\{P_a, \frac{\beta}{\beta-1}\theta\right\}, \quad \textit{where} \quad \beta = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} > 1, \quad \textit{and} \qquad (2)$$

$$W(P_a;\theta) = D(P_a; P^*(\theta))(P^*(\theta) - \theta), \quad \textit{where} \quad D(P;P') = \left(\frac{P}{P'}\right)^\beta \quad \textit{for} \quad P \leq P'. \qquad (3)$$

Note that $D(P_t; P')$ corresponds to the time-t price of an Arrow-Debreu security that pays one dollar when the first moment threshold $P' \geq P_t$ is reached. The option value of the

[17] See, for example, McDonald and Siegel (1986) and Dixit and Pindyck (1994).

project is simply the total value of Arrow-Debreu securities that replicate the payoff of the investment option at exercise.

Bidder i's private valuation is then $W(P_a;\theta_i)-X$, which decreases in θ_i, and his bidding strategies are the same as those in standard cash FPAs and SPAs. A cutoff type exists for participation $\theta_c=\min\{\overline{\theta},\theta_{BE}\}$, where the break-even type θ_{BE} solves $W(P_a;\theta)-X=0$ and is given explicitly as $\theta_{BE}=(\beta-1)(P_a^{\beta}\beta^{-\beta}X^{-1})^{\frac{1}{\beta-1}}1_{\{P_a>\beta X\}}+(P_a-X)1_{\{P_a\leq\beta X\}}$. Types with costs higher than θ_c do not participate. Because post-auction investments are not distorted, FPAs and SPAs generate equivalent revenues to the seller, and efficiently allocate the project to type $\theta_{(1)}$ if $\theta_{(1)}\leq\theta_c$, where $\theta_{(j)}$ is the jth lowest realized θ. The cases with a reserve price or entry fee are similar.

Optimal Investments Post-auction

Next I characterize the optimal investment strategy taking as given the standard security bid in equilibrium.

Suppose the winning bidder of type θ pays standard security $\{C,S(P_t)\}$ when project is invested at time $t\geq t_a$. His private valuation at t_a is

$$\tilde{V}(C,S(\cdot),\theta)=\max_{\tau\geq t_a}\mathbb{E}_P[e^{-r(\tau-t_a)}(P_\tau-S(P_\tau)-\theta)]-X-C, \tag{4}$$

where τ is any stopping time. $S(P)$, being of general form, distinguishes this problem from traditional real-options models. Whether $V(\theta)$ is well-defined a priori is unclear, but the following lemma dispels such concern.

Lemma 1. *A threshold investment strategy exists that is optimal among all stopping times. Moreover, the valuation $\tilde{V}(C,S(\cdot),\theta)$ is continuously decreasing in θ.*

The strategy generally involves both upper and lower thresholds that are dependent on P_a—auction timing clearly matters.[18] **The remainder of this paper assumes the following for standard security bids $S(P)$:**

[18]For notational simplicity, except in the discussion of auction timing, I do not explicitly write P_a as an argument for the valuation.

Assumption (Conv): *For any type, $P^{-\beta}[P - S(P) - \theta]$ of P is quasi-concave with a maximum achieved at some $\tilde{P}(\theta)$. For $P \geq \tilde{P}(\theta)$, $S(P)$ is piecewise twice-differentiable with only positive jumps in $S'(P)$, and $S''(P)$ (when exists) satisfies $PS''(P) \geq (1-\beta)[1-S'(P)]$.*

Condition (Conv) requires that the security is not too "concave" at large P, and is non-restrictive since it holds for most securities used in real life and in equilibria in this paper, such as equities, call options, and the optimal securities derived later. As the following lemma shows, (Conv) allows us to focus on simple threshold strategy that is standard and often implicitly assumed in the real-options literature:

Lemma 2. *With assumption (Conv), the optimal investment follows an upper threshold strategy with threshold $\tilde{P}(\theta)$. If, in addition, $P - S(P)$ is non-decreasing in P, $\tilde{V}(C, S(\cdot), \theta)$ is non-decreasing in P_a.*

Compared to the investment threshold $\frac{\beta}{\beta-1}\theta$ in Equation (2), the threshold with security payment may be higher because intuitively, the bidder faces an additional cost $S(P)$. On the other hand, the sensitivity of the security payment to cash flow implies a smaller option premium.[19] Depending on which effect dominates, the threshold could be either higher or lower, whereas in prior literature agency conflicts mostly delay investments. I show in Appendix A.3 that the direction of distortion depends on whether the cash flow elasticity of winning security bid (CES) $E_S = \frac{PS'(P)}{S(P)}$ is bigger than β, which depends on equilibrium bidding that I analyze next.

3 Formal Auctions

The knowledge of optimal investment strategies allows bidders to value the real option. This section continues to analyze bidding equilibria in formal auctions. Besides extending the analysis in DKS and Board (2007b) to the standard real-options setting, I also show how equilibrium security bids can both delay and accelerate investments and how conventional results in security-bid auctions are modified. Then I prove the main theorem of the section in which all formal auctions are inefficiently delayed, before deriving the optimal auction design

[19]As in Grenadier (2002) and Grenadier and Malenko (2011), option premium is the NPV of investment at the moment of exercise divided by the total cost: $OP(\theta) = \frac{\tilde{P}(\theta)-S(\tilde{P}(\theta))-\theta}{S(\tilde{P}(\theta))+\theta}$.

that can be implemented using standard securities combining cash and royalty payments. I conclude the section by discussing extension to interdependent value settings and relating it to the auctions and development of oil and gas tracts.

In formal auctions, the seller times the auction and commits to a pre-specified, well-ordered set of allowed bids, which in real life are ranked by simple, easily implementable rules. A variant of the definition in DKS formalizes this notion of well-orderedness:

DEFINITION An ***ordered set of securities*** ranked by index s is defined by a left-continuous map $\Pi(s) = \{C(s), S(s, \cdot)\}$ from $[s_L, s_H] \subset \mathbb{R}$ to the set of standard security bids such that for each voluntary participant of type θ, $\boldsymbol{V(s,\theta) \equiv \tilde{V}(C(s), S(s,\cdot), \theta)}$ is non-negative and non-increasing in s on $[s_L, \tilde{s}]$ and negative on $(\tilde{s}, s_H]$ for some $\tilde{s} \in [s_L, s_H)$.

This definition simply requires that in addition to being standard, an ordered set of securities admits one-dimensional ranking with index s for any payoff from the project, and permissible bids cover a wide range such that each participant earns a non-negative profit by bidding low enough but earns no profit by bidding too high. Any such sets can be represented by the mapping defined above up to an order-preserving transformation of the index. The seller commits to allocating the project to the bidder with the highest index. The winning bidder pays a security using the highest-bid index in FPAs or the next-highest-bid index in SPAs.

This notion of an ordered set of securities subsumes a definition based on securities' values to the seller, because the latter is necessarily an ordered set here. In fact, most security bids I examine including the optimal security derived later are also well-ordered in terms of their values to the seller. Moreover, this definition is more inclusive of contingent bids used in real life: s could be the fraction of shares α in a pure equity auction $\{C(\alpha) = 0, S(\alpha, P) = \alpha P\}$, the (negative) strike price k in a call-option auction $\{C(-k) = 0, S(-k, P) = \max\{P-k, 0\}\}$, or the bonus b in a bonus-bid auction with royalty rate ϕ fixed $\{C(b) = b, S(b, P) = \phi P\}$.[20] M&As, VC contracts, and lease auctions routinely uses such securities, and indeed the bidder offering the highest s wins.[21]

[20] One could equivalently define an ordered set based on monotonicity of security values to the seller and derive almost all the results, but that approach rules out auctions with commonly used security bids, such as equity bids without a minimum share retention.

[21] In M&As with the acquirer's stocks as bids, C simply corresponds to the value of the acquirer's cash flows that are independent of the acquisition, X corresponds to the opportunity cost of incorporating the

3.1 Bidding Strategies in Formal Auctions

Throughout the paper, I assume that the standard "single-crossing" (as described in Lemma 3) for formal security-bid FPAs, and that buyers resolves any indifference in bidding by bidding higher s. Lemma 1 allows me to characterize equilibrium bidding strategies similar to those in DKS.

Lemma 3. *(**DKS Lemma 3**) In FPAs, when $\ln V(s,\theta)$ is absolutely continuous in s with the derivative (when exists) decreasing in θ, a unique symmetric Bayesian Nash equilibrium exists that is decreasing, differentiable, and is characterized by:*

$$s'(\theta) = \frac{(N-1)f(\theta)}{1-F(\theta)} \frac{V(s(\theta),\theta)}{V_1(s(\theta),\theta)} \tag{5}$$

for $\theta \leq \hat{\theta}$ with the boundary condition $s(\hat{\theta}) = \sup\{s \in [s_L, s_H] \mid V(s,\hat{\theta}) = 0\}$. The cut-off type for participation is $\hat{\theta} = \sup\{\theta \leq \overline{\theta} \mid \max_s V(s,\theta) \geq 0\}$.

Lemma 4. *(**DKS Lemma 2**) In SPAs, the unique Bayesian Nash equilibrium in weakly undominated strategies is for type θ to bid $s(\theta) = \sup\{s \in [s_L, s_H] \mid V(s,\theta) \geq 0\}$, which is decreasing in θ. The cut-off type for participation is $\hat{\theta} = \sup\{\theta \leq \overline{\theta} \mid \max_s V(s,\theta) \geq 0\}$.*

Because the bidding strategies are monotone, the investment option is allocated, if at all, to a bidder with the lowest cost. Moreover, the level of participation is the same for FPAs and SPAs, and is weakly smaller than that in cash auctions. In addition, it can be shown that bidders bid more aggressively (weakly greater s for all types, and strictly greater s for a positive measure of types) in FPAs as N increases or X decreases, or if V and V/V_1 are increasing in P_a, as P_a increases. They bid more aggressively in SPAs as X decreases or if V is increasing in P_a, as P_a increases.

Given the existence of bidding equilibria and optimal exercise of the real option, I next illustrate that investments can be both delayed and accelerated.

Investment Delays and Accelerations

Lemmas 3 and 4 hold for equity auctions with $S(\alpha, P) = \alpha P$ where α is the shares bid, and $C(\alpha) = C$ is the reserve price. The equilibrium $\alpha(\theta)$ is continuous and decreasing, and

target firm, and P is the payoff from the acquired assets and projects, and the synergy created.

the winning bidder invests when cash flow first reaches $P^{equity}(\theta) = \max\left\{P_a, \frac{\beta\theta}{(\beta-1)(1-\alpha)}\right\} \geq P^*(\theta)$. Investments are therefore inefficiently delayed.

Standard security bids can also lead to investment accelerations. Consider call-option auctions with $S(-k, P) = \max\{P - k, 0\}$ and $C(-k) = 0$, where k is the strike price. In both FPAs and SPAs, if a bidder of type θ bids a strike less than $X + \theta$, with non-trivial probability he wins with a required strike $k < X + \theta$ and fails to break even. If he bids a strike greater than $P^*(\theta)$, he always invests with the threshold $P^*(\theta)$ and the call is never exercised. But he could bid lower k to increase the chance of winning. Therefore, a bidder of type θ always bids $k \in [X + \theta, P^*(\theta)]$, and upon winning, invests when the cash flow first reaches $P^{call}(\theta) = \max\{P_a, k\} \leq P^*(\theta)$. Inefficiency thus lies in the potential *acceleration* of investments. The seller never makes a profit if $P_a < k$ for the winner. Nonetheless, as long as $P_a > \max\{X + \underline{\theta}, \frac{\beta}{\beta-1}\underline{\theta}\}$, i.e., some real options are sufficiently in the money, the seller expects to raise positive revenue. Whereas existing real-option models with agency, such as that in Grenadier and Wang (2005), often predict decreased or delayed investments, the security choice in selling real options could be an alternative to empire-building-based explanations of overinvestments.[22]

Figure 2 illustrates how different security leads to different investment thresholds and timings. The figure includes another common form of security: friendly debt $S(B, P) = \min(P, B)$, where B is a fixed promise of payment.[23]

Number of Bidders

The literature has established that in private-value auctions increasing the number of bidders enhances the seller's revenue (e.g. Bulow and Klemperer (1996)), but more bidders implies more aggressive bidding, as seen earlier, resulting in greater moral hazard. Simulations in Figure 3 illustrate in the spirit of Samuelson (1985) that revenue and welfare could vary in almost any way with N. The impact of competition clearly depends on the security design. The result generalizes to auctions with standard securities such as friendly debts and call options (see Figure 5), and is robust to distributional assumptions and endogenous

[22] Gryglewicz and Hartman-Glaser (2014) underscore this point in a setting with dynamic agency.

[23] Friendly debts are esssentially debts without interests, also known as Qard/Qardul hassan in Islamic finance. They are popular in contractual agreements in Islamic banking and microfinance, and are equivalent to granting the winning bidder instead of the seller call options - the exact opposite situation to that for call option bids.

entries with entry costs. Since the expected social welfare and revenue to the seller need not increase with the number of potential bidders, limiting participation may improve revenue or welfare. This channel is different from that in Samuelson (1985) and consistent with that sellers in real life restrict the number of bidders *even absent entry fees* (see Hansen (2001) and French and McCormick (1984)).

The number of bidders also matters for security choice. Prior studies indicate that security bids usually perform better than cash bids. Rhodes-Kropf and Viswanathan (2000) show that any securities auction generates higher expected revenue to the seller than a cash auction. But since the linkage advantage of security bids lies in the extraction of the winning bidder's rent, it decreases in expectation when N increases. Yet moral hazard persists with many standard securities. In appendix A.6, I define M-regular securities that include or can closely approximate most common securities, and show cash bids dominate M-regular securities in FPAs and SPAs in terms of expected revenue and social welfare, as the number of bidders gets large. The size of the bidders' market is thus an important consideration in security choice. The result also suggests security bids are rarely used when the number of bidders is large.

3.2 Security Design and Auction Timing

DKS show that "steeper" securities yield higher revenues for the seller. This ranking breaks down due to post-auction moral hazard: a "steeper" security extracts more from the winning bidder's information rent, but it also reduces his incentive to invest efficiently post-auction. This subsection approaches security ranking from a mechanism-design perspective and allows general structures of security payments. I first demonstrate that there is no "one size fits all" regarding ranking standard security bids; in particular, the ranking depends on the auction timing. I then show the optimal mechanism in Board (2007b) extends to this setting and can be implemented using a standard security combining cash and royalty payments. Most importantly, I show optimal auction timing exists and is inefficiently late.

The direct revelation principle allows us to focus on a truth-telling mechanism. Suppose the seller times the auction at $t_a$and specifies the security choice of the general form $S(\tilde{\theta}_i, \theta_{-i}, \mathcal{I}_t)$ at time $t \geq t_a$, where $\tilde{\theta}_i$ is the reported type by i, θ_{-i} are other participants' reported types, and $\mathcal{I}_\tau$ is the set of contractible information up to time t.[24] ***I assume for the remainder of the paper that*** $z(\theta) = \theta + F(\theta)/f(\theta)$ ***is increasing***.[25]

Ranking Security Design

Through deriving an integral form of the bidders' incentive compatibility conditions, I first extend Board (2007b)'s analysis to standard real-options settings with standard security bids and interdependent values.

Proposition 1. *The seller's revenue in formal FPA and SPA held at t_a with standard security bids is given by*

$$\mathbb{E}\left[1_{\{\theta_{(1)} \leq \hat{\theta}\}}\left[e^{-r(\tau_1^* - t_a)}(P_{\tau_1^*} - z(\theta_{(1)})) - X - Y\right]\right], \tag{6}$$

where $\theta_{(1)}$ is the smallest realized cost, and τ_1^* is the bidder's corresponding optimal stopping time for investment according to Lemmas 1, 3, and 4, with the cutoff type $\hat{\theta}$ given therein.

The seller's payoff thus depends on the "virtual valuation" of the best type rather than the actual valuation.[26] The seller essentially owns the best type's real option with an additional stochastic cost. In general, the winning bidder's optimal investment timing differs from the seller's. This proposition, together with the bidding equilibria for formal auctions derived earlier, allow computational ranking of various security designs in either FPAs or SPAs.

Importantly, security ranking depends on auction timing as seen in Figure 4(a): among several pure contingent securities, equity gives the highest expected revenue and call option the lowest at $P_a = 280$, whereas call option is the highest and debt is the lowest at $P_a = 360$. The worst security design at $P_a = 300$ more than doubles the revenue from the best security design at $P_a = 220$. Welfare is similarly affected (Figure 4(b)). In this regard, strategic timing is as important as security design. Security ranking also depends on parameters such

[24]Standard security bids is a special case under this general form. Though written in flow payment, S could be a lump-sum payment when it is a Delta function. For standard security bids, $\mathcal{I}_\tau$ contains cash flow from project P_τ when invested at τ, but in general $\mathcal{I}_t$ could include the history of P up to t, and t itself if they are contractible.

[25]This assumption is standard in the auctions literature, for example, see Krishna (2009). One sufficient condition is the "inverse hazard function" $F(\theta)/f(\theta)$'s being non-decreasing. The general intuition still applies without this assumption, though one has to introduce "ironing" techniques which complicates the discussion.

[26]This payoff is equivalent to the expected marginal revenue (MR), see Bulow and Roberts (1989).

as N and σ, such as shown in Figure 5. In sum, security ranking has to be considered in conjunction with potential misalignment of incentives, timing of auctions, and the number of bidders.

Endogenous Auction Timing

In formal auctions, a seller also chooses time t_a to hold an auction to maximize

$$\mathbb{E}\left[e^{-rt_a}1_{\{\theta_{(1)}\leq\hat{\theta}\}}\left[e^{-r\tau_1^*}(P_{\tau_1^*}-z(\theta_1))-X-Y\right]\right], \tag{7}$$

where $\hat{\theta}$ and τ_i^* depend on both auction timing and security design.

Theorem 1. *Given a security design and allocation rule, an optimal threshold strategy for timing a formal auction exists. The auctioneer inefficiently delays the auction and never sells the project when she expects no chance of immediate investment.*

Intuitively, option values erode as P_a increases, thus it would not be optimal to postpone the auction indefinitely. But because the seller effectively bears $X+Y$, she can profitably delay the incidence of this cost, especially if she expects no bidder to invest right away and she can wait for greater participation. The bigger $X+Y$ is, the more the seller endogenously delays the auction.[27] As the seller does not get the information rent, she faces lower virtual valuation the auction surplus than a social planner, and inefficiently delays the auction. The inefficient delay depends on the specific security used, as the latter affects auction timing through τ_1^*, yet the result holds even in cash auctions, because as long as virtual valuation differs from true option value, changing auction timing leads to substantial variations in revenue. Figure 6 illustrates this effect by plotting time zero present values of the expected revenues and welfare from cash auctions held when P_t first reaches P_a. Hence, for regulators concerned with welfare, auction timing is as important a consideration as market power.

Optimal Auction

Optimal auction involves both security design and auction timing. Despite the complexity in security ranking, Theorem 2 in Board (2007b) extends to the current setting, and the optimal security design can be implemented using an auction with standard security bids.

[27] $X+Y<0$ clearly implies immediate auctions to avoid missing optimal investments.

Proposition 2. *An optimal auction design exists and FPAs using well-ordered securities indexed by s implement it: denote $\hat{\theta}$ as the solution to $P_a^\beta(\beta-1)^{\beta-1} = (X+Y)\beta^\beta z(\theta)^{\beta-1}$,*

$$C(s) = \begin{cases} \frac{s}{1-\beta}\left[\frac{P_a}{P^*(z(-s))}\right]^\beta - X - \int_{s_L}^{s}\left[\frac{1-F(-s')}{1-F(-s)}\right]^{N-1}\left[\frac{P_a}{P^*(z(-s'))}\right]^\beta ds', & \text{if } s \in [s_L, -\underline{\theta}] \\ C(-\underline{\theta}) + \underline{\theta} + s, & \text{if } s > -\underline{\theta} \end{cases} \quad (8)$$

$$S(s,P) = \phi(s)P, \quad \text{where } \phi(s) = \frac{F(-s)1_{\{s\in[s_L,-\underline{\theta}]\}}}{F(-s) - sf(-s)}, \quad \text{and} \quad s_H = \infty,\ s_L = \max\{-\overline{\theta}, -\hat{\theta}\}.$$

In equilibrium, type θ bids $s = -\theta$. Recall $P^{bonus} = \frac{\beta}{\beta-1}\frac{\theta}{1-\phi} = P^*(z(\theta_i))$, which implies that when the bidder equates his marginal benefit of waiting to his marginal cost of waiting, he and the seller face the same optimization problem. This is exactly Board (2007b)'s insight, but instead of using revenue-independent strike payment $\frac{F(-s)}{f(-s)}$ for option exercises, the seller can use royalty payments to align incentives, and the delays in investments depends on the optimal auction timing, as well as the optimal security. The interpretation of the optimal security as a cash down payment plus a royalty payment relates to common discussions on security-bid auctions, and the contingent payment satisfies limited liability $S(\cdot,P) \in [0,P]$ and double monotoniticy ($S(P)$ and $P - S(P)$ being non-decreasing), as is typically required in security design.[28] Variable royalties with upfront cash are indeed frequently observed in the sales of licensing or marketing rights and contracts in publishing or movie production. These results also show that McAfee and McMillan (1987a)'s optimal linear incentive contracts are robust to time discounting, despite the fact that the discounted project payoff is actually decreasing in contractible output P.

Note that the royalty rate is increasing in θ if and only if $\frac{\theta}{z(\theta)}$ is decreasing in θ. The type with smaller actual cost relative to virtual cost thus pays less upfront cash and higher royalty rate. This result makes sense as higher royalty rate is needed to make him invest as if he bears the virtual cost. This prevents him from mimicking others, lest he pays more cash, and has a contingent residual that is more sensitive to his investment timing, which is more distorted in equilibrium. Optimal security thus involves negatively correlated cash down payments and contingent royalty payments, a novel and testable prediction that is of interests for empirical studies.

In Appendix A.9, I show Theorem 1 also generalizes to the case where we do not constraint

[28] See Hart and Moore (1995), DeMarzo and Duffie (1999), and DeMarzo, Kremer, and Skrzypacz (2005).

the security design and allocation rule:

Corollary 1. *Optimal formal auction occurs later than a constrained-efficient formal auction.*

These results relate to Myerson (1981)'s analysis in a static setting on the wedge between the seller's revenue and welfare. In addition to bidder exclusion, option exercises are inefficiently delayed under the current settings to increase revenue (see Board (2007b) and Proposition 2). Moreover, auction timing leads to several distinct features in an optimal auction. Although the seller still excludes bidders, the auction is held under better market conditions (higher P_a), which encourages participation and mitigates the exclusion. This implies that in real life one may not see sellers excluding bidders as much using entry fees or reserve prices, because she has the alternative tool of choosing a more propitious time to hold the auction. For example, an entrepreneur selling a startup seldom excludes potential acquirers, but rather waits for the product to have a higher valuations before going onto the market. Inefficiencies in dynamic settings are thus multi-dimensional.

3.3 Implication for Oil Lease Auctions

One key application of the model is the sales of natural resources, such as oil lease auctions. In many countries, the predominant design for auctioning oil leases involves fixing a royalty rate ϕ and having contractors bid up-front "bonus" in FPAs.[29]. As such auctions are typically modeled as auctions with interdependent values, I generalize in Appendix A.10 Propositions 1 and 2 to interdependent settings and show that the inefficient delay in drilling is increasing and convex in volatility and royalty rate, and has positive cross partials. The social cost of the investment lag due to the distortion associated with royalty is increasing and convex in the royalty rate ϕ.

These predictions are consistent with available empirical evidence. The US Department of the Interior experimented with royalty auctions in 1978–1983, where the government fixed a small up-front "bonus" payment and allowed the bidders to compete on royalty rates.

[29] In the United States, the Minerals Lands Leasing Act prescribes the base share of royalty rate at 1/8 the value of production for onshore leases, and the Outer Continental Shelf Lands Act used 1/6 for offshore leases. The offshore rate for leasing beginning in 2008 is set at 18.75%. See Hendricks, Porter, and Tan (1993) and Haile, Hendricks, and Porter (2010) for more details.

Many bidders bid extremely high royalty rates and the tracts were never drilled.[30] Oil price and volatility were indeed extremely high during that period. Moreover, Humphries (2009) reports that the royalty relief programs in the 1990s significantly increased interest in deep-water leases, and oil production increased sharply. Opaluch, Grigalunas, Anderson, Trandafir, and Jin (2010) and Cong (2013) also conclude that increased royalty rates would have a net negative effect on the social value of offshore development. The distortion gives a potential explanation for why large tracts of land remain idle.[31] Without prescribing detailed policy changes, this paper suggests that any useful policy recommendations should first focus on reducing the post-auction moral hazard that is inimical to both the revenue and social welfare. Moreover, instead of uniformly raising the royalty rate, allowing bidders to self select into differential rates as described in Proposition 2 could be a more effective way in to increase revenue to the government, in both private-value and common-value frameworks.

4 Informal Auctions

Many economic interactions such as corporate takeovers, competition for supply contracts, and talent recruitment have characteristics of auctions because buyers are competing with one another to make offers.[32]. Yet unlike formal auctions in which the sellers restrict security bids to a pre-specified ordered set, bidders often come up with their own offer terms that sellers typically cannot ignore, essentially leaving the security design of the auction to the bidders who can bid any contingent payment. A seller would consider all bids and choose the most desirable one ex post. Because of the lack of commitment to explicit auction design, I follow DKS and call these transactions informal auctions.[33] For example, in Shire's acquisition of NPS Pharmaceutical, Shire repeatedly proposed various deal terms and NPS

[30] See Dougherty and Lohrenz (1980) and Binmore and Klemperer (2002).

[31] In reality, many other strategic interactions among the bidders complicate the issue. For example, Beshears (2011) shows alliances in oil and gas drilling perform better than solo bidders; Hendricks and Porter (1996) attributes the delays in exploratory drilling to free-rider problem and war of attrition. The above analysis complements these studies.

[32] When an asset of a Delaware corporation is for sale, the Revlon rule imposes upon directors a duty to solicit competitive bids to maximize shareholders' value. It may seem that many takeovers occur after one-on-one negotiations, but as demonstrated in Aktas, De Bodt, and Roll (2010), even in such cases latent competition such as the threat of sale to a rival buyer is significant.

[33] DKS do not consider offer adjustments from winning and losing bidders as I do in ascending informal auctions defined later.

was considering deals with other pharmaceutical companies as well.

Investments are always efficient conditional on auction timing when the seller cannot commit to pre-specified security design, and in equilibrium, every bid is equivalent to cash. This strengthens the conclusion in DKS: cash is not only the cheapest way for a better type to separate from worse types; it is also the most efficient way. This section further shows the seller times the auctions inefficiently late, and when the bidders can initiate in an ascending informal auction, they always do so in equilibrium. Because auction timing, bidding, and investment involve sequential actions, the equilibrium concept for informal auctions is **Perfect Bayesian Equilibrium**.

4.1 A Signaling and Timing Game

If the seller commits to neither a pre-specified timing of the auction nor a bidding and allocation rule, she holds the auction at the most opportune time, and then chooses the bid that gives her the highest expected payoff based on her beliefs regarding the type of each bidder at the time the auction is held. A **first-price informal auction** therefore exhibits features of a signaling and timing game of the following form:

1. The seller initiates the auction at some time $t_a \geq 0$, the bidders learn the support $[\underline{\theta}, \overline{\theta}]$ and their own types.

2. Participating bidders submit informal bids simultaneously. An informal bid Π^i by bidder i is a cash payment C^i and a standard security payment $S^i(P)$.

3. The seller chooses the winning bidder rationally according to the valuation function $R(\Pi^i) = C^i + \mathbb{E}[R_\theta(S^i)|\Theta(\Pi^i)]$ provided she values the bid more than the reservation value Y. $\Theta(\Pi^i)$ is her belief of bidder i's type upon seeing the bid and all available information, and $R_\theta(S^i) = \mathbb{E}[e^{-r\tau_\theta^i} S^i(P_{\tau_\theta^i})]$, where τ_θ^i is the optimal stopping rule for type θ when bidding Π^i, that is, $\tau_\theta^i = \operatorname{argmax}_{\tau \geq t_a} \mathbb{E}[e^{-r\tau}(P_\tau - S^i(P_\tau) - \theta)] - X - C^i$.

4. The winning bidder i pays the upfront cash C^i and the initial cost X at t_a, and then invests rationally at $\tau_{\theta_i}^i$ and makes the contingent payment.

Note the seller's valuation $R(\Pi^i)$ is not necessarily the same as the value of the security to bidder i, $C^i + R_{\theta_i}(S^i)$. One may question if the setup of the game misses out on any informal offers, such as contracting on the timing of investment when feasible. The results are robust to additional side contracts because one can enlarge the security space from $S(P_t)$ to $S(\mathcal{I}_t)$ where $\mathcal{I}_t$ is the entire contractible information set, as long as limited liabilities hold. The proofs apply with minor changes in notations.[34]

I work backward to solve for the bidding equilibrium before deriving the seller's endogenous timing of the auction.

Lemma 5. *The seller and a participating bidder i have the same valuation for bid Π^i, that is, $R(\Pi^i) = C^i + R_{\theta_i}(S^i)$.*

If this were not the case, at least one bidder would find the seller values his security payment less than he does, and would rather pay the seller's valuation in cash, which keeps his marginal probability of winning the same.

Lemma 6. *In a bidding equilibrium, a participating bidder i has $\tau^i_{\theta_i} = \tau^*_i$, where τ^*_i is the stopping time corresponding to the threshold strategy with investment trigger $P^*(\theta_i)$.*

The intuition is that if a bidder does not invest efficiently upon winning, he can always deviate to a bid that results in efficient investment, and offer more cash to the seller to increase his marginal probability of winning without reducing the payoff upon winning.

Lemma 7. *Informal auctions only admit fully-separating equilibria.*

Because every bidder upon winning invests efficiently, a better type generates greater social surplus and can offer more to separate from worse types. This lemma also implies that no two bidders place the same bid. These results lead to the next key result of the paper:

Theorem 2. *An essentially unique bidding equilibrium exists for an informal FPA, which is equivalent, in terms of allocation outcome and expected payoffs, to a first-price cash auction with reserve price Y. In particular, post-auction investment is efficient.*

In equilibrium, the bids are all cash-like, that is, their values are independent of beliefs on bidders' types. A better type finds it cheaper to use a security that is less sensitive to the true type and creates more social surplus. For example, using equities to separate from worse types not only inefficiently delays investment, but also costs better types more

[34] The results also hold for non-standard security bids as long as optimal investment strategy exists.

because their α shares are worth more than the worse types'. Cash-like securities ensure efficient investment and cheap separation and are thus most attractive. Moreover, because a better type is indifferent to mimicking a marginally worse type in equilibrium, all bidders must be using cash-like securities.

Given Proposition 2, and the revenue equivalence between FPAs and SPAs in cash, the seller's auction timing problem is equivalent to the strategic timing of a first-price or second-price cash auction. From Theorem 1,

Corollary 2. *The seller inefficiently delays holding a first-price informal auction.*

The intuition is the same as in timing cash auctions. A welfare maximizer initiates only when the auction payoff exceeds the cost of ownership transfer by a certain threshold. But the seller faces the additional cost in the form of information rent paid to the winning bidder; thus, her option value starts to erode only with higher P_a, commanding a higher option premium for holding the auction. Compared to formal auctions, informal auctions are inefficient solely due to auction timing.

4.2 Ascending Informal Auctions

As McAdams and Schwarz (2007) point out, in real life committing to a sealed-bid auction is hard, especially in corporate acquisitions.[35] The board of directors of a target firm has to disclose all bids to shareholders, and considers subsequent offers to avoid shareholder lawsuits. In reality, informal auctions either entail sellers and buyers' engagement in multiple rounds of negotiations and repeated communications, or manifest themselves in two-stage auctions used in privatization, takeover, and merger and acquisitions.[36] The former resembles an informal English auction in which buyers raise their bids until one winner emerges, Perry, Wolfstetter, and Zamir (2000) show the latter are typically robust mechanisms equivalent to an English auction. This calls for the definition of an **ascending informal auction**:

1. The seller initiates the auction at some time t_a and all agents enter the bidding stage.

[35] Even in formal auctions, such a commitment is difficult to maintain. in "Lawsuit Seeks to Block Sale of G.M. Building", New York Times, September 20, 2003, Charles Bagli documents how General Motors entertained a late offer after auctioning its Manhattan building in a first-price auction.

[36] For example, see Frankel (2011).

2. The seller gradually increases a numerical score R from $R = Y$, and a bidder remains in the auction if he can deliver an informal bid from a "feasible set" $\{\Pi : R(\Pi) \geq R\}$. The auction ends when only one bidder is left, and he chooses an informal bid from the final "feasible set."

3. The winning bidder i pays the upfront cash C^i and the initial cost X at t_a, and then invests rationally at $\tau^i_{\theta_i}$ and makes the contingent payment $S^i(P_{\tau^i_{\theta_i}})$, where C^i and S^i are given by his chosen final bid.

Note this variant of the English auction is equivalent to SPAs, in which bidders bid a score they generate and the winner pays the second-highest score bid.[37] This a priori is different from SPAs in which the winning bidder pays the informal bid corresponding to the second highest score. This distinction is important because the same security bids generally cost the buyers differently. In what follows, I show this ascending informal auction leads to the same auction timing and expected revenue for the seller, and is not subject to renegotiation from the bidders.

Lemma 5 applies to the winner's final bid and the losers' drop-out bids, and since bidder i can bid a maximum value of $W(P_a; \theta_i) - X$, bidding until the score reaches this value is an undominated strategy. This implies Lemma 7 is true for ascending informal auctions. Finally, Lemma 6 applies to the winner's bid; otherwise, a profitable deviation using cash exists.

Theorem 3. *An ascending informal auction has an essentially unique bidding equilibrium that is equivalent to a second-price cash auction with reserve price Y. Post-auction investment is efficient, and the optimal auction timing strategy is the same as in cash auctions and first-price informal auctions.*

Theorems 2 and 3 establish useful benchmark outcomes for endogenous timing, informal bidding, and post-auction investment. They do not imply bidders only use cash in informal auctions, but when bidders are not liquidity constrained, they would use contingent securities that result in the same outcomes as using cash. Moreover, other complicating factors exist

[37] Defining an ascending auction with multiple security bids is challenging, and Gorbenko and Malenko (2014), another pioneering study to formalize an English auction with security bids (both cash and equity), is closely related to this paper.

in reality, such as the seller's informational advantage, or risk-sharing and tax considerations that would alter the results. For example, Malmendier, Opp, and Saidi (2012) show that when an acquirer firm's stocks are overvalued, the acquirer tends to use stocks as a payment, but when such issues are absent, it indeed uses cash - consistent with this paper's findings. This paper abstracts from these considerations and focuses on signaling through contingent securities and post-auction moral hazard of investment timing - key issues in the literature of security-bid auctions.

4.3 Bidder Initiation

So far, we have assumed the bidders have no knowledge of the investment option until the auction, which is realistic when the seller possesses key proprietary information about the asset that is costly to reveal or communicate prior to the auction. Yet, in real life, especially in M&As and patent sales, we often see bidders who possess information about the asset and can initiate the auction.[38] This section analyzes the case in which "informed" bidders know their types and the support of types prior to the auction and both seller and bidders can initiate. The fact that all parties dynamically update their beliefs about the distribution of types complicates the game. Although a bidding equilibrium with first-price informal auction can be derived that involves mixing strategies by the initiating bidder, it does not survive bidders' offer adjustments and renegotiations. Therefore I focus on the more realistic ascending informal auctions and on symmetric equilibria with a weakly monotone threshold for initiation; that is, if $\theta < \theta'$, the initiation threshold for bidder θ is weakly lower than that for θ'.

To avoid extended discussions of off-equilibrium-path beliefs and bidders' signaling to the seller, let us restrict our attention to cases in which the seller does not observe the initiating bidder's identity, which is the case in oil and gas auctions.[39] The ensuing analysis would also apply to cases in which initiations involve cash-like offers only or the seller does not form beliefs about the initiating type. Basically, this assumption allows us to isolate the learning and timing aspect of bidders' initiation.[40] I also assume indifference in timing is resolved by

[38] See Fidrmuc, Roosenboom, Paap, and Teunissen (2012), and Gorbenko and Malenko (2014).

[39] Prior to the introduction of Area Wide Leasing (AWL) in May 1983, energy firms could nominate oil and gas tracts to be auctioned but the seller does not use the nomination record. Firms thus could initiate an auction without being identified.

[40] This setup still allows signaling to other bidders. If the bidder could also signal his identity to the seller

initiating later.[41]

Proposition 3. *With informed bidders, an ascending informal auction admits an essentially unique auction timing equilibrium*[42]*, whereby bidders always initiate with threshold* $\max\{P_I(\theta), P_0\}$ *where* $P_I(\theta)$ *is weakly increasing in* θ *and uniquely solves*

$$\int_{\theta}^{\hat{\theta}} \frac{d}{dP} \left. \frac{W(P;\theta) - W(P;\theta')}{P^{\beta}} \right|_{P=P_I} f(\theta')[1-F(\theta')]^{N-2} d\theta' = 0, \tag{9}$$

and $\hat{\theta}$ *solves* $W(P_I;\hat{\theta}) = X + Y$.

The intuition is that if the auction has not been initiated at P, everyone updates their beliefs about types that are present. The seller times the auction to maximize the second-highest valuation, whereas type $P_I^{-1}(P)$ times the auction to maximize the present value of informational rent (difference between his valuation and the second-highest valuation). The latter starts to erode earlier than the former as the initiation threshold P_a increases. Therefore, the seller always waits in such an equilibrium.

The prediction that bidders initiate when informed is broadly consistent with empirical evidence. For example, the above result predicts that a bidder initiates only when his real option is in the money, which implies an investment option is, on average, exercised more quickly when bidders initiate than when a seller initiates strategically or randomly. Cong (2013) uses the data on leasing and exploration of oil and gas tracts in the Gulf of Mexico with over 20,000 leases to test this implication, and by estimating a Cox proportional hazards model with time-varying covariates for the window of 1978-1989, finds that when bidders could initiate before the implementation of Area-Wide Leasing (AWL) in May 1983, they did so and explore-drilled at least 10% faster than after AWL. This difference translates into waiting time being doubled after AWL. Another example is that patent holders rarely organize an auction and instead are often approached by acquirers when details of the patent are public information. Also, because strategic bidders are more likely to have

via initiation, the bidding strategy would depend on auction initiation. In the context of ascending informal auction, we would need complicated restrictions of off-equilibrium-path beliefs that are beyond the scope of this paper. Gorbenko and Malenko (2014) consider such signaling effect in details.

[41] This assumption can be formally justified by a small initiation cost, then taking the cost to zero.

[42] This is a Markovian Perfect Bayesian Equilibrium if we use both P_t, and $\hat{P} := \sup\{P_{t'}, t' \leq t\}$ as state variables, though on the equilibrium path they coincide

information regarding valuation than financial bidders, acquisitions by strategic bidders in informal negotiations are primarily bidder-initiated, whereas private equity deals are more often target-initiated.[43] The analysis here can be extended to situations, in which the bidders are informed with certain probabilities, in which case the seller initiates the auction if the informed bidders with highest valuations are absent. This result implies that when the seller initiates, the surplus generated tends to be smaller, consistent with the fact that the adjusted acquisition premium is lower for target-initiated deals (see Fidrmuc, Roosenboom, Paap, and Teunissen (2012)).

5 Conclusion

Auctions of real options are ubiquitous, involve tremendous financial resources, and have policy implications. To better understand these business transactions, this paper extends prior studies on security-bid auctions and real options to incorporate endogenous auction timing and moral hazard of option exercise. I find that both endogenous auction initiation and the seller's commitment to the auction design significantly influences equilibrium outcomes and are integral to selling real options. Commitments to common security designs lead to inefficient and often sub-optimal accelerations or delays in investment, but without such commitments, the bidding equilibria are equivalent to cash auctions, and post-auction investments are efficient. Most auctions are inefficiently timed, and informed bidders always initiate when feasible. I also find that optimal auction design involves combinations of cash and royalty payments in real life, and entails inefficient sales and investments. Taken together, the results of the paper challenge earlier approaches that analyze auction initiation, security design, and corporate investments separately: the interactions of these factors in dynamic settings provide a rich interplay that is not accessible otherwise, and as a consequence, many conventional beliefs should be revised.

As an initial attempt to capture the salient features of auctions of real options under various settings, this paper adds insights to security bids, endogenous initiation, and agency issues in the real-options framework, and helps understand real-life observations. More work

[43]For example, Fidrmuc, Roosenboom, Paap, and Teunissen (2012) document almost 80% are bidder-initiated. Note the current model is more applicable to strategic acquisitions, in which bidders are more likely to have private information regarding valuation than in financial acquisitions.

is clearly needed, in particular, selling real options with renegotiation and resales is worth exploring further. Incorporating sellers; private information is also important in many applications, especially M&As. Moreover, some of the novel predictions are consistent with stylized facts, which is reassuring, and further empirical examinations may reveal more quantitative relations.

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Appendix: Derivations and Proofs

A.1 Proof of Lemma 1

Proof. First note $\tilde{V} \in [-X - C, W(P_a;\theta) - X - C]$, thus the valuation is finite. The value function of the optimal stopping is thus the infimum of a class of $\mathcal{C}^2$ functions with non-positive drift that majorize $P - S(P) - \theta$, and the stopping time is first-hitting. (Proposition 5.8, and 5.10 in Harrison (2013)). Therefore,

$$\tilde{V}(C, S(\cdot), \theta) = D(P_a; P_L, P_U)\,[P_L - S(P_L) - \theta] + D(P_a; P_U, P_L)\,[P_U - S(P_U) - \theta] - C - X,$$

where $D(P_a; P_L, P_U)$ is the Arrow-Debreu security that pays one dollar when P first hits before P_L before hitting P_U, and $D(P_a; P_U, P_L)$ is similarly defined. And $P_L \in [0, P_a]$ and $P_U \in [P_a, \infty]$ are the optimal lower and upper thresholds for investment. $D(P_a; P_L, P_H)$ satisfies $\frac{1}{2}\sigma^2 P^2 D_{PP} + \mu P D_P - rD = 0$ with the boundary conditions $D(P_L; P_L, P_U) = 1$ and $D(P_U; P_L, P_U) = 0$. The solution is $D(P_a; P_L, P_U) = \left(P_a^\beta - P_a^\gamma P_U^{\beta-\gamma}\right)\left(P_L^\beta - P_L^\gamma P_U^{\beta-\gamma}\right)^{-1}$, and similarly, $D(P_a; P_U, P_L) = \left(P_a^\beta - P_a^\gamma P_L^{\beta-\gamma}\right)\left(P_U^\beta - P_U^\gamma P_L^{\beta-\gamma}\right)^{-1}$, where β is given in (2) and $\gamma = 1 - 2\mu/\sigma - \beta < 0$. The optimal P_L and P_H are obviously independent of X and C and are functions of θ and P_a in general.

Finally, type θ can always do strictly better than $\tilde{\theta} > \theta$ by using $\tilde{\theta}$'s strategy, thus $\tilde{V}(C, S(\cdot), \theta)$ is decreasing in θ. For P_t in the exercise region, $\tilde{V}(C, S(\cdot), \theta) = P_t - S(P_t) - \theta - C - X$ is obviously continuous in θ. For P_t in the continuation region, consider a change of $\Delta\theta > 0$, $0 < \tilde{V}(C, S(\cdot), \theta) - \tilde{V}(C, S(\cdot), \theta + \Delta\theta) \le \Delta\theta$ because type $\theta + \Delta\theta$ does weakly better than simply mimicking θ's strategy. As $\Delta\theta \to 0$, $\tilde{V}(C, S(\cdot), \theta + \Delta\theta) \to \tilde{V}(C, S(\cdot), \theta)$. The case of $\Delta\theta < 0$ is similar. Continuity in θ follows. $\square$

A.2 Proof of Lemma 2

Proof. Since an upper threshold strategy has payoff $\left(\frac{P_a}{P}\right)^\beta [P - S(P) - \theta]$ for $P \ge P_a$, threshold $\tilde{P}$ is optimal among all upper threshold strategies. I now verify that it is optimal among all stopping times by showing the expected value following any stopping time is bounded above by the expected value associated with the $\tilde{P}$-threshold strategy.

Let $x_t = e^{-rt}\hat{W}(P_t)$, where $\hat{W}(P_t) = D(P_t; \hat{P})[\hat{P} - S(s, \hat{P}) - \theta]$ and $\hat{P} = \max\{P_t, \tilde{P}\}$. For $P \le \tilde{P}$, using an extended version of Itô's formula (as, for example, in Karatzas and Shreve (1988), page 219), $dx_t = e^{-rt}[\mathcal{D}\hat{W}(P_t) - r\hat{W}(P_t)]dt + e^{-rt}\hat{W}_P(P_t)\sigma P_t dB_t$, where $\mathcal{D}\hat{W}(P) = \hat{W}_P(P)\mu P + \frac{1}{2}\hat{W}_{PP}(P)\sigma^2 P^2$. $\hat{W}_P$ is bounded as seen by direct computation, thus by Proposition 5B in Duffie (2009) (also found in Protter (2004)), the last term in dx_t is a martingale under the current measure. The drift is $\mathcal{D}\hat{W}(P) - r\hat{W}(P) = 0$ by the definition of β in (2). For $P > \tilde{P}$, apply Tanaka's Formula (Revuz and Yor (1999), also Karatzas and Shreve (1988)), the drift $\mathcal{D}\hat{W}(P) - r\hat{W}(P) = \mu P[1 - S'(P)] - r[P - S(P) - \theta] - \frac{1}{2}\sigma^2 P^2 S''(P) < [\mu + \frac{1}{2}(\beta - 1)\sigma^2]P[1 - S'(P)] - r[P - S(P) - \theta] < [\mu\beta + \frac{1}{2}\sigma^2\beta(\beta - 1) - r][P - S(P) - \theta] = 0$, using (Conv) and the definition of β. Since to the discounted occupancy measure, there is a discounted local time l (Stokey (2009), Theorems 3.6 and 3.7), the additional local time term in dx_t when $S'(P)$ jumps is $\frac{1}{2}\sigma^2 \int_{\mathbb{R}+} l(P, t, r)\nu(dP)$, where $\nu((a, b]) \equiv \hat{W}'(b) - \hat{W}'(a)$, is non-positive due to (Conv). Therefore, x_t is a super-martingale, implying for any stopping time τ, $\hat{W}(P_0) = x_0 \ge \mathbb{E}[x_\tau] = \mathbb{E}[e^{-r\tau}\hat{W}(P_\tau)] \ge \mathbb{E}[e^{-r\tau}(P_\tau - S(s, P_\tau) - \theta)]$. The equality holds for the first-hitting time with threshold $\tilde{P}$, establishing its optimality. Finally, when $P - S(P)$ is non-decreasing in P, $\tilde{V}$ is non-decreasing in P_a since the optimal exercise involves upper-threshold only. $\square$

A.3 Characterization of Investment Distortion

Relative to what is socially efficient ex post the auction, **(a)** a project rationally invested at P is weakly delayed if $\beta S(P) - PS'(P) > 0$, and weakly accelerated if $\beta S(P) - PS'(P) < 0$, regardless of the winning bidder's type; **(b)** a winning bidder of type θ will invest weakly late if $(\beta-1)S\left(\frac{\beta}{\beta-1}\theta\right) > \theta S'\left(\frac{\beta}{\beta-1}\theta\right)$, and weakly early if $(\beta-1)S\left(\frac{\beta}{\beta-1}\theta\right) < \theta S'\left(\frac{\beta}{\beta-1}\theta\right)$.

These results follow directly from Lemma 2 and (Conv). If a project rationally invested at P is accelerated, then $\beta\theta - (\beta-1)P > 0$. If in addition $\beta S(P) - PS'(P)$ exists and is positive, waiting for a slightly higher P for exercise would be better by (Conv), contradicting the project is rationally invested. Therefore $\beta S(P) - PS'(P) > 0$ implies a project rationally invested cannot be accelerated. Similarly, $\beta S(P) - PS'(P) < 0$ implies a project rationally invested cannot be delayed. Next if $(\beta-1)S\left(\frac{\beta}{\beta-1}\theta\right) > \theta S'\left(\frac{\beta}{\beta-1}\theta\right)$, a winning bidder of type θ would prefer delaying further at $P = \frac{\beta}{\beta-1}\theta$. With (Conc), the optimal threshold must be weakly higher than $P = P^*(\theta)$, hence the investment is weakly delayed. The rest of result (b) follows a similar argument.

Basically if the contingent payment as a fraction of total cash flow grows too quickly (slowly) as cash flow increases, the bidder may choose invest earlier (later) than what is socially efficient.

A.4 Proof of Lemma 3

Proof. For $s_1 < s_2$ and $\theta_1 < \theta_2$, because $V(s,\theta)$ is absolutely continuous with derivative in s decreasing in θ,

$$\ln\left(\frac{V(s_1,\theta_1)V(s_2,\theta_2)}{V(s_1,\theta_2)V(s_2,\theta_1)}\right) = \int_{s_1}^{s_2} \frac{\partial V(s',\theta_2)}{\partial s} ds' - \int_{s_1}^{s_2} \frac{\partial V(s',\theta_1)}{\partial s} ds' < 0 \tag{10}$$

i.e., $V(s,\theta)$ is log-submodular, and thus strictly submodular. Let $Q(s)$ be the probability of winning. Because $s(\theta) \in argmax_s Q(s)V(s,\theta) = argmax_s \ln(Q(s)V(s,\theta))$, by Topkis (1978), $s(\theta)$ is non-increasing in θ. If $s(\theta) < s_H$ were constant on an interval, the bidder with the lower θ can increase his bid marginally and increase his probability of winning (thus his payoff) by a discrete amount. Therefore $s(\theta)$ must be decreasing in type for types bidding less than s_H. Therefore, $Q(s(\theta)) = [1-F(\theta)]^{N-1}$. Note s is also continuous in θ, lest a type right below a discontinuity could lower his bid marginally without affecting the chance of winning.

Next, by direct revelation, $\theta \in \operatorname{argmax}_{\theta' \in [\underline{\theta},\overline{\theta}]} Q(s(\theta'))V(s(\theta'),\theta)$. For any $\theta' < \theta$,

$$Q(s(\theta))V(s(\theta),\theta) \geq Q(s(\theta'))V(s(\theta'),\theta) = Q(s(\theta'))[V(s(\theta),\theta) + V_1(s^*,\theta)[s(\theta') - s(\theta)]]$$

for some s^* between $s(\theta')$ and $s(\theta)$. Since $V_1 < 0$, the above expression can be written as

$$\frac{Q(s(\theta')) - Q(s(\theta))}{\theta' - \theta} \frac{V(s(\theta),\theta)}{-Q(s(\theta'))V_1(s^*,\theta)} \geq \frac{s(\theta') - s(\theta)}{\theta' - \theta}$$

Similarly, exchanging θ and θ', for some s^{**} between $s(\theta)$ and $s(\theta')$,

$$\frac{Q(s(\theta')) - Q(s(\theta))}{\theta' - \theta} \frac{V(s(\theta'),\theta')}{-Q(s(\theta))V_1(s^{**},\theta')} \leq \frac{s(\theta') - s(\theta)}{\theta' - \theta}$$

Taking the limit we get (5).

As $V(s,\theta)$ is continuous in s over $[s_L, s_H]$ and decreasing in θ, $\max_s V(s,\theta)$ exists and $\sup\{\theta \leq \overline{\theta} | [1 -$

$F(\hat{\theta})]^{N-1} \max_s V(s, \hat{\theta}) \geq 0\}$ gives the cutoff type. In equilibrium, $s(\hat{\theta}) = \sup\{s \in [s_L, s_H] | V(s, \hat{\theta}) \geq 0\}$, otherwise bidding slightly more increases the winning probability discretely from zero while still breaking even upon winning. As $V(s(\hat{\theta}), \hat{\theta}) \leq W(P_a; \hat{\theta}) - X$ and $W(P_a; \hat{\theta}) - X = 0$ in cash auctions, the cutoff type for security bids is in general weakly smaller than that in cash auctions. With the absolute continuity assumption in the proposition, the cutoffs are the same as in cash auctions.

This establishes uniqueness of the equilibrium, whose existence follows from the sufficiency of bidders' F.O.C. - the quasiconcavity of $\ln(Q(s)V(s,\theta))$. For any $s' \in (s(0), s(\theta))$, $\exists \theta' \in (0, \theta)$ such that $s(\theta') = s'$. Submodularity of V implies $\frac{\partial}{\partial s} \ln[Q(s')V(s',\theta)] > \frac{\partial}{\partial s} \ln[Q(s')V(s',\theta')] = 0$. Similarly, $\frac{\partial}{\partial s} \ln[Q(s')V(s',\theta)] < 0$ for $s' \in (s(\theta), s(\underline{\theta}))$. Therefore for every θ, there exists a unique s maximizing $Q(s)V(s,\theta)$. □

A.5 Proof of Lemma 4

Proof. Since Π is a left-continuous map, $V(s,\theta)$ is left-continuous in s by an argument similar to the one in Lemma 1 for $V(s,\theta)$ to be continuous in θ. Therefore $s(\theta)$ is well-defined. Suppose a participating bidder of type θ bids $s > s(\theta)$, he benefits from decreasing s to reduce the states of the world in which he wins but receives negative payoff. Similarly, he wants to increase s when $s < s(\theta)$, assuming any indifference in bidding is resolved by bidding higher. As $V(s,\theta)$ is decreasing in θ, for $\theta' > \theta$, $V(s(\theta),\theta') < V(s(\theta),\theta) = 0 = V(s(\theta'),\theta')$. Thus $s(\theta) > s(\theta')$, leading to $s(\theta)$ being decreasing. The cut-off type is the same as in FPAs by an argument similar to that in the proof of Lemma 3. □

A.6 Discussion of Cash Dominating M-regular Securities

Proof. Suppose $\underline{s}$ is the security the type $\underline{\theta}$ bids, without loss of generality, $b_i(\underline{s}) \leq b_j(\underline{s})$ if $i \leq j$. I define ***M-regular security*** to be a class of contingent securities in the form $\sum_{i \in I} a_i(s)[P - b_i(s)]^+$, where I is a countable set and $\sum_i a_i(s) \leq 1 \; \forall s$, such that for $M > 0$, and $\frac{\beta}{\beta-1}\underline{\theta} \in [b_m, b_{m+1})$,

$$\min\left(\left|\frac{\beta}{\beta-1}\underline{\theta} - b_m\right|, \left|\frac{\beta}{\beta-1}\underline{\theta} - b_{m+1}\right|, \left|\sum_{i \leq m} a_i b_i - \underline{\theta}\sum_{i \leq m} a_i\right|\right) > M. \tag{11}$$

Most common securities are M-regular securities or can be closely approximated by M-regular securities. For example, equity corresponds to $a_1 = \alpha(\underline{\theta})$, $a_2 = b_1 = 0$, $b_2 = \infty$. For any $M > 0$, cash bids dominate M-regular securities in FPAs and SPAs in terms of expected revenue and social welfare, as the number of bidders gets large.

To show this, consider pure contingent securities. Extension to include cash is straightforward. Conditional on an auction timing, cash auctions lead to efficient investments and obviously dominate in terms of welfare. For the seller's revenue, first consider SPAs. The revenue is $\mathbb{E}[e^{-r\tilde{\tau}} S(s(\theta_{(2)}), P_{\tilde{\tau}}) 1_{\{\theta_{(2)} \leq \hat{\theta}\}}]$ $= \mathbb{E}[\big(e^{-r\tilde{\tau}}(P_{\tilde{\tau}} - \theta_{(1)}) - U(\tilde{\tau}, s(\theta_{(2)}), \theta_{(1)})\big) 1_{\{\theta_{(2)} \leq \hat{\theta}\}}] \leq \mathbb{E}[\big(e^{-r\tilde{\tau}}(P_{\tilde{\tau}} - \theta_{(1)}) - X\big) 1_{\{\theta_{(1)} \leq \hat{\theta}\}}] \equiv R_0$, where $\tilde{\tau} = argmax_\tau U(\tau, s(\theta_{(2)}), \theta_{(1)})$ and $U(\tau, s, \theta) = \mathbb{E}[e^{-r\tau}(P_\tau - S(s, P_\tau) - \theta)]$. Similarly in FPAs, the revenue is bounded above by R_0 with $\tilde{\tau} = argmax_\tau U(\tau, s(\theta_{(1)}), \theta_{(1)})$. Let s_w denote the index the winning bidder pays in general. Then in FPAs and SPAs, the revenue is bounded above by R_0 with $\tilde{\tau} = argmax_\tau U(\tau, s_w, \theta_{(1)})$

The revenue from cash auction would be the expected second highest valuation $R_2 \equiv \mathbb{E}[(W(P_a; \theta_{(2)}) - X)1_{\{\theta_{(2)} \leq \hat{\theta}\}}]$. When $N \to \infty$, $\theta_{(2)} - \theta_{(1)} \overset{a.s.}{\to} 0$. Thus $W(P_a; \theta_{(2)}) - W(P_a; \theta_{(1)}) \overset{a.s.}{\to} 0$. Now $1_{\{\theta_{(2)} \leq \hat{\theta}\}}$ and the above are bounded, by bounded convergence, R_2 converges a.s. to $R_1 \equiv \mathbb{E}[(W(P_a; \theta_{(1)}) - X)1_{\{\theta_{(1)} \leq \hat{\theta}\}}]$.

If $R_1 - R_0$ converges to a quantity bounded below by a positive constant, the claims follow. First note $U(\tau, s_w, \theta_{(1)})$ admits an optimal stopping solution involving threshold strategies. To see this, write $U(\tau, s_w, \theta_{(1)}) = D(P_a; P)[P - \theta_{(1)} - \sum_{i \in I} a_i(s_w)[P - b_i(s_w)]^+]$, which admits a maximizer $\tilde{P}(\theta_{(1)})$. Then use that as an investment trigger and apply the standard verification argument. Next, as $\theta_{(1)} - \underline{\theta} \overset{a.s.}{\to} 0$, the investment trigger in cash auctions converges to $P^* = \frac{\beta}{\beta-1}\underline{\theta}$, and $\tilde{P}(\theta_{(1)})$ to $\tilde{P}^* = \tilde{P}(\underline{\theta})$. Whether $\tilde{P}^* \in [b_m, b_{m+1})$ or not, $|\tilde{P}^* - P^*| \geq M$. Since P^* is the optimal trigger for $\mathbb{E}[e^{-r\tau}(P_\tau - \underline{\theta})]$, $R_1 - R_0 \overset{a.s.}{\to} \epsilon$ for some $\epsilon > f(M)$, where $f(M)$ is a function of M that is positive and independent of N. Therefore as N becomes big, R_2 converges to R_1 which dominates R_0 in the limit. Thus cash auctions yield higher revenue than the security-bid auctions. □

A.7 Proof of Proposition 1

Proof. Let $Q(\tilde{\theta}_i, \theta_{-i})$ be the probability of allocating the project to bidder i, who has investment cost $K(\theta_i, \theta_{-i})$, where K is symmetric in other bidders' report types, and has positive derivative in θ_i denoted by K_1 that is uniformly bounded by a constant $A > 0$. In the main model of this paper, $K(\theta_i, \theta_{-i}) = \theta_i$, but this specification allows other cases with interdependent values such as common-value auctions.

The expected utility at time zero to type θ_i upon participating and optimally investing is

$$U(\theta_i, \tilde{\theta}_i) = \mathbb{E}_{\theta_{-i}} \left[Q(\tilde{\theta}_i, \theta_{-i}) \max_{\tau \geq t_a} \mathbb{E}_P \left[e^{-r\tau}(P_\tau - K(\theta_i, \theta_{-i})) - \int_{t_a}^{\infty} e^{-rt} S(\tilde{\theta}_i, \theta_{-i}, \mathcal{I}_t) dt - e^{-rt_a} X \right] \right].$$

As $S(\tilde{\theta}_i, \theta_{-i}, \mathcal{I}_t)$ could be artificially constructed that an optimal stopping time for exercising the real option may not exist, it is reasonable to focus attention on the set of $S(\tilde{\theta}_i, \theta_{-i}, \mathcal{I}_t)$ such that an optimal stopping time exists for all types under a direct mechanism. With this restriction, let $\tau^*(\theta_i, \tilde{\theta}_i, \theta_{-i})$ denote the optimal stopping time that is almost surely bigger than t_a, and $\tau_i^* = \tau^*(\theta_i, \theta_i, \theta_{-i})$. Incentive compatibility requires $U(\theta_i) \equiv U(\theta_i, \theta_i) \geq U(\theta_i, \tilde{\theta}_i)$ and the individual rationality requires $U(\theta_i) \geq 0$.

The IC constraint can be written as $\theta_i \in \text{argmax}_{\tilde{\theta}_i \in [\underline{\theta}, \overline{\theta}]} U(\theta_i, \tilde{\theta}_i)\ \forall\ i$. Let $a = (\tau, \tilde{\theta})$ denote the action pair of reporting $\tilde{\theta}$ and rationally exercise following the stopping time τ. Let

$$g(a, \theta) = Q(\tilde{\theta}, \theta_{-i}) \mathbb{E}_P \left[e^{-r\tau}(P_\tau - K(\theta_i, \theta_{-i})) - \textstyle\int_{t_a}^{\infty} e^{-rt} S(\tilde{\theta}, \theta_{-i}, \mathcal{I}_t) dt - e^{-rt_a} X \right]$$

Then following the argument in Milgrom and Segal (2002), for any θ',$\theta'' \in [\underline{\theta}, \overline{\theta}]$ with $\theta' < \theta''$,

$$\begin{aligned}
|U(\theta') - U(\theta'')| =& \mathbb{E}_{\theta_{-i}} \left[\left| \sup_{a'} g(a', \theta') - \sup_{a''} g(a'', \theta'') \right| \right] \\
\leq& \mathbb{E}_{\theta_{-i}} \left[\sup_a |g(a, \theta') - g(a, \theta'')| \right] = \mathbb{E}_{\theta_{-i}} \left[\sup_a \left| \int_{\theta''}^{\theta'} g_\theta(a, \theta) d\theta \right| \right] \\
\leq& \mathbb{E}_{\theta_{-i}} \left[\int_{\theta'}^{\theta''} \sup_a |g_\theta(a, \theta)|\, d\theta \right] \leq A|\theta'' - \theta'|
\end{aligned}$$

This implies $U(\theta)$ is absolutely continuous, and thus differentiable everywhere. $U(\theta) = U(\overline{\theta}) - \int_\theta^{\overline{\theta}} U'(\theta') d\theta'$. By Theorem 1 in Milgrom and Segal (2002), $U'(\theta) = g_\theta(a^*, \theta)$. Writing it in the integral form gives that any incentive compatible and individually rational mechanism satisfies

$$U(\theta_i) = \mathbb{E}_{\theta_{-i}}\left[\int_{\theta_i}^{\overline{\theta}} Q(\theta_j,\theta_{-i})\mathbb{E}_P[e^{-r\tau_j^*}]K_1(\theta_j,\theta_{-i})d\theta_j\right] + U(\overline{\theta}) \tag{12}$$

where $U(\overline{\theta}) \geq 0$. Moreover $\tau_i \geq t_a$ $\forall i$ for time consistency.

The ex-ante social welfare is $N\mathbb{E}_{\vec{\theta}}[Q(\theta_i,\theta_{-i})\left(\mathbb{E}_P[e^{-r\tau_i^*}(P_{\tau_i^*} - K(\theta_i,\theta_{-i}))] - e^{-rt_a}(X+Y)\right)]$, and the seller's *ex ante* revenue is the social welfare less the agents' ex-ante utilities: $N\mathbb{E}_{\vec{\theta}}[Q(\theta_i,\theta_{-i})(\mathbb{E}_P[e^{-r\tau_i^*}(P_{\tau_i^*} - K(\theta_i,\theta_{-i}))] - e^{-rt_a}(X+Y))] - N\mathbb{E}_{\vec{\theta}}[U(\theta_i)]$. Using (12) and taking expectations over the winning bidder's type, it becomes

$$N\mathbb{E}_{\vec{\theta}}[Q(\theta_i,\theta_{-i})(\mathbb{E}_P[e^{-r\tau_i^*}(P_{\tau_i^*} - K(\theta_i,\theta_{-i}) - F(\theta_i)/f(\theta_i))] - e^{-rt_a}(X+Y))] - NU(\overline{\theta}) \tag{13}$$

When $K(\theta_i,\theta_{-i}) = \theta_i$, this simplifies to $N\mathbb{E}_{\vec{\theta}}[Q(\theta_i,\theta_{-i})(\mathbb{E}_P[e^{-r\tau_i^*}(P_{\tau_i^*} - z(\theta_i))] - e^{-rt_a}(X+Y))] - NU(\overline{\theta})$. With standard securities, a participant with the least cost wins, the proposition follows. □

A.8 Proof of Proposition 2

Proof. To maximize seller's revenue, for every realization of the types and any allocation rule, the seller wants winner θ_i to invest when P first hits $P^*(z(\theta_i))$. The proposed contingent payment achieves this outcome because given the royalty rate for type θ, the investment threshold is $P^{bonus} = \frac{\beta}{\beta-1}\frac{\theta}{1-\phi} = \frac{\beta}{\beta-1}z(\theta)$. Moreover, $U(\overline{\theta}) = 0$ and the project is only allocated to types that contribute positively to the revenue. z is increasing in θ leads to the unique cutoff type $\hat{\theta}$ proposed and allocation to a participant with the smallest θ. With interdependent values, for the same set of realized types, assume $K(\theta_i,\theta_{-i}) \leq K(\theta_j,\theta_{-j})$ if $\theta_i \leq \theta_j$. Then the cutoff type is well defined and the type with the smallest θ gets allocated the real option, if at all.

That $U(\theta_i)$ is decreasing in θ_i implies any mechanism satisfying the above meets IR of all types. Suppose $\theta_i < \tilde{\theta}_i$, (12) leads to $U(\theta_i,\tilde{\theta}_i) = U(\tilde{\theta}_i) - \int_{\theta_i}^{\tilde{\theta}_i} U_1(\theta,\tilde{\theta}_i,\tau^*(\theta,z_i(\tilde{\theta}_i,\theta_{-i},\cdot)))d\theta \leq U(\tilde{\theta}_i) - \int_{\theta_i}^{\tilde{\theta}_i} U_1(\theta,\theta,\tau^*(\theta,z_i(\theta,\theta_{-i},\cdot)))d\theta$ $= U(\theta_i)$, where the inequality follows from the differential form of (12) and the fact that reporting a higher investment cost leads to a lower probability of winning and a later investment. Similarly, $U(\theta_i,\tilde{\theta}_i) \leq U(\theta_i)$ for $\theta_i > \tilde{\theta}_i$. Thus incentive compatibility holds if (12) holds, which requires the $C(\theta_i,\theta_{-i})$ given in the proposition. □

A.9 Proof of Theorem 1

Proof. Denote $P_{\tau^*(\theta_{(1)},\theta_{(2)})}$ as P_{τ^*}. The seller's expected utility for holding auction when P_a is first reached can be written as

$$D(P_0;P_a)\int_{\underline{\theta}}^{\hat{\theta}}\int_{\theta}^{\hat{\theta}} \frac{N(N-1)}{2} f(\theta)f(\theta')[1-F(\theta)]^{N-2}[D(P_a;P_{\tau^*})[P_{\tau^*} - \theta - \chi\frac{F(\theta)}{f(\theta)}] - X - Y]d\theta' d\theta. \tag{14}$$

where $\chi = 1$, $P_{\tau^*}(\theta)$ is the winning bidder's investment threshold according to Lemma 2, and $\hat{\theta}$ is the cutoff type. The derivative w.r.t. P_a is

$$\frac{D(P_0;P_a)}{P_a}\int_{\underline{\theta}}^{\hat{\theta}}\int_{\theta}^{\hat{\theta}} \frac{N(N-1)}{2} f(\theta)f(\theta')[1-F(\theta)]^{N-2}\left[\beta(X+Y) + \mathbf{I}_{\{P_a > P_{\tau^*}\}}\left[\beta z(\theta) - (\beta-1)P_a\right]\right]d\theta, \tag{15}$$

where I have used the fact that marginal revenue from an interior cutoff type is zero. This expression is continuous in P_a with derivative positive for $P_a < \underline{P}$ where $\underline{P} = \min_{\theta,\theta'} P_{\tau^*}(\theta)$ and negative for a big enough constant $\overline{P}$ Thus there exists P_a in the compact region $[\underline{P}, \overline{P}]$ that maximizes (14). This proves the existence of optimal threshold strategy for auction timing, and the fact that the seller never holds the auction when no bidder would exercise immediately.

Now apply the above argument to welfare, an efficient threshold strategy exists, and the derivative of social surplus w.r.t. P_a is

$$\frac{D(P_0; P_a)}{P_a} \int_{\underline{\theta}}^{\hat{\theta}} \int_{\theta}^{\hat{\theta}} \frac{N(N-1)}{2} f(\theta) f(\theta')[1 - F(\theta)]^{N-2} \left[\beta (X + Y) + \mathbf{I}_{\{P_a > P_{\tau^*}\}} \left[\beta\theta - (\beta - 1)P_a\right]\right] d\theta, \tag{16}$$

which is smaller than (15) for every P_a. At the optimal threshold P_{opt}, integrating (15) over $[P_{opt}, P_a]$ must be negative for any $P_a > P_{opt}$. Thus integrating (16) over $[P_{opt}, P_a]$ must also be negative for any $P_a > P_{opt}$, implying the efficient threshold $P_{eff} \leq P_{opt}$. Another way to see this is that Equation (14) with $\chi = 0$ corresponds to welfare, and the equation is supermodular in (P_a, χ). Thus a seller optimally delays the auction beyond the socially efficient threshold given the same security design and allocation rule.

Finally, if the security design is not fixed, the above argument combined with the Envelop Theorem give the result in Corollary 1. □

A.10 An Extension to the "Mineral Rights Model"

In addition to being analytically tractable when analyzing auction timing, the private-value framework is not unrealistic in the sense that the dispersion of bidder types over the common component, such as signals on the amount of oil reserve has decreased in recent years due to technological improvement, and the government typically provides as much information as possible to the buyers. By contrast, firms often have private drilling technologies, and retail and transportation contractors, which fit private-value settings.

Given that the literature has usually adopted the "Mineral Rights Model" with interdependent values, it is important discuss how the key insights in the optimal design generalize in such settings. To do this, let the investment cost is $K(\theta_i, \theta_{-i})$, where K is symmetric in other bidders' report types, and has positive derivative in θ_i denoted by K_1 that is uniformly bounded by a positive constant. In the main model of this paper, $K(\theta_i, \theta_{-i}) = \theta_i$, but this specification allows other cases with interdependent values such as common-value auctions where $K(\theta_i, \theta_{-i}) = \frac{1}{N}\sum_j \theta_i$.

In "bonus-bid" auctions, the winning bidder owns a fraction $1 - \phi$ of the project and has a real option value $L(\theta, \theta_{-1}) = \max_\tau \mathbb{E}[e^{-r(\tau - t_a)}((1 - \phi)P_\tau - K(\theta_i, \theta_{-i}))] - X$. Scaling the cash flow in Eq.(2) gives the optimal investment threshold $P^{bonus}(\theta) = \max\{P_a, \frac{\beta}{\beta-1}\frac{K(\theta_i,\theta_{-i})}{1-\phi}\} \geq P^*(K(\theta_i, \theta_{-i}))$, thus investment is inefficiently delayed. The equilibrium bidding strategies are standard from Proposition 6.3 in Krishna (2009):

$$C(\theta) = \int_{\theta}^{\hat{\theta}} \frac{E[L(\theta', \theta'_{-1})|\theta'_{(1)} = \theta']}{1 - F(\theta)} f(\theta') d\theta' \tag{17}$$

where $\theta'_{(1)}$ is the type with least cost among the remaining bidders, and $\hat{\theta}$ is the cut-off type for participation.

In the proofs of Propositions 1 and 2, I have generalized Propositions 1 and 2 to interdependent-value settings. I show that when K is such that $\theta_i \leq \theta_j$ implies $K(\theta_i, \theta_{-i}) \leq K(\theta_j, \theta_{-j})$, the optimal security design is still a combination of cash and royalty payment, with the equilibrium royalty rate for type θ_i

modified to

$$\phi(\theta_i, \theta_{-i}) = \frac{F(\theta_i)}{f(\theta_i)K(\theta_i, \theta_{-i}) + F(\theta_i)} \tag{18}$$

Thus a bonus-bid auction with a uniform royalty rate would not generate the highest revenue. Moreover, investments are inefficiently delayed with both bonus-bid auction and auction with optimal security design.

Using the properties of the Wald distribution, the expected inefficient time delay when a royalty rate ϕ is used is $\Gamma = -\ln(1-\phi)/[\mu - \frac{\sigma^2}{2}]$, where we have assumed $\mu - \frac{\sigma^2}{2} > 0$ for the expectation to exist.[44] Moreover, $\frac{\partial \Gamma}{\partial \phi} > 0$, $\frac{\partial \Gamma}{\partial \mu} < 0$, $\frac{\partial \Gamma}{\partial \sigma} > 0$, $\frac{\partial^2 \Gamma}{\partial \phi^2} > 0$, $\frac{\partial^2 \Gamma}{\partial \sigma^2} > 0$, $\frac{\partial^2 \Gamma}{\partial \sigma \partial \phi} > 0$. Not only do more volatile markets or high royalty rates result in longer delays, but they are mutually reinforcing, with increasing marginal effects. What is the social cost of the investment lag? It can be shown that the option value is a fraction $(1 - \phi + \phi\beta)(1 - \phi)^{\beta}$ of the socially efficient value, and the fractional loss L satisfies $\frac{\partial L}{\partial \phi} > 0$, $\frac{\partial^2 L}{\partial \phi^2} > 0$. Again, royalty rate has a compounding effect on social cost.

A.11 Proof of Lemma 5

Proof. This is obviously true if only one type uses Π^i. If more than one type use this bid, either it holds or one of the types θ_1 has $R_{\theta_1}(S^i) + C^i \neq R(\Pi^i)$. Then $\exists\ \theta_2$ (potentially $= \theta_1$) s.t. $R(\Pi^i) < C^i + R_{\theta_2}(S^i)$. Consider the deviation for bidder 2 in the subgame to a cash bid equal to $R(\Pi^i)$ and invest efficiently. This deviation is profitable because he creates weakly greater social surplus, pays less, and has the same marginal probability of winning. Thus by contradiction $R(\Pi^i) = C^i + R_{\theta_i}(S^i)$ always. □

A.12 Proof of Lemma 6

Proof. Now suppose $\tau^i_{\theta_i} \neq \tau^*_i$, consider deviating to a cash bid $C = R(\Pi^i)$. The payoff from deviation $\mathbb{E}[e^{-r\tau^*_i}(P_{\tau^*_i} - \theta_i)] - R(\Pi^i)$ dominates the original payoff $\mathbb{E}[e^{-r\tau^i_{\theta_i}}(P_{\tau^i_{\theta_i}} - \theta_i - S^i(P_{\tau^i_{\theta_i}}))] - C^i = \mathbb{E}[e^{-r\tau^i_{\theta_i}}(P_{\tau^i_{\theta_i}} - \theta_i)] - R_{\theta_i}(S^i) - C^i$. Thus the deviation is profitable and the claim follows. □

A.13 Proof of Lemma 7

Proof. Suppose a non-singleton set Θ_p of types pool to bid Π in FPAs, or have the same drop-out bid in SPAs. The claim follows if there is always a profitable deviation by a type in this set.

From Lemma 6, a type θ in expectation pays $C + D(P_a; P^*(\theta))S(P^*(\theta))$. Let $\theta_k = \text{argmax}_{\theta' \in \Theta_p} R_{\theta'}(S)$ where $R_{\theta'}(S) = D(P_a; P^*(\theta'))S(P^*(\theta'))$. Then $R(\Pi) \leq C + D(P_a; P^*(\theta_k))S(P^*(\theta_k))$. If the inequality is strict, type k can profitably deviate to cash bid $R(\Pi)$. Otherwise, $R_{\theta_i}(S) = R_{\theta_j}(S) = R(\Pi) - C$, for some $\theta_i < \theta_j$ both in Θ_p, but there is still a profitable deviation:

We first argue that Θ_p contains a positive measure of types. For any $\theta_n \in (\theta_i, \theta_j) \bigcap \Theta_p^c$, call his bid $\tilde{\Pi}$. Let Q and $\tilde{Q}$ be the probability of winning when bidding Π and $\tilde{\Pi}$. Since θ_i does not want to deviate to cash bid $R(\tilde{\Pi})$, $Q[W(P_a; \theta_i) - R(\Pi) - X] \geq \tilde{Q}[W(P_a; \theta_i) - R(\tilde{\Pi}) - X]$. Similarly, $Q[W(P_a; \theta_j) - R(\Pi) - X] \geq \tilde{Q}[W(P_a; \theta_j) - R(\tilde{\Pi}) - X]$. As $\theta_i \neq \theta_j$, the equality signs cannot hold simultaneously. Thus for $\theta_n \in (\theta_i, \theta_j)$, $Q[W(P_a; \theta_n) - R(\Pi) - X] > \tilde{Q}[W(P_a; \theta_n) - R(\tilde{\Pi}) - X]$. This means θ_n can profitably deviate to cash bid $R(\Pi)$. Therefore, it has to be that $[\theta_i, \theta_j] \in \Theta_p$.

[44] If $\mu < \sigma^2/2$, the median lag M can be considered instead.

Next, note $W(P_a;\theta_i) - X - R_{\theta_i}(S) - C > W(P_a;\theta_j) - X - R_{\theta_j}(S) - C \geq 0$. Type θ_i can deviate profitably to cash bid $\epsilon + R(\Pi)$ which reduces his payoff by ϵ upon winning but increases his marginal chance of winning by a discrete amount (because he separates from a positive measure of types). □

A.14 Proof of Theorem 2

Proof. Consider the bidding strategy from a FPA in cash. The valuations for the bids are simply the cash amounts. I show there exists a belief that supports an equilibrium with this bidding strategy in the informal auction. First, there would not be any deviation to another cash amount since the bidding strategy comes from the equilibrium in FPA cash auction. Next, for beliefs such that upon seeing an out-of-equilibrium bid Π^i, the auctioneer believes it comes from $\tilde{\theta}_i = \text{argmin}_{\theta \in [\underline{\theta}, \overline{\theta}]}[R_\theta(S^i) + C^i]$ and gives it a valuation $\tilde{R}$. If bidder i finds this deviation attractive (yielding an expected payoff more than the original amount after cash payments), then he also finds deviating to cash bid $\tilde{R}$ weakly more attractive, contradicting the fact that no deviation to another cash amount is profitable. Thus the equilibrium from a first-price cash auction is an equilibrium in the informal auction. The argument also applies to cash-like bid Π such that $R(\Pi)$ is independent of the seller's belief on the bidders' types.

Next I show any bidding equilibrium in the informal auction has the same allocation outcome and expected payoffs as cash auctions. The seller forms correct beliefs about types since Lemma 7 rules out pooling. Bidder i's bid S^i can be replaced by an equivalent cash bid. This would not change the marginal probability of winning by Lemma 5, neither does it change the payoff upon winning as Lemma 6 implies the total surplus is the same. Since the bidders face the same maximization problem as in a FPA with cash, almost every bid is cash-like in terms of its expected payoff. □

A.15 Proof of Theorem 3

Proof. In equilibrium, the proof of Lemma 5 goes through for the winner's final bid and each type's drop-out bids, otherwise there must be multiple types using the same bid and for some type, its bid is undervalued and he can profitably deviate to bidding cash. Since by bidding cash, bidder i has a value of $W(P_a;\theta_i) - X$, so he would remain in the game before the score surpasses this value. Since $W(P_a;\theta_i) - X \neq W(P_a;\theta_j) - X$ for $\theta_i \neq \theta_j$, different types drop out at different score values, resulting in a separating equilibrium strategy of dropping out. When there is one bidder remaining, he has to pay a score at which the second last bidder drops out. If the bid does not lead to efficient investment, the winning bidder can simply bid cash equal to the score, and increase his own profit by investing efficiently. Therefore Lemma 6 holds.

With these results, the allocations, payoffs, and investment outcomes in a bidding equilibrium are identical to those in an ascending second price auction in cash. By revenue equivalence theorem, they are equivalent to those in first-price and second-price cash auctions too. The auction timing strategy by the seller is thus the same as that in a first-price informal auction. □

A.16 Proof of Proposition 3

Proof. Conjecture that in equilibrium bidder θ initiates the auction with threshold $P_I(\theta)$, which is increasing, and the seller initiates with a threshold P_S. Let $\theta(P_a) = sup\{\theta : P_I(\theta) \leq P_a\}$. The expected payoff to the bidder θ following initiation threshold $P_a \leq P_S$ is

$$\int_{\underline{\theta}}^{\theta(P_a)} d\theta'(N-1)f(\theta')[1-F(\theta')]^{N-2}\left(\frac{P_0}{P_I(\theta')}\right)^{\beta}\left[[W(P_I(\theta');\theta)-X-Y]^{+}-[W(P_I(\theta');\theta')-X-Y]^{+}\right]^{+}$$
$$+\int_{\theta(P_a)}^{\overline{\theta}} d\theta'(N-1)f(\theta')[1-F(\theta')]^{N-2}\left(\frac{P_0}{P_a}\right)^{\beta}\left[[W(P_a;\theta)-X-Y]^{+}-[W(P_a;\theta')-X-Y]^{+}\right]^{+} \tag{19}$$

where θ' is basically the first-order statistic of the remaining $N-1$ bidders; similarly the payoff when $P > P_S$ is

$$\int_{\underline{\theta}}^{\theta(P_S)} d\theta'(N-1)f(\theta')[1-F(\theta')]^{N-2}\left(\frac{P_0}{P_I(\theta')}\right)^{\beta}\left[[W(P_I(\theta');\theta)-X-Y]^{+}-[W(P_I(\theta');\theta')-X-Y]^{+}\right]^{+}$$
$$+\int_{\theta(P_S)}^{\overline{\theta}} d\theta'(N-1)f(\theta')[1-F(\theta')]^{N-2}\left(\frac{P_0}{P_S}\right)^{\beta}\left[[W(P_S;\theta)-X-Y]^{+}-[W(P_S;\theta')-X-Y]^{+}\right]^{+} \tag{20}$$

Denote the solution to $W(P_a,\theta) = X+Y$ by $\hat{P}(\theta)$. Note when $P_a \leq P_S$, $(P_0/P_a)^{\beta}\left([W(P_a;\theta)-X-Y]^{+}-[W(P_a;\theta')-X-Y]^{+}\right)$, if positive, is decreasing when $P_a > P^*(X+Y+\theta)$, increasing at $\hat{P}$, and constant for $P_a < \hat{P}$. Differentiating (19) w.r.t. P_a and applying Leibniz's formula gives that in equilibrium $\hat{P} \leq P_I(\theta) \leq P^*(\theta+X+Y)$. Now for the seller, if she uses threshold P_a, the expected payoff is,

$$\int_{\underline{\theta}}^{\theta(P_a)} d\theta' N(N-1)f(\theta')F(\theta')[1-F(\theta')]^{N-2}\left(\frac{P_0}{P_I(\theta')}\right)^{\beta}[W(P_I(\theta');\theta')-X-Y]^{+}$$
$$+\int_{\theta(P_a)}^{\overline{\theta}} d\theta' N(N-1)f(\theta')F(\theta')[1-F(\theta')]^{N-2}\left(\frac{P_0}{P_a}\right)^{\beta}[W(P_a;\theta')-X-Y]^{+}. \tag{21}$$

Suppose $P_a < P_I(\overline{\theta})$. For any $\theta' > \theta(P_a)$, the earlier argument leads to $P_a < P^*(\theta'+X+Y)$, for otherwise θ' would initiate earlier than P_a - a contradiction. The derivative of (21) is thus positive for any P_S unless $P_S = P_I(\overline{\theta})$. Thus almost surely the seller never initiates.

Now the bidder's problem is reduced to expression (19). The derivative at P_a has the same sign as

$$\int_{\theta(P_a)}^{\overline{\theta}} d\theta' f(\theta')[1-F(\theta')]^{N-2}\frac{d}{dP}\left[\frac{[W(P;\theta)-X-Y]^{+}-[W(P;\theta')-X-Y]^{+}}{P^{\beta}}\right]^{+}\Bigg|_{P=P_a}, \tag{22}$$

which is positive at $\hat{P}(\theta)$ and non-positive at $P^*(\theta+X+Y)$. The integrand is weakly monotone in P_a path-by-path, thus (22) changes sign at a unique $P_a = P_I(\theta)$.

Given (19) is concave in P_a with non-negative cross-partial in P_a and θ, and there exists unique maximizer $P_I(\theta)$, Implicit Function Theorem gives that $P_I(\theta)$ is indeed non-decreasing. A similar argument would rule out a decreasing equilibrium in which the initiator always loses. This ensures (22) is continuous, establishing the optimality of P_I and the FOC in the proposition. There could be multiple equilibria with different initiation thresholds below P_0, but in terms of initiation outcome and payoffs, they are all equivalent, making the proposed equilibrium essentially unique.

Given that a bidder's threshold for holding the auction is lower than his threshold if he were maximizing social welfare, the initiation is accelerated in the ex post sense. Moreover, he would invest in the project right away, making the exercise of the real option faster than in seller-initiated auctions where the realized winning type might still wait after the auction. □

Tables and Figures

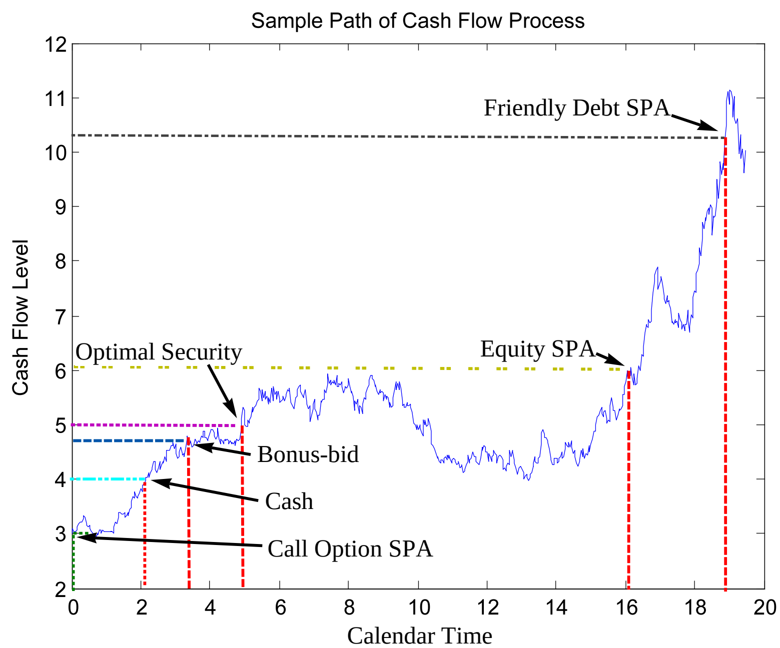
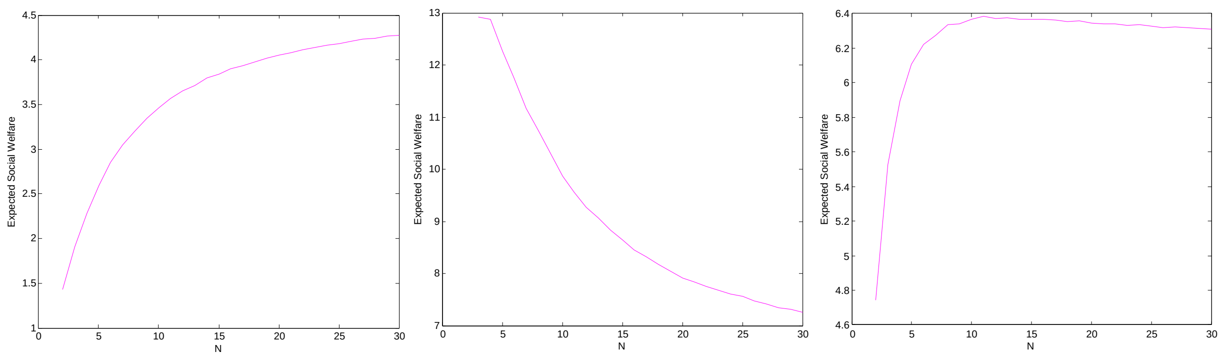


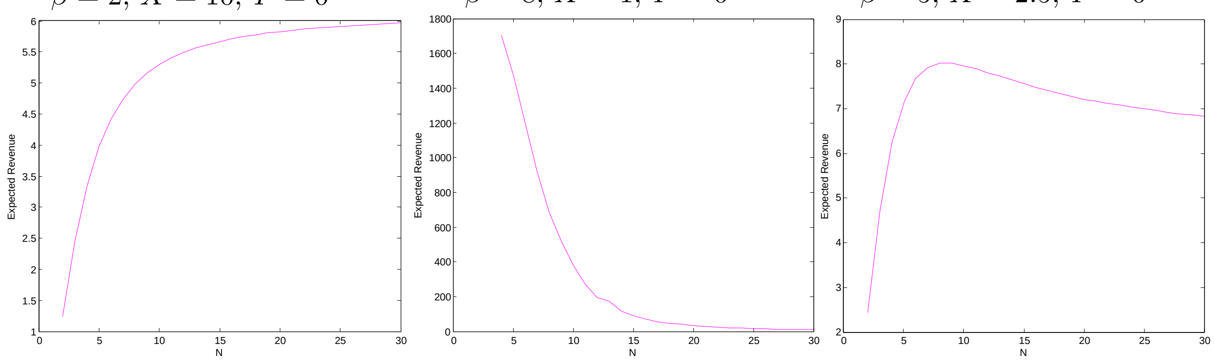
Figure 2: Investment thresholds under various security designs. Simulated with $\mu = 0.06$, $\sigma = 0.2$, $r = 0.16$, $\theta \sim Unif[1.5, 5]$, $X = 0.4$, $Y = 0$, $P_a = 3$.



(a) Unif$[20, 50]$, $P_a = 35$, $\beta = 2$, $X = 10$, $Y = 0$

(b) Unif$[20, 50]$, $P_a = 35$, $\beta = 8$, $X = 1$, $Y = 0$

(c) Unif$[30, 60]$, $P_a = 45$, $\beta = 5$, $X = 2.5$, $Y = 0$

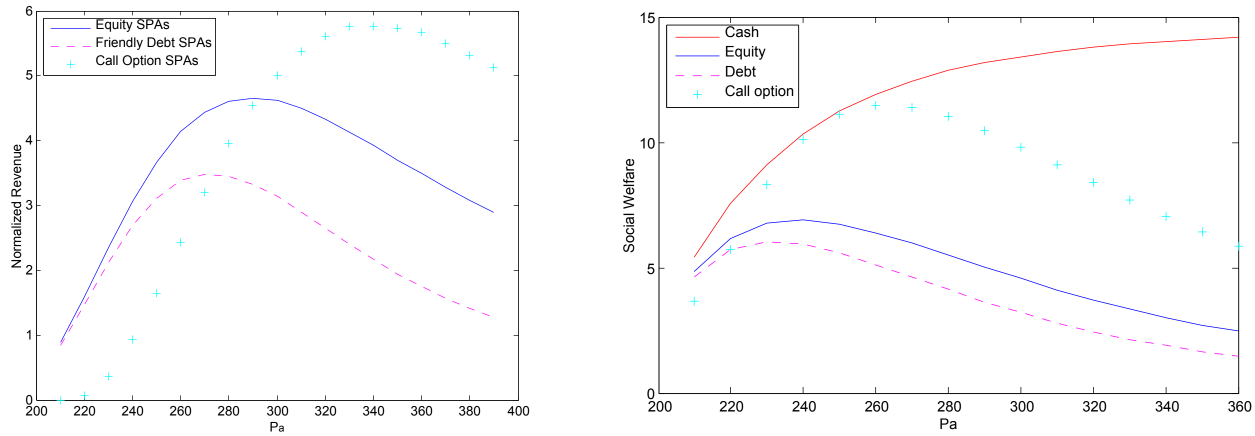


(d) Unif$[30, 60]$, $P_a = 45$, $\beta = 5$, $X = 2.5$, $Y = 0$

(e) Unif$[20, 50]$, $P_a = 35$, $\beta = 25$, $X = 0.1$, $Y = 0$

(f) Unif$[20, 50]$, $P_a = 35$, $\beta = 8$, $X = 1$, $Y = 0$

Figure 3: Plots of expected social welfare (a)(b)(c) and seller's revenue (d)(e)(f) against number of bidders N. One million simulations in SPA with equity bids and uniformly distributed θ. For exposition, β is specified instead of the primitives r, μ, and σ.



(a) $\beta = 5$, $X = 10$, $Y = 0$, $N = 5$ (b) $\beta = 6$, $X = 8$, $Y = 0$, $N = 30$

Figure 4: Plots of expected seller's revenue and social welfare against the auction threshold for SPAs with equities, friendly debts as defined in Section 3.1, and call options. One million simulations for θ uniformly distributed in $[200, 500]$, $P_0 = 210$. For exposition, β is specified instead of the primitives r, μ, and σ.

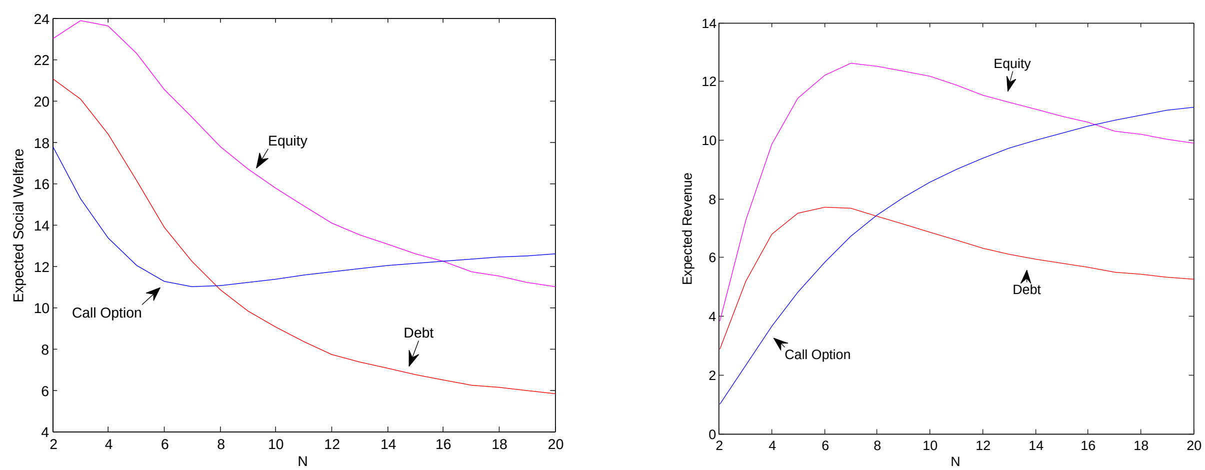


Figure 5: Plots of expected social welfare and seller's revenue against number of bidders N for SPAs with equities, friendly debts as defined in Section 3.1, and call options. One million simulations with θ uniformly distributed in $[20, 50]$, $P_a = 35$, $r = 0.123$, $\mu = 0.001$, $\sigma = 0.05$, and $X + Y = 1$.

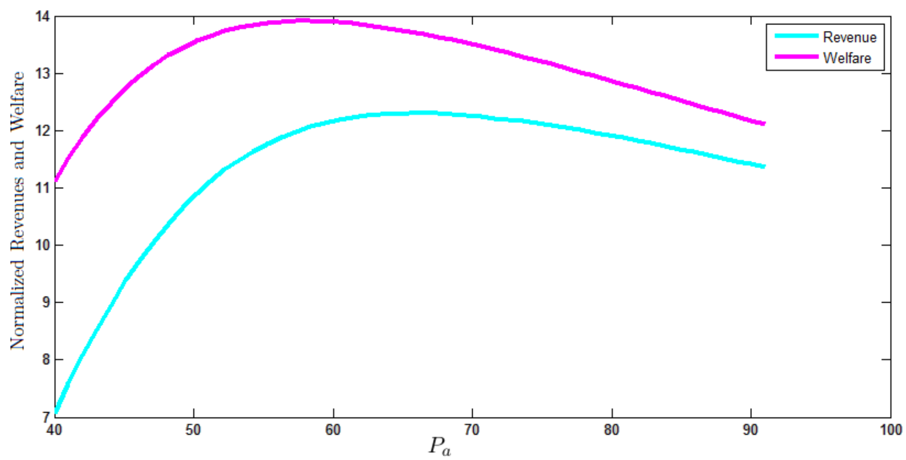


Figure 6: Revenues and Welfare for cash auctions following threshold timing P_a. 200,000 simulations for $\theta \sim Unif[10, 40]$, $r = 0.06$, $\mu = 0.01$, $\sigma = 0.2$, $X = 15$, $Y = 0$, $N = 7$, $P_a = 40$. Welfare-maximizing auction timing threshold is lower than revenue-maximizing timing.