Dynamic Noisy Rational Expectations Equilibrium with Information Production and Beliefs-Based Speculation*

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Abstract

A continuous time economy with information production and belief-based speculation is studied. All agents, informed, uninformed and speculators, have constant absolute risk aversion. In contrast to the model with exogenous noise trading, the economy with beliefs-based speculation has a noisy rational expectations equilibrium. Equilibrium is in closed form, except for a coefficient satisfying a Riccati equation. Costly information production generates asynchronous private and public information flows. Private information dissemination reduces price volatility. Intertemporal hedging amplifies this decrease. Asynchrony between information flows increases volatility over time. A decomposition of ex-ante and interim utilities, identifying the sources of welfare, is obtained. Necessary and sufficient conditions for Pareto efficiency of equilibrium are derived. Under these conditions speculation is socially beneficial. A contingent information trading tax enforces the Pareto optimal equilibrium and is shown to be implementable.

JEL Classification: JEL Classification : G14

Keywords: Intertemporal noisy rational expectations equilibria, costly information production, asynchronous information flows, competition, speculation, beliefs, constant absolute risk aversion, hedging, volatility, market price of risk, welfare, Pareto optimality, regulation, heterogeneous risk tolerances, differential information, dividend surprises.

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1 Introduction

Information and speculation are fundamental motives underlying securities trading. Individuals with superior information have different assessments of returns, therefore find it beneficial to trade. Individuals with different beliefs resulting from fundamental differences in views about the future, also have their own assessments of returns. They find it optimal to speculate on the difference between their beliefs and those of others. These motives for trade have long been sources of interest as well as concerns. Regulatory limits on informed trading have been enacted in a variety of countries to curb the rents extracted by certain types of informed agents. Current debates also question the role of speculation in the recent financial crisis and regulatory proposals seeking to limit such activity have been circulated. This paper seeks to shed light on these issues in a setting with endogenous information flows. It examines the effects of information and speculation on the dynamic properties of financial market equilibria. In particular, it attempts to identify the welfare benefits and costs associated with regulations affecting information-based trading or speculation.

The model under consideration has two key features, costly information production infrastructure and speculation based on utility maximizing noise trading. Costly information infrastructure implies that information arrives at different frequencies depending on its nature. Public information is costless. Its arrival rate is therefore dictated by the frequency of news releases by the exogenous underlying source. Private information, in contrast, is under the control of the acquirer. It is usually difficult to extract and does not update automatically. It requires meeting the managers of a company or poring over documents and data to extract a few bits of useful information regarding future prospects. The costly nature of the information gathering technology implies that the rate of arrival is dictated by decisions of the acquirer. This paper captures this basic asymmetry between public and private information flows. It shows, in particular, that private information stabilizes the market by reducing volatility and that the endogenous asynchrony in news arrivals leads to an increasing volatility pattern over time. It also reveals that the information infrastructure, by determining the optimal frequency of news, dictates the strength of the stabilizing role of private information and the extent of associated welfare gains.

Speculation is a powerful motive underlying a fraction of trades in financial markets. Beliefs-fueled speculation has long been recognized as an important ingredient for market models. This paper develops a model where some individuals speculate by trading on noise (Black (1986)).[1] Such individuals have the same conditional beliefs as informed agents, but instead of conditioning on factual private information, they base their views on irrelevant noise. In other respects, they behave rationally and maximize preferences by choosing the best allocation among the assets available. This noise trading model enables us to endogenize the noise trading demand and conduct a meaningful welfare analysis. It has the property that noise traders do not systematically lose money by speculating on noise. It also resolves a problem of non-existence of noisy rational expectations equilibria (NREE) when noise trading is exogenous. Speculative behavior shapes the informational properties

[1] "Noise trading is trading on noise as if it were information. People who trade on noise are willing to trade even though from an objective point of view they would be better off not trading. Perhaps they think the noise they are trading on is information. Or perhaps they just like to trade." Black (1986).

of equilibrium and incentives to collect information. It plays a critical role for the welfare of society.

Our study is based on a continuous time setting in which the firm pays a liquidating dividend at the terminal date and information flows are asynchronous due to costly acquisition. Public information, regarding the fundamental underlying the dividend, arrives continuously. Private information, about the terminal dividend, arrives at discrete dates determined by the endogenous acquisition technology. In the base model, it arrives once, at the initial date, which can be rationalized as the endogenous outcome of the infrastructure selection process. This asynchrony has multiple effects. When private information disseminates, expectations become less responsive to public news, which lowers volatility. The subsequent frequent arrival of public information exerts conflicting pressures on the local value of private information, the market price of risk and the incentives of investors. A prominent effect is that it reduces the usefulness of the private information collected by the informed at the outset and of the endogenous information extracted from equilibrium by the uninformed. This basic mechanism puts upward pressure on volatility as time passes. As private information becomes more stale, the sensitivity of the stock price to public news increases, leading to a volatility increase. This is the sole effect when investors are myopic. Investors with CARA utility, however, care about fluctuations in the opportunity set, therefore hedge intertemporally (Merton (1971)). Hedging tames the response of optimal portfolios to shocks, thereby reducing the sensitivity of the equilibrium price to shocks. It is the source of a further reduction in the volatility level, uniformly over time.

The dissemination of private information has two effects on welfare. The first is due to the immediate change in the stock price (price impact), the second to the gains from trade associated with the improved information of agents (trading impact). Asynchrony has multiple effects on these components. One set of effects arises through volatility, which influences both the price and the trading impacts. Another effect arises through the hedging demand of an agent, distinct from the effect of hedging on volatility. This hedging effect is an important determinant of the trading impact.

In the absence of hedging, welfare improves uniformly across agents upon dissemination of private information, when risk tolerance is either sufficiently low or sufficiently high. The immediate decrease in volatility is accompanied by an increase in the stock price which raises the value of the initial allocation of shares, leading to a positive price impact. At the same time, it reduces the market price of risk, therefore the size of stock holdings and the resulting gains from trade. The sum of these two effects is positive. The immediate increase in informational efficiency works in the opposite direction as far as gains from trade are concerned. It fine-tunes the risk taking behavior of an agent, leading to an increase in gains from trade. Volatility effects are tamed by (inversely related to) risk tolerance. Trading effects are enhanced by (positively related to) risk tolerance. When risk tolerance is low, volatility effects dominate, leading to a welfare improvement. At the other end of the spectrum, when risk tolerance is high, the informational efficiency gains dominate, also leading to a welfare gain. In both cases, these universal benefits associated with the dissemination of private information offset the informational disadvantage of uninformed agents.

Hedging affects both the price and trading impacts. As explained, it further decreases volatility, hence magnifies the welfare improvements for low risk tolerance. But, it also curbs the agents' risk-taking behavior, hence reduces the gains from trade associated with the dissemination of information.

Nevertheless, when risk tolerance is high, the hedging motive weakens so that the previous conclusions remain valid. Informational efficiency gains dominate, leading to a welfare improvement. In these instances, it is Pareto optimal (PO) to permit trading based on private information.

Speculation plays a critical role for the welfare of market participants. In its absence, incentives to collect information vanish. An equilibrium with homogeneous information flow, determined by the fundamental, then prevails. In this equilibrium, volatility is driven by the volatility of the fundamental, hence exceeds the volatility in the NREE. Information is also less precise than in the NREE, worsening investment decisions. Both effects are sources of welfare reductions. Under the conditions for Pareto optimality, prohibiting speculation leads to a Pareto dominated equilibrium.

The existence of cases where the NREE is or is not PO is a challenge for regulation. Ideally, the regulator would like to encourage (deter) informed trading when the NREE is (not) PO. We show that a properly designed contingent Tobin tax, the Information Trading Tax (ITT), enforces the PO equilibrium. Moreover, the ITT is implementable based solely on fundamental and price information.

When the skilled (informed) investor selects the information gathering infrastructure and the cost is an increasing function of the number of signals produced, it is optimal to choose a finite sampling frequency under mild conditions. Two opposite forces determine the outcome. On the one hand, an increase in the number of signals produced increases the precision of the information collected, which improves decision-making and welfare. On the other hand, the cost increases. As long as the asymptotic precision of the private information is finite and the marginal cost of sampling is bounded away from zero, the second effect will eventually dominate. In this instance, public and private information flow at different frequencies. Asynchrony ensures that the fundamental effects and trade-offs documented in the base model will arise. Moreover, at private information arrival times, the stock volatility experiences downward jumps. This phenomenon reinforces the stabilizing role of private information documented in the base model with an endogenously single private signal.

Aside from the questions discussed above, the model also has practical implications for the fund management industry. Academic debates regarding the skills of fund managers figure prominently in the literature.[2] Evidence seems to suggest that a number of funds lack skill. Our model of speculative noise traders captures one possible view of unskilled management, namely the implicit suggestion of knowledge and skill through the promotion of sophisticated investment strategies, but the actual lack of true information extraction skill and the absence of superior information in the implementation of these strategies. Speculative noise traders, in our setting, behave exactly as truly informed (skilled) agents because they seek to be identified as skilled, but they operate on the basis of noise. Our model shows that this behavior serves useful purposes. The existence of an active unskilled sector effectively clouds the price system, thus provides incentives for information collection. The information conveyed by the price system furthermore allows uninformed (retail) investors to improve their decision-making, which is a source of welfare.

It is important to stress that our dynamic model with speculative noise trading is essential for the results described above. With exogenous noise trading, a NREE does not exist. As a result, there

[2] See, for instance, Jensen (1968), Ferson and Schadt (1996) and Jagannathan, Malakhov and Novikov (2010).

are no incentives for information acquisition and the stock price volatility equals the fundamental volatility. These properties show that basic insights from static models cannot be straightforwardly extrapolated to intertemporal settings.

The paper also contributes on the methodological front. Equilibrium is constructed based on the private information price of risk (PIPR).[3] The approach has three steps. The first determines the structure of the PIPR and the candidate equilibrium information flows. This step does not require knowledge of the hedging demands and stock price. The second derives a system of equations for the hedging demands and stock price, conditional on the information flows identified. The final step solves the system of equations and verifies the informational content of equilibrium. This approach is constructive. It relies on the fact that the PIPR is the ultimate source of all effects in the model.

1.1 Related Literature

Classical studies pertaining to informational efficiency are based on static models. The seminal articles of Grossman (1976, 1978) and Grossman and Stiglitz (1980) identify basic determinants of efficiency in competitive markets. Non-competitive behavior is examined by Hellwig (1980), Kyle (1989) and Leland (1992). The last study focuses more specifically on insider trading and properties of equilibrium in a static model with production and monopolistic insider. It finds that private information increases the average stock price, decreases the return's expectation and variance for the uninformed, reduces the liquidity of the market and can increase or decrease welfare.[4]

Dynamic models with asymmetric hierarchical information and competitive behavior were pioneered by Wang (1993, 1994). In these settings, the stock is an infinitely-lived asset that pays dividends continuously/periodically through time. Informed investors observe the state variable driving the expected future dividend. Uninformed investors do not, but they learn through dividends and prices. Noise trading injects supply stochasticity and prevents full revelation. Wang (1993) derives a stationary competitive noisy rational expectations equilibrium (NREE). Asymmetric information is shown to increase the stock's long run risk premium. It can also increase the price volatility and enhance negative serial correlation. Asymmetric information can therefore have a destabilizing effect. Wang (1994) focuses on issues pertaining to the volume of trade in a similar setting. The article highlights the relation between volume and price changes. Further insights are provided by He and Wang (1995) in a model with terminal dividend, differential private information signals across agents, hence non-hierarchical information flows, and a common frequency of private and public information arrivals as well as trades. Assuming the existence of a competitive NREE, they study the relation between information flow and trading volume. The common frequency implies that there are no

[3]This notion is introduced in Detemple and Rindisbacher (2013) in a portfolio problem with private information.

[4]Extensions of Leland (1992) can be found in Bhattacharya and Spiegel (1991), Bernhardt, Hollifield and Hughson (1995), Repullo (1999) and Vives and Medrano (2004). Earlier studies documenting negative effects of insider information include Manove (1989), Ausubel (1990) and Fishman and Hagerty (1992). Possible negative effects of information disclosure in various economic settings are documented in Angeletos and Pavlan (2007), Amador and Weill (2010), Peress (2010) and Kurlat and Veldkamp (2015).

dynamic effects between arrival times of private information signals.[5,6]

Albuquerque and Miao (2014) extend Wang (1994) by allowing for private advance information about future dividends. Time is discrete and advance information pertains to the temporary component of the dividend paid at the next date. Myopic agents derive utility from next period wealth, ignoring future consumption. They solve for the stationary equilibrium and study the effects of information on the stock price and risk premium.

Brennan and Cao (1996) develop the only model with (exogenous) asynchronous information flows. An initial private signal is followed by a sequence of public signals at later dates. Information arrival dates are also trading dates. In other respects, the setting is similar to He and Wang (1995). Assuming the existence of a NREE, they examine the impact of an increase in the number of trading dates and the effects of financial innovation.

The present model builds on this literature. The main economic differences are threefold, (i) the information production infrastructure is endogenous, (ii) noise traders are expected utility maximizers who trade on noise (iii) the NREE exists and is obtained in closed form. Costly information infrastructure implies that information flows endogenously arrive at different frequencies. This asynchrony is the source of novel effects documented. Utility maximization enables us to endogenize the demand of noise traders and conduct a comprehensive welfare analysis. It opens up the possibility of Pareto ranking equilibria. The specific beliefs model developed reflects Black's (1986) idea that some investors trade on noise. The existence of a NREE is a consequence of the optimizing behavior of the noise traders. The structure of their demand ensures that the residual demand cannot be inverted to recover the private signal of the informed.

Other differences pertain to the scope of the analysis or the nature of the results. In that regard, it could first be noted that equilibrium coefficients are explicit functions of the horizon. The timing effects identified are driven by the asynchrony in information flows and the hedging motives of agents. Second, the nature of the informational advantage of the informed differs. Relative to Wang (1993, 1994), this advantage is finitely lived in our setting. Relative to Albuquerque and Miao (2014), it is non-transitory. This follows, because the time between the reception of the private signal and the resolution of the uncertainty pertaining to the dividend payment is a finite interval. This has a critical impact on optimal behavior and pricing. Third, a detailed welfare analysis is carried out. A novel decomposition of the welfare of each agent into two parts, one associated with the value of the initial share endowment, the other with the dynamic gains from trade, is derived. This decomposition is particularly useful to analyze the impact of regulation governing private information usage and speculation.[7] Fourth, a novel solution method introduced. The approach relies on the construction

[5]Effects of imperfect competition and asymmetric information on dynamic properties of prices and liquidity are examined in Vayanos and Wang (2012). In a three-period model, they show that asymmetric information and imperfect competition can have opposite effects on ex-ante expected returns.

[6]A vast microstructure literature also deals with non-competitive informed trading. Fundamental contributions are in Kyle (1985) and Glosten and Milgrom (1985). In these models, risk neutral market makers extract private information from the aggregate order flow and set the price so as to break even on average. This pricing rule does not account for the endogenous interactions between risk, price appreciation and price level. The absence of diversification benefits implies that trading is purely informational. The price evolution is typically determined by the exogenous noise trading behavior and is locally orthogonal to fundamental risk.

[7]The dynamic welfare results obtained extend the static analysis in Leland (1992), as they identify a new and

of the PIPR, which isolates the effects of private information. Its structure can be used to formulate natural conjectures about the informational content of the price. More significantly, it does not depend on the intertemporal hedging behavior of agents.

Recently, Banerjee and Green (2015) develop a model where noise traders also trade on a signal believed to be informative. Uninformed investors, unsure about whether they face an informed or a noise trader, learn over time. Uncertainty about the identity of the counterparty generates a nonlinear equilibrium price. The dynamic version of their model considers successive generations of mean-variance investors, each living for two dates. They show numerically that the model can produce expected return predictability and volatility clustering. Our framework shares aspects of the noise traders' specification. It differs as the identity of parties is common knowledge. It also differs in that we consider long-lived agents with CARA utility. The implied hedging behavior amplifies the volatility impact of private information trading and enhances the related welfare gains for low risk tolerance levels. The endogeneity of information flows and the focus on welfare also differ. Finally, the explicit nature of equilibrium enables us to derive analytical results.

Our analysis also builds on a literature dealing with dynamic portfolio selection and costly information acquisition; see Detemple and Kihlstrom (1987), Huang and Liu (2007), Hasler (2012), and Andrei and Hasler (2014). The present paper considers a related costly information production problem, but in an equilibrium setting with heterogeneously informed agents. It studies the relation between the endogenous price/volatility and the asynchrony of news. It also addresses welfare and regulatory questions pertaining to the use of private information and the relevance of speculation.[8]

The notion of speculation based on differences in beliefs has been widely discussed. Foundations appear in Working (1962) and Hirshleifer (1975). Black (1986) introduces the view that "noise trading is trading on noise as if it were information." Beliefs of noise traders therefore differ from those of truly informed individuals. Beliefs-based speculation has received recent attention in the context of financial reforms pertaining to derivatives. Posner and Weil (2012) argue for regulation, on the ground that such an activity reduces consumption smoothing and is therefore socially harmful. Duffie (2014) discusses various challenges to beliefs-based regulation and stresses the importance to base policies on social welfare principles. The analysis carried out here sheds light on some of these issues in the context of a stock market model. It shows that a ban on beliefs-based speculation could increase price volatility and reduce informational efficiency, leading to welfare losses. It provides explicit conditions under which stock market speculation is PO.

The regulation of insider trading has been a long standing issue. Arguments in favor of the deregulation of insider trading can be found in Manne (1966). Medrano and Vives (1994) study the effect of a disclose-or-abstain rule in a static Leland (1992) setting. In the present paper, we show that a contingent Tobin tax can be effective at enforcing PO equilibria as well as implementable.

Section 2 describes a benchmark model with asynchronous information flows and endogenous acquisition at the initial date. Section 3 presents the demand functions. Section 4 shows the non-existence of a NREE when noise trading is exogenous. Section 5 studies the NREE with endogenous

significant source of welfare gains. The analysis is comprehensive in that it encompasses all agents in the model.

[8]For information acquisition in static NREE see Verrechia (1982), Diamond (1985), and Goldstein and Yang (2015).

noise trading. Section 6 examines welfare properties. The optimal information infrastructure with endogenous frequency is analyzed in Section 8. Extensions are in Section 9 and conclusions follow. Proofs are in the appendix.

2 The Economy

This section describes benchmark model. The financial market is presented in Section 2.1, agents and their information sets in Section 2.2 and candidate stock price processes in Section 2.3. Preferences and optimal demands are in Sections 2.4 and 2.5. Equilibrium is defined in Section 2.6.

2.1 Assets and Markets

A riskless asset and a risky stock are available for trade. The riskless asset is a money market account paying interest at the rate r. In the absence of intertemporal consumption, which will be assumed, the interest rate can be set at zero ($r = 0$). The risky stock pays a liquidating dividend D_T at the terminal date T. The dividend is the terminal value of the process $dD_t = \mu^D dt + \sigma^D dW_t^D$, where μ^D is a constant drift coefficient and σ^D is a constant and positive volatility coefficient. W^D is a Brownian motion process with filtration $\mathcal{F}^D_{(\cdot)}$, defined on a probability space $\left(\Omega, \mathcal{F}^D, P\right)$. The process D can be viewed as a fundamental factor that eventually determines the terminal dividend.

One share of stock, perfectly divisible, is outstanding. It trades at an endogenously price S. Trading takes place in continuous time. There are no restrictions on stock holdings or borrowing.

2.2 Agents, Noise and Information Signal

Three groups of investors operate in the financial market, informed, uninformed and noise traders. The respective fractions of the three groups in the population are ω^i, ω^u and ω^n, with $\omega^i + \omega^u + \omega^n = 1$. Each group is treated as a homogeneous entity with a representative individual.

The (representative) informed investor is a skilled individual, able to extract at cost C information about the future stock payoff D_T. In the benchmark model, information extraction is carried out at the initial date $t = 0$ and generates the noisy signal $G = D_T + \zeta$, where $\zeta \sim \mathcal{N}\left(0, \left(\sigma^\zeta\right)^2\right)$. Skill is measured by the precision $v_\zeta = \left(\sigma^\zeta\right)^{-2}$ of the signal. When $\left(\sigma^\zeta\right)^2$ increases, precision falls and the informational content of the signal decreases. Thus, skill decreases. In the limit $\left(\sigma^\zeta\right)^2 \to \infty$, the signal becomes pure noise and skill vanishes. The "informed" investor effectively becomes unskilled (uninformed). The optimality of an information infrastructure producing a single signal at $t = 0$ is discussed in Section 8 (see Remark 11). Throughout sections 3-6, we will assume $C = 0$, unless explicitly stated.

The uninformed investor does not have extraction ability. He/she observes prices and other quantities that are in the public information set. Let $\mathcal{F}^m_{(\cdot)}$ be the public information filtration. The uninformed filtration is the public filtration $\mathcal{F}^u_{(\cdot)} = \mathcal{F}^m_{(\cdot)}$. The last investor trades on noise. His/her beliefs are similar to the beliefs of the informed, but depend on an independent random variable ϕ as opposed to true private information. A precise description is provided below.

The three groups are identically endowed at the outset. The distribution of initial stock shares in the population is $(\omega^i, \omega^u, \omega^n)$.

2.3 Stock Price and Information Flows

The opportunity set of investors depends on the stock price structure. In this environment, there are two sources of uncertainty, W^D associated with fundamental information and ϕ with noise trading speculation. Standard arguments can be invoked to write any candidate price process as,

$$dS_t = \mu_t^S dt + \sigma_t^S dW_t^S, \qquad S_T = D_T. \tag{1}$$

In this structure, W^S is a Brownian motion relative to the public information filtration $\mathcal{F}_{(\cdot)}^m$. It is endogenous and, ultimately, relates to the underlying source of fundamental uncertainty W^D. The coefficients (μ^S, σ^S) of the price process are also endogenous and adapted to $\mathcal{F}_{(\cdot)}^m$. The uninformed observes the stock price, hence can retrieve the volatility coefficient from its quadratic variation. The Brownian motion $dW_t^S = (\sigma_t^S)^{-1}(dS_t - \mu_t^S dt)$ is an innovation process in their filtration. The information filtration $\mathcal{F}_{(\cdot)}^S$ generated by S is in the public information flow $\mathcal{F}_{(\cdot)}^m$. That is, $\mathcal{F}_{(\cdot)}^S \subseteq \mathcal{F}_{(\cdot)}^m$.

The information flow of the informed is augmented by the private signal G. Private information is carried by the enlarged filtration $\mathcal{F}_{(\cdot)}^i \equiv \mathcal{F}_{(\cdot)}^m \vee \sigma(G)$. As private information modifies the perception of the risk-reward trade-off, the fundamental source of risk W^D is no longer Brownian motion relative to the enlarged filtration. Instead, the translated process,

$$dW_t^G = dW_t^S - \theta_t^{G|m}(G)\,dt \qquad \text{where} \qquad \theta_t^{G|m}(G)\,dt \equiv E\left[dW_t^S \middle| \mathcal{F}_t^i\right]$$

becomes a Brownian motion. The translation $\theta_t^{G|m}(G)$ is the private information price of risk (PIPR), which is a function of the private signal G. Relative to private information, the stock price evolution is $dS_t = \left(\mu_t^S + \sigma_t^S \theta_t^{G|m}(G)\right) dt + \sigma_t^S dW_t^G$. The superior information is reflected in the private information premium $\sigma_t^S \theta_t^{G|m}(G)$. Given that public information $\mathcal{F}_{(\cdot)}^m$ is endogenous, the private information premium is endogenous as well.

2.4 Informed and Uninformed Preferences

Throughout the paper, superscripts i and u are used to distinguish the informed (i) from the uninformed (u) investor. Let X_t^j denote the wealth of investor j at time t, $j \in \{i, u\}$. Conditional preferences have the von Neumann-Morgenstern representation,

$$\mathcal{U}^j\left(\mathcal{F}_0^j\right) = E\left[u\left(X_T^j\right) \middle| \mathcal{F}_0^j\right] \text{ for } j \in \{i, u\} \tag{2}$$

where the utility function has constant absolute risk aversion, $u(X) = -\Gamma \exp(-X/\Gamma)$. The parameter $\Gamma > 0$ is the common absolute risk tolerance coefficient. Preferences of the informed (resp. uninformed) are conditional on private (resp. public) information. For welfare comparisons, it is useful to consider expected utility conditional on public information at date zero, called interim expected utility. Interim expected utility is $\mathcal{U}^i = E\left[\mathcal{U}^i\left(\mathcal{F}_0^i\right)\middle|\,\mathcal{F}_0^m\right]$ for the informed agent and $\mathcal{U}^u = E\left[\mathcal{U}^u\left(\mathcal{F}_0^u\right)\middle|\,\mathcal{F}_0^m\right] = \mathcal{U}^u\left(\mathcal{F}_0^u\right)$ for the uninformed. The last equality holds because $\mathcal{F}_0^u = \mathcal{F}_0^m$.

Let N^j be the number of shares held. Investors maximize (2) subject to the dynamics of wealth,

$$dX_t^j = \begin{cases} N_t^i\left(\left(\mu_t^S + \sigma_t^S\theta_t^{G|m}(G)\right)dt + \sigma_t^S dW_t^G\right) & \text{for } j = i \\ N_t^u\left(\mu_t^S dt + \sigma_t^S dW_t^S\right) & \text{for } j = u \end{cases} \tag{3}$$

and the informational constraint that N^j be adapted to $\mathcal{F}_{(\cdot)}^j$ for $j \in \{i, u\}$.

2.5 Noise Trading: Beliefs and Preferences

The decisions of the informed are based on the conditional distribution of states given the realization of the private signal,

$$P^i(d\omega) \equiv P(d\omega|\,G = x)_{|x=G} = \left(\frac{P(G \in dx|\,\omega)}{P(G \in dx)}\right)_{|x=G} P(d\omega).$$

Informed beliefs are based on the conditional Wiener measure $P(d\omega|\,G = x)$. As the unconditional and conditional distributions of the signal, $P(G \in dx)$ and $P(G \in dx|\,d\omega)$, as well as the distribution of states $P(d\omega)$ are common knowledge, the conditional measure $P(d\omega|\,G = x)$ is known.

The noise trader is aware of the availability of private information and can calculate the conditional beliefs $P(d\omega|\,G = x)$, but does not observe the realization of the private signal. Instead, he/she relies on a forecast of this realization, $x = \phi$, and evaluates conditional beliefs at ϕ. Thus,

$$P^n(d\omega) \equiv P(d\omega|G = x)_{|x=\phi} = \left(\frac{P(G \in dx|\,\omega)}{P(G \in dx)}\right)_{|x=\phi} P(d\omega)$$

$$\frac{P(G \in dx|\,\omega)}{P(G \in dx)} = \exp\left(\int_0^T \theta_t^{G|m}(x)\,dW_t^S - \frac{1}{2}\int_0^T \theta_t^{G|m}(x)^2\,dt\right).$$

The distribution of the forecast ϕ can be arbitrary. In the sequel, ϕ is assumed to be independent and normally distributed with mean μ^ϕ and standard deviation σ^ϕ. If $\mu^\phi = E[G]$ and $\left(\sigma^\phi\right)^2 = VAR[G]$, the forecast is an unbiased and identically distributed estimate of G. Irrespective of the distributional structure, ϕ is noise.

Ultimately, the noise trader is an agent with bounded rationality who attempts to replicate the behavior of the informed, but without the benefit of observing the private signal. The noise trader and the informed share conditional beliefs, therefore have the same demand structures. The realized demands nevertheless differ, because one is truly informed, while the other is not. The noise trader, effectively, speculates by trading on noise (Black (1986)).

The noise trader's conditional preferences are $\mathcal{U}^n(\phi) = E^n\left[u\left(X_T^n\right)\middle|\,\mathcal{F}_0^m\right] = E_0\left[u\left(X_T^n\right)\middle|\,G = x\right]_{|x=\phi}$, where the expectation $E^n\left[\cdot|\,\mathcal{F}_0^m\right]$ is with respect to the beliefs P^n evaluated at a given realization of the forecast ϕ and the utility function is CARA with absolute risk tolerance parameter Γ. Under the beliefs P^n, the stock price evolves as $dS_t = \left(\mu_t^S + \sigma_t^S\theta_t^{G|m}(x)\right)dt + \sigma_t^S dW_t^x$, where $W_\cdot^x = W_\cdot^D - \int_0^\cdot \theta_t^{G|m}(x)\, dt$ is a $\mathcal{F}_{(\cdot)}^m$-Brownian motion under the conditional Wiener measure $P(d\omega|G = x)$, evaluated at $x = \phi$. The associated stock price of risk is $\theta_t^\phi \equiv \theta_t^m + \theta_t^{G|m}(x)$. Interim expected utility is $\mathcal{U}^n = E[\mathcal{U}^n(\phi)|\mathcal{F}_0^m] = E[\mathcal{U}^n(G)\mathcal{L}_{\phi,G}(G|\mathcal{F}_0^m)|\mathcal{F}_0^m]$, where,

$$\mathcal{L}_{\phi,G}(x|\mathcal{F}_0^m) \equiv \frac{P(\phi \in dx|\mathcal{F}_0^m)}{P(G \in dx|\mathcal{F}_0^m)}$$

is a density measuring the beliefs distortion relative to the informed investor. If $\mu^\phi = E[G]$ and $(\sigma^\phi)^2 = VAR[G]$, then $\mathcal{L}_{\phi,G}(x|\mathcal{F}_0^m) = 1$ and the beliefs distortion vanishes.

Given $x = \phi$, the noise trader maximizes $\mathcal{U}^n(\phi)$ over the $\mathcal{F}_{(\cdot)}^m$-progressively measurable number of shares N^n, subject to the dynamic budget constraint under conditional beliefs, $dX_t^n = N_t^n\left(\left(\mu_t^S + \sigma_t^S\theta_t^{G|m}(x)\right)dt + \sigma_t^S dW_t^x\right)$. Note that even in the absence of a beliefs distortion, i.e., when $\mathcal{L}_{\phi,G}(x|\mathcal{F}_0^m) = 1$, the noise trader exhibits bounded rationality as the optimal policy is chosen adapted to the public information flow $\mathcal{F}_{(\cdot)}^m$, given a fixed realization ϕ, rather than adapted to the enlarged filtration $\mathcal{F}_{(\cdot)}^m \bigvee \sigma(\phi)$. Full rationality fails, as the joint information conveyed by the realized forecast ϕ and the public information flow $\mathcal{F}_{(\cdot)}^m$ is ignored.[9]

2.6 Equilibrium

A competitive rational expectations equilibrium (REE) for the economy under consideration is a triplet of demands (N^u, N^i, N^n) and a price process $dS_t = \mu_t^S dt + \sigma_t^S dW_t^S$, $S_T = D_T$, such that (i) Individual rationality: N^j is optimal for agent $j \in \{u, i, n\}$ taking the price process as given, and (ii) Market clearing: $\omega^u N^u + \omega^i N^i + \omega^n N^n = 1$. The competitive REE is noisy (NREE) if the informed and uninformed filtration differs, $\mathcal{F}_{(\cdot)}^u \subset \mathcal{F}_{(\cdot)}^i$.

3 Optimal Stock Demands and Residual Demand

The next two propositions describe the stock demands of the various agents in the economy.

Proposition 1 *The optimal number of shares held by the uninformed and informed investors are,*

$$N_t^u = \Gamma\frac{\theta_t^m + h_t^u\sigma^D}{\sigma_t^S} \qquad \text{and} \qquad N_t^i = \Gamma\frac{\theta_t^m + \theta_t^{G|m}(G) + h_t^i(G)\sigma^D}{\sigma_t^S}$$

where θ^m is the price of risk for the uninformed and, with $E_t[\cdot] \equiv E_t[\cdot|\mathcal{F}_t^m]$,

$$h_t^u = -\frac{1}{2}\int_t^T \partial_{D_t} E_t\left[\xi_{t,v}^m(\theta_v^m)^2\right] dv$$

$$h_t^i(x) = h_t^u - \frac{1}{2}\int_t^T \partial_{D_t} E_t\left[\xi_{t,v}^m\left(\theta_v^{G|m}(x)^2 + 2\theta_v^m\theta_v^{G|m}(x)\right)\right] dv \equiv h_t^u + h_t^{G|m}(x).$$

[9] The beliefs distortion and the absence of Bayesian updating is similar to the bounded rational behavior studied in Dumas, Kurshev and Uppal (2009). Alternatively, under the assumption of uncertainty about the counterparty and learning, the uncertainty structure becomes similar to the one in Banerjee and Green (2015).

The terms $\Gamma h_t^u \sigma^D / \sigma_t^S, \Gamma h_t^i \sigma^D / \sigma_t^S$ *are the intertemporal hedging demands for the uninformed and informed. The informed holds more shares than the uninformed if and only if the private information premium exceeds the adjusted difference in hedging terms,* $\sigma_t^S \left(\theta_t^{G|m}(G) + h_t^{G|m}(G) \sigma^D \right) > 0$.

An investor with CARA utility seeks to hedge stochastic fluctuations in the opportunity set (Merton (1971)). The optimal stock demand has a mean-variance component as well as a dynamic hedging component. The demands in Proposition 1 have this basic structure in common.

The fundamental difference between the two investors resides in their evaluation of the expected stock return. The informed evaluates the return on the basis of private information as well as public information. The resulting expected return has two components. The first one, $\mu_t^S = \sigma_t^{S,D} \theta_t^m$, is the expected return based on public information. The second one, $\sigma_t^S \theta_t^{G|m}(G)$, is the additional premium calculated on the basis of private information. This premium is affine in the PIPR $\theta_t^{G|m}(G)$, i.e., the private information price of risk (Detemple and Rindisbacher (2013)). The PIPR is the incremental price of risk assessed in light of information that is not revealed by public information sources. It represents the private information price of risk conditional on public information. Thus, the informed mean-variance component has a public information part, $\Gamma \theta_t^m / \sigma_t^S$, and a private information part, $\Gamma \theta_t^{G|m}(G) / \sigma_t^S$. The uninformed mean-variance demand has only a public information part, $\Gamma \theta_t^m / \sigma_t^S$.

The difference in assessed expected returns induces a difference in hedging behavior. The informed agent hedges stochastic fluctuations in the market price of risk as well as in the PIPR. The total hedging demand can be decomposed as $h_t^i(G) = h_t^u + h_t^{G|m}(G)$, where $h_t^u, h_t^{G|m}(G)$ capture the two separate motives. The uninformed hedging demand stems entirely from the stochastic behavior of the price of risk. It is also of interest to note that the hedging demand depends on the information set of the agent considered. For the informed, the hedge is conditional on the private information filtration, hence parameterized by the private signal. For the uninformed, the hedge depends on the public filtration. The hedging demand formulas are valid for any given, but arbitrary, diffusion opportunity set. The specific functional form depends on the structure of the latter.

Proposition 2 *The optimal number of shares held by the noise trader is,*

$$N_t^n = \Gamma \frac{\theta_t^m + \theta_t^{G|m}(\phi) + h_t^n(\phi) \sigma^D}{\sigma_t^S} = \Gamma \frac{\theta_t^m + \theta_t^{G|m}(\phi) + h_t^i(\phi) \sigma^D}{\sigma_t^S} \tag{4}$$

where θ^m *is the uninformed price of risk,* $\theta_t^{G|m}(\phi)$ *is a speculative premium/discount reflecting the departure from rationality and* $\Gamma h_t^n \sigma^D / \sigma_t^S$ *is the intertemporal hedging demand. The hedging demand of the noise trader has the same functional form as that of the informed, but evaluated at* ϕ *instead of* G. *The noise trader holds more (resp. less) shares than the uninformed if and only if the speculative premium exceeds the adjusted difference in hedging terms,* $\sigma_t^S \left(\theta_t^{G|m}(\phi) + h_t^{G|m}(\phi) \sigma^D \right) > 0$.

The optimal noise trading demand has a mean-variance component and a hedging component. The mean-variance component has two parts. The first part, $\Gamma \theta_t^m / \sigma_t^S$, is the usual mean-variance demand of an uninformed rational agent. It reflects a demand based on public information. The second part, $\Gamma \theta_t^{G|m}(\phi) / \sigma_t^S$, is a speculative demand associated with an informational signal consisting of pure noise. The hedging component can also be split in two parts, $h_t^n(\phi) = h_t^{n,mpr}(\phi) + h_t^{n,pipr}(\phi)$, reflecting hedging motives stemming from stochastic fluctuations in the MPR and in the PIPR. Because the noise trader and the informed seek to hedge the same objects, the structures of their hedges are the same. The only difference is the conditioning factor, consisting of the true signal G for the informed and the independent random variable ϕ for the noise trader. Thus, $h_t^n(\phi) = h_t^i(\phi)$, $h_t^{n,mpr}(\phi) = h_t^u$ and $h_t^{n,pipr}(\phi) = h_t^{G|m}(\phi)$. In the end, the noise trader demand mimics the informed demand. It effectively corresponds to the demand of an investor with randomized beliefs, i.e., an unskilled active investor.

Remark 1 *The combined demand of the informed and noise trader, i.e., the residual demand, is,*

$$N_t \equiv \omega^i N_t^i + \omega^n N_t^n = \Gamma \frac{\omega\theta_t^m + \omega^i\left(\theta_t^{G|m}(G) + h_t^i(G)\,\sigma^D\right) + \omega^n\left(\theta_t^{G|m}(\phi) + h_t^n(\phi)\,\sigma^D\right)}{\sigma_t^S}$$

where $\omega = \omega^i + \omega^n$. The residual demand is an affine function of the weighted average price of risk (WAPR) $\Theta_t\left(G,\phi;\omega^i,\omega^n\right) \equiv \omega^i\theta_t^{G|m}(G) + \omega^n\theta_t^{G|m}(\phi)$ and of the weighted average normalized hedge (WANH) $h_t\left(G,\phi;\omega^i,\omega^n\right) \equiv \omega^i h_t^i(G) + \omega^n h_t^n(\phi)$. If the PIPR and hedges are also affine functions, the residual demand depends on $\Theta_t(Z;\omega) = \theta_t^{G|m}(Z;\omega)$ and $h_t(Z;\omega) = h_t^i(Z;\omega)$, which are functions of the signal $Z \equiv \omega^i G + \omega^n\phi$ and the combined population weight $\omega = \omega^i + \omega^n$.

4 Non-Existence of NREE with Exogenous Noise Trading

In order to put our noise trading model in perspective, it is useful to consider the traditional approach with exogenous noise trading (or stochastic supply). This section shows that a competitive NREE does not exist in such an economy.

Suppose that noise trading is exogenous and given by $N_t^n = \phi$ where ϕ is an independent random variable with Gaussian distribution.[10] Throughout the section, assume that $\sigma^D > 0$, $\sigma^\zeta < \infty$ and $\omega^i, \omega^n \in (0,1)$. The next proposition gives a necessary condition for the existence of a NREE in this setting with exogenous Gaussian noise trading. To state the result, let N be an arbitrary stochastic process and let $\nabla_M N_t$ for $M \in \{G,\phi\}$ be the perturbation of the random variable N_t with respect to the random variable M. The stochastic process $\{\nabla_M N_t : t \in [0,T]\}$ is the *first variation process* of N with respect to M (e.g., Kunita (1990)).

Proposition 3 *Consider the model with exogenous noise trading described above. A necessary condition for the existence of a NREE is,*

$$\nabla_G N_t^i \nabla_\phi N_s^n = \nabla_G N_s^i \nabla_\phi N_t^n; \quad P\otimes leb\otimes leb - a.e.\ on\ \Omega\times[0,T]\times[0,T] \tag{5}$$

[10] The setting with exogenous noise trading is observationally equivalent to a model with stochastic supply ϕ^s where $\phi^s = 1 - \omega^n\phi$ or a setting without noise trading where the informed holds a non-traded asset with terminal payoff $D_T\phi^i$ where $\phi^i \in \mathcal{F}_0^i$ is a privately known random variable independent of fundamental information $\mathcal{F}_{(\cdot)}^D$ and private signal G, and is given by $\phi^i = -\left(\omega^n/\omega^i\right)\phi$. The optimal informed portfolio is $N_t^{i,p} = N_t^i - \phi^i$ where N_t^i is the portfolio without the private asset. The residual demand becomes $N_t^{a,p} = \omega^i N_t^i - \omega^i\phi^i = N_t^a$. Models where the source of noise is the availability of a correlated private asset are considered by Medrano and Vives (2004) in the static case and Albuquerqe and Miao (2014) in the dynamic case under the assumption of myopia.

where leb is the Lebesgue measure, $\nabla_G N_t^i$ is the first variation process of the optimal informed demand with respect to G and $\nabla_\phi N_t^n$ the first variation process of the exogenous noise trading with respect to ϕ. As $\nabla_\phi N_t^n = 1$, (5) can be restated as $\nabla_G N_t^i = \nabla_G N_s^i$ on $\Omega \times [0,T] \times [0,T]$.

Condition (5) is tied to the informational content of the residual demand function, given by,

$$N_t \equiv \omega^i N_t^i + \omega^n N_t^n = \omega^i \Gamma \frac{\theta_t^m + h_t^u \sigma^D + \theta_t^{G|m}(G) + h_t^{G|m}(G)\sigma^D}{\sigma_t^S} + \omega^n \phi$$

in the model with exogenous noise trading. The residual demand function is observed by the uninformed, therefore belongs to the public information flow (see Kreps (1977)). To prevent revelation, it must be that observations at different times do not reveal the private signal. That is, the vector (N_t, N_s) cannot be inverted for any pair of times $t, s \in [0,T]$. Invertibility fails if and only if the determinant of the Jacobian of (N_t, N_s) is null, which leads to condition (5).

Proposition 4 *A competitive NREE does not exist in the model with exogenous noise trading.*

The reason for non-existence is that condition (5) fails, ensuring the revelation of the private signal G. The accumulation of information and the evolving volatility of the stock price lie at the core of this revelation property. These features ensure that the informational content of the residual demand is carried by $Z_t = \omega^i \delta(t) G + \omega^n \phi$ for some function of time $\delta(t)$. The time dependence of $\delta(t)$ ensures full revelation.

Remark 2 *(i) It has become standard practice to solve for equilibrium in asymmetric information models by postulating a linear equilibrium price function. Propositions 3 and 4 show that it is essential to verify the information flow generated by the residual demand. Equilibrium filtrations in these models can be non-Markovian in the price and contain the private signal, i.e., residual demands can be fully revealing. (ii) The NREE described in Brennan and Cao (1996) does not exist for arbitrary coefficients of the underlying processes. Generically, the price can be inverted to recover the private signal.*

The results above show that the model with exogenous noise trading and terminal dividend does not have a NREE. This provides further economic motivation for the study of speculative noise trading behavior, carried out next.

5 The NREE with Beliefs-Based Speculation

The competitive NREE is described in Section 5.1. Properties of the PIPR and the WAPR are examined in Section 5.2. Price and return properties are discussed in Section 5.3.

5.1 Equilibrium Structure

In order to present the main result, define the combined share of the informed and the noise trader $\omega = \omega^i + \omega^n$ and the functions of time,

$$\hat{\alpha}(t) \equiv \alpha(t) + \alpha^h(t), \qquad \hat{\beta}(t) \equiv \beta(t) + \beta^h(t), \qquad \hat{\gamma}(t) \equiv \gamma(t) + \beta^h(t) \tag{6}$$

where $\left(\alpha^h, \beta^h, \gamma^h\right)$ are associated with the aggregate hedging behavior of the agents and defined in the Appendix, and

$$\alpha(t) = \frac{1-\kappa_t\omega}{H(t)}\sigma^D, \qquad \beta(t) = -\omega\frac{1-\kappa_t\omega^i}{H(t)}\sigma^D, \qquad \kappa_t = \frac{\omega^i H(t)}{M(t)} \tag{7}$$

$$\gamma(t) = -\omega\frac{\left(1-\kappa_t\omega^i\right)\mu^D(T-t) - \omega^n\kappa_t\mu^\phi}{H(t)}\sigma^D, \qquad \lambda(t,s) = \frac{\omega^i\left(\sigma^D\right)^2(s-t)}{M(t)}, \ s \in [t,T] \tag{8}$$

$$H(t) = \left(\sigma^D\right)^2(T-t) + \left(\sigma^\zeta\right)^2, \qquad M(t) = \left(\omega^i\right)^2 H(t) + \left(\omega^n\right)^2\left(\sigma^\phi\right)^2. \tag{9}$$

The function $H(t) = Var\left(G|\mathcal{F}_t^D\right)$ is the conditional variance of the private signal G given fundamental information at t. The function $M(t) = Var\left(Z|\mathcal{F}_t^D\right)$ is the conditional variance of an endogenous signal $Z \equiv \omega^i G + \omega^n\phi$ given fundamental information at t. The coefficients $\kappa_t = \frac{COV(G,Z|\mathcal{F}_t^D)}{VAR(Z|\mathcal{F}_t^D)}$ and $\lambda(t,s) = \frac{COV(D_s,Z|\mathcal{F}_t^D)}{VAR(Z|\mathcal{F}_t^D)}$ are regression coefficients. The next proposition presents the NREE.

Proposition 5 *A competitive NREE exists. The equilibrium stock price is,*

$$S_t = \hat{A}(t)Z + \hat{B}(t)D_t + \hat{F}(t) \qquad \text{where} \qquad Z = \omega^i G + \omega^n\phi \tag{10}$$

$$\hat{B}(t) = B(t)B^h(t), \quad B(t) = \left(\frac{H(T)}{H(t)}\right)^\omega\left(\frac{M(T)}{M(t)}\right)^{1-\omega}, \quad B^h(t) = e^{\sigma^D\int_t^T\beta^h(v)dv} \tag{11}$$

$$\hat{A}(t) = \lambda(t,T) + \sigma^D\left(\int_t^T\hat{B}(s)\left(\hat{\alpha}(s) + \hat{\beta}(s)\lambda(t,s)\right)ds\right) \tag{12}$$

$$\hat{F}(t) = \hat{B}(t)\mu^D(T-t) - \frac{\left(\sigma^D\right)^2}{\Gamma}\int_t^T\hat{B}(s)^2 ds + \sigma^D\int_t^T\hat{B}(s)\hat{\gamma}(s)ds - \omega^n\hat{I}(t)\mu^\phi \tag{13}$$

$$\hat{I}(t) = \lambda(t,T) + \sigma^D\int_t^T\hat{B}(s)\hat{\beta}(s)\lambda(t,s)ds \tag{14}$$

and $\left(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \lambda\right)$ *as in (6)-(9). The coefficients of the equilibrium stock price process (1) are,*

$$\mu_t^S = \frac{\left(\sigma_t^S\right)^2}{\Gamma} - \sigma_t^S\left(\Theta_t(Z) + h_t^m(Z)\sigma^D\right), \qquad \sigma_t^S = \hat{B}(t)\sigma^D \tag{15}$$

$$\Theta_t(Z;\omega) = \alpha(t)Z + \beta(t)D_t + \gamma(t) \tag{16}$$

$$h_t^m(Z;\omega)\sigma^D = \alpha^h(t)Z + \beta^h(t)D_t + \gamma^h(t) \tag{17}$$

where $\Theta_t(Z;\omega) \equiv \omega^i \theta_t^{G|m}(G) + \omega^n \theta_t^{G|m}(\phi)$ *is the endogenous WAPR and* $h_t^m(Z;\omega) \equiv h_t^u + h_t^i(Z;\omega)$ *is the endogenous aggregate hedging. The innovation in the uninformed filtration is* $dW_t^S = dW_t^D - \theta_t^{D|m}dt$*, an* $\mathcal{F}_{(\cdot)}^m = \mathcal{F}_{(\cdot)}^{D,Z}$*-Brownian motion, where,*

$$\theta_t^{D|m} = \frac{E\left[dW_t^D|\mathcal{F}_t^m\right]}{dt} = \frac{\omega^i\sigma^D}{M(t)}\left(Z - \omega^i\left(D_t + \mu^D(T-t)\right) - \omega^n\mu^\phi\right). \tag{18}$$

The competitive equilibrium price in (10) is an affine function of the fundamental D and of the random variable Z. This random variable is a noisy translation of the private information signal G. It provides information about the terminal dividend, but is less informative than the private signal. Both the price S and the fundamental D are in the public information set $\mathcal{F}_{(\cdot)}^m$. It follows that Z is publicly observed as well. Thus, $Z \in \mathcal{F}_{(\cdot)}^m$ and $\mathcal{F}_{(\cdot)}^{D,Z} \subseteq \mathcal{F}_{(\cdot)}^{D,S} \subseteq \mathcal{F}_{(\cdot)}^m$. Conversely, the pair (D,Z) reveals the price S, i.e., $\mathcal{F}_{(\cdot)}^S \subseteq \mathcal{F}_{(\cdot)}^{D,Z}$. Thus, $\mathcal{F}_{(\cdot)}^{D,S} = \mathcal{F}_{(\cdot)}^{D,Z} \subseteq \mathcal{F}_{(\cdot)}^m$.

In equilibrium, the uninformed extracts the noisy signal Z from the pair (D,S). The uninformed also observes the residual aggregate demand function $\omega^i N_t^i + \omega^n N_t^n$, described in Remark 1. At equilibrium, the residual demand is also affine in D and Z. It therefore fails to reveal any information beyond what is already contained in (D,S): condition (5) holds. In the end, the equilibrium public information set consists of the pair (D,Z). That is, $\mathcal{F}_{(\cdot)}^{D,S} = \mathcal{F}_{(\cdot)}^{D,Z} = \mathcal{F}_{(\cdot)}^m$. The equilibrium uninformed filtration is $\mathcal{F}_{(\cdot)}^u = \mathcal{F}_{(\cdot)}^m = \mathcal{F}_{(\cdot)}^{D,S} = \mathcal{F}_{(\cdot)}^{D,Z}$. The equilibrium informed filtration is strictly more informative, $\mathcal{F}_{(\cdot)}^i = \mathcal{F}_{(\cdot)}^m \vee \sigma(G) \supset \mathcal{F}_{(\cdot)}^m = \mathcal{F}_{(\cdot)}^u$. The equilibrium is a NREE.

The specific impact of intertemporal hedging on equilibrium can be identified by comparing to the solution above with that for $\left(\alpha^h, \beta^h, \gamma^h\right) = (0,0,0)$. This benchmark case corresponds to a model in which all agents behave myopically and have traditional mean-variance demands. As revealed by Proposition 5, intertemporal hedging modifies the coefficients of the equilibrium relationships, not their overall structure. A surprising aspect is that the information content of equilibrium does not change. Indeed, the endogenous signal $Z = \omega^i G + \omega^n \phi$ is the same as in the benchmark pure mean-variance setting. This follows, because neither the PIPR, nor the WAPR, are affected by the hedging activities of agents. Both only depend on the public and private information structure.

Remark 3 *(Limit economy with small informed) Consider the limit economy with an infinitesimal informed population (*$\omega^i \to 0$ *and* $\omega^n \to 1-\omega^u = \omega$*). The limit equilibrium price is* $S_t^{si} = \hat{A}^{si}(t) Z^{si} + \hat{B}^{si}(t) D_t + \hat{F}^{si}(t)$*, where* $Z^{si} = \omega^n \phi$ *and,*

$$\mu_t^{S,si} = \frac{\left(\sigma_t^{S,si}\right)^2}{\Gamma} - \sigma_t^{S,si}\left(\Theta_t^{si}\left(Z^{si};\omega\right) + h_t^{m,si}\left(Z^{si};\omega\right)\sigma^D\right), \qquad \sigma_t^{S,si} = \hat{B}^{si}(t)\sigma^D$$

$$\Theta_t^{si}\left(Z^{si};\omega\right) = \alpha^{si}(t) Z^{si} + \beta^{si}(t) D_t + \gamma^{si}(t), \quad h_t^{m,si}\left(Z^{si};\omega\right)\sigma^D = \alpha^{h,si}(t) Z^{si} + \beta^{h,si}(t) D_t + \gamma^{h,si}(t)$$

with coefficients defined in (44)-(55). The limit WAPR is $\Theta_t^{si}\left(Z^{si};\omega\right) = \omega\theta_t^{G|m,si}(\phi)$*. Innovations in the uninformed filtration vanish* $dW_t^S = dW_t^D$ *because* $\theta_t^{D|m} \to 0$ *when* $\omega^i \to 0$*. The limit equilibrium fails to reveal any private information. If, in addition, there is no noise trader (*$\omega^i, \omega^n, \omega \to 0$*), the* *equilibrium collapses to a no-trade equilibrium where,*

$$S_t^{si,0} = D_t + \left(\mu^D - \frac{\left(\sigma^D\right)^2}{\Gamma}\right)(T-t), \qquad \sigma_t^{S,si,0} = \sigma^D, \qquad \mu_t^{S,si,0} = \frac{\left(\sigma^D\right)^2}{\Gamma}.$$

In the absence of hedging, price volatilities in the two limit economies and the economy of Proposition 5 rank as $\sigma_t^S < \sigma_t^{S,si} < \sigma_t^{S,si,0} = \sigma^D$ *for* $t < T$*. Intertemporal hedging can alter this ranking. Indeed, with a small informed, the market price of risk becomes more sensitive to the fundamental* ($-\beta^{si} > -\beta > 0$)*. Hedging amplifies this sensitivity* ($-\beta^{h,si} > \beta^h > 0$)*. Volatility, which is negatively related to the sensitivity of the market price of risk, decreases. Thus* $\sigma_t^S \gtrless \sigma_t^{S,si}$ *depending on parameter values. In all cases,* $\max\left(\sigma_t^S, \sigma_t^{S,si}\right) < \sigma_t^{S,si,0} = \sigma^D$ *for* $t < T$*. As the payment date approaches, the volatilities converge,* $lim_{t\to T}\sigma_t^S = lim_{t\to T}\sigma_t^{S,si} = lim_{t\to T}\sigma_t^{S,si,0} = \sigma^D$*. Informed trading increases the informational efficiency of the market. It stabilizes the market by reducing the stock's exposure to fundamental shocks if the hedging effect does not dominate (i.e.,* $\hat{B}(t) < \hat{B}^{si}(t)$*).*

Remark 4 *(Limit economy with small uninformed) Consider the limit economy with an infinitesimal uninformed population* ($\omega^u \to 0$ *and* $\omega^i \to 1-\omega^n$)*. The limit equilibrium price is* $S_t^{su} = \hat{A}^{su}(t) Z^{su} + \hat{B}^{su}(t) D_t + \hat{F}^{su}(t)$*, where* $Z^{su} = (1-\omega^n) G + \omega^n \phi$ *and,*

$$\mu_t^{S,su} = \frac{\left(\sigma_t^{S,su}\right)^2}{\Gamma} - \sigma_t^{S,su}\left(\Theta_t^{su}(Z^{su};1) + h_t^{m,su}(Z^{su};1)\sigma^D\right), \qquad \sigma_t^{S,su} = \hat{B}^{su}(t)\sigma^D$$

$$\Theta_t^{su}(Z^{su};1) = \alpha^{su}(t) Z^{su} + \beta^{su}(t) D_t + \gamma^{su}(t), \quad h_t^{m,su}(Z^{su};1)\sigma^D = \alpha^{h,su}(t) Z^{su} + \beta^{h,su}(t) D_t + \gamma^{h,su}(t)$$

with coefficients defined in (56)-(69). If, in addition, there is no noise trader ($\omega^i \to 1, (\omega^u, \omega^n) \to 0$)*, the equilibrium collapses to a no-trade equilibrium where,*

$$S_t^{su,0} = \hat{A}^{su,0}(t) G + \hat{B}^{su,0}(t) D_t + \hat{F}^{su,0}(t), \qquad Z^{su} = G$$

$$\mu_t^{S,su,0} = \frac{\left(\sigma_t^{S,su,0}\right)^2}{\Gamma}, \qquad \sigma_t^{S,su,0} = \sigma_t^{S,su} = \hat{B}^{su}(t)\sigma^D, \qquad \Theta_t^{su}(Z^{su};1) = h_t^{m,su}(Z^{su};1) = 0$$

with $\left(\hat{A}^{su,0}, \hat{B}^{su,0}, \hat{F}^{su,0}\right)$ *as in (70)-(75). The pair* $\left(D, S^{su,0}\right)$*, in the limit economy, is fully revealing. Stock price volatilities in the three equilibria rank as* $\sigma_t^{S,su} = \sigma_t^{S,su,0} < \sigma_t^S < \sigma^D$ *for* $t < T$*. As the payment date approaches,* $lim_{t\to T}\sigma_t^{S,su} = lim_{t\to T}\sigma_t^{S,su,0} = lim_{t\to T}\sigma_t^S = \sigma^D$*. Equilibrium prices in economies with small uninformed (large informed) populations are less sensitive to fundamental shocks and have lower volatility.*

Remark 5 *(Limit economy with small noise trader) Consider the limit economy with an infinitesimal noise trader population* ($\omega^i \to 1-\omega^u$ *and* $\omega^n \to 0$)*. The limit equilibrium price is* $S_t^{sn} = \hat{A}^{sn}(t) Z^{sn} + \hat{B}^{sn}(t) D_t + \hat{F}^{sn}(t)$*, where* $Z^{sn} = G$ *and,*

$$\mu_t^{S,sn} = \frac{\left(\sigma_t^{S,sn}\right)^2}{\Gamma} - \sigma_t^{S,sn}\left(\Theta_t^{sn}\left(Z^{sn};\omega^i\right) + h_t^{m,sn}\left(Z^{sn};\omega^i\right)\sigma^D\right), \qquad \sigma_t^{S,sn} = \hat{B}^{sn}(t)\sigma^D$$

$$\Theta_t^{sn}\left(Z^{sn};\omega^i\right)=\alpha^{sn}(t)\,Z^{sn}+\beta^{sn}(t)\,D_t+\gamma^{sn}(t)\,,\quad h_t^{m,sn}\left(Z^{sn};\omega^i\right)\sigma^D=\alpha^{h,sn}(t)\,Z^{sn}+\beta^{h,sn}(t)\,D_t+\gamma^{h,sn}(t)$$

with coefficients defined in (76)-(86). The pair (D,S^{sn}), in the limit economy, is fully revealing. Stock price volatilities in the three equilibria rank as $\sigma_t^{S,sn}<\sigma_t^S<\sigma^D$ for $t<T$. As the payment date approaches, $lim_{t\to T}\sigma_t^{S,sn}=lim_{t\to T}\sigma_t^S=\sigma^D$. The equilibrium price in the economy with small noise trader population is less sensitive to fundamental shocks and has lower volatility.

5.2 PIPR and WAPR Properties

To provide further insights about the structure of equilibrium, it is instructive to start with the PIPR. The PIPR is the (negative of the) instantaneous volatility of the growth rate of the conditional density of the private information signal given public information. In equilibrium, with $\mathcal{F}_t^m=\mathcal{F}_t^{D,Z}$,

$$\theta_t^{G|m}(G)=vol\left(\frac{dp_t^G(G)}{p_t^G(x)}\right)=\frac{G-\mu_t^{G|D,Z}}{\left(\sigma_t^{G|D,Z}\right)^2}vol\left(\mu_t^{G|D,Z}\right)=\frac{G-\mu_t^{G|D,Z}}{\left(\sigma_t^{G|D,Z}\right)^2}\left(1-\kappa_t\omega^i\right)\sigma^D.$$

In the model under consideration, given the linearity of the endogenous signal Z revealed, the conditional density is normal. The conditional mean alone depends on the dividend. The conditional variance is a function of time. The PIPR therefore reduces to the volatility of the conditional mean suitably normalized. It is affine in the private signal. As noted in Remark 1, it follows that the WAPR becomes $\Theta_t\left(G,\phi;\omega^i,\omega^n\right)\equiv\Theta_t(Z;\omega)$ and that the residual demand is an affine function of $\Theta_t(Z;\omega)$. The equilibrium risk premium inherits this affine structure. Moreover, the equilibrium residual demand, being affine in $\Theta_t(Z;\omega)$, also reveals the signal $Z=\omega^iG+\omega^n\phi$.

The next corollary describes the behavior of the endogenous PIPR.

Corollary 1 *The equilibrium PIPR is,*

$$\theta_t^{G|m}(G)=\frac{G-\mu_t^{G|D,Z}}{\left(\sigma_t^{G|D,Z}\right)^2}\left(1-\kappa_t\omega^i\right)\sigma^D=\alpha_1(t)\,G+\alpha_2(t)\,Z+\beta_0(t)\,D_t+\gamma_0(t)$$

$$\alpha_1(t)\equiv\frac{\sigma^D}{H(t)},\qquad\alpha_2(t)\equiv-\frac{\kappa_t\sigma^D}{H(t)}=-\frac{\omega^i\sigma^D}{M(t)},\qquad\beta_0(t)=\frac{\beta(t)}{\omega},\qquad\gamma_0(t)=\frac{\gamma(t)}{\omega}\tag{19}$$

where $\omega=\omega^i+\omega^n$ and $\beta(t),\gamma(t)$ are defined in (7)-(9). The coefficients $\alpha_1(t),\alpha_2(t)$ and $\beta(t)$ are the sensitivities with respect to the private signal G, the endogenous public signal Z and the fundamental D_t. The coefficient $\gamma(t)$ is a translation factor. The following properties hold,
(i) Sensitivity to information: $\alpha_1(t)>0,\alpha_2(t)<0,\beta(t)<0$.
(ii) Dynamic behavior: $\frac{\partial\alpha_1(t)}{\partial t}>0$, $\frac{\partial\alpha_2(t)}{\partial t}<0$, $\frac{\partial\beta(t)}{\partial t}<0$

The reaction of the equilibrium PIPR to news is intuitive. Indeed, a larger private signal indicates a greater terminal dividend, thus provides more valuable information. In contrast public information, be it endogenous or exogenous, reduces the local value of private information.

The evolution of these sensitivities is also intuitive. The reaction to private information $\alpha_1(t)$ is tamed by the unconditional variance of the signal $H(t)$ in the denominator. Over time, the informed observes the fundamental and updates the content of the private signal. Effectively, the residual private information is $G - D_t$. This residual signal becomes more informative over time, as uncertainty resolves, thereby enhancing the value of information. For the same reason, the precision of the endogenous public signal increases. This reduces the (negative) sensitivity of the PIPR to the endogenous signal, which decreases the value of private information. The reaction to fundamental information reflects the same effect. Its decrease further reduces the value of information.

The WAPR is closely related to the PIPR and inherits most of its properties.

Corollary 2 *The equilibrium WAPR is given by (16). The coefficients $\alpha(t)$ and $\beta(t)$ are the sensitivities with respect to the endogenous public signal and the fundamental information. The coefficient $\gamma(t)$ is a translation factor. The properties of $\beta(t)$ are the same as those of $\beta_0(t)$ in Corollary 1. The behavior of $\alpha(t)$ differs in the following respects,*
(i) Sensitivity to information: $\alpha(t) > 0$ if and only if $(\sigma^\phi)^2 > sH(t)$.
(ii) Dynamic behavior: $\alpha(t)$ increases with time if and only if $\kappa_t^2 < 1/\omega^i\omega$.

The behavior of $\alpha(t) = \alpha_1(t) + \alpha_2(t)\omega$ is more intricate because $\alpha_1(t), \alpha_2(t)$ have different, sometimes opposite properties. The evolution of $\alpha(t)$ is especially noteworthy. If $\omega^i\omega\kappa_0^2 < 1$, the coefficient increases over time. If $\omega^i\omega\kappa_0^2 > 1$ and $\omega^i\omega\kappa_T^2 < 1$, it initially decreases, then increases. If $\omega^i\omega\kappa_0^2 > 1$ and $\omega^i\omega\kappa_T^2 > 1$, it decreases throughout. The possibility of a U-shaped pattern reflects conflicting effects on $\alpha_1(t)$ and $\alpha_2(t)$. Under the conditions stated, the decrease in $\alpha_2(t)$ dominates early on, then is overtaken by the increase in $\alpha_1(t)$. An illustration is in Figure 1.

5.3 Price and Return Properties

Fundamental information accumulates with the passage of time, providing more precise estimates of the next dividend payment. Information accumulation affects the properties of equilibrium. The next corollary describes these dynamic aspects of the price and return components. For transparency, it first presents the pure mean-variance case, then describes the incremental effects of hedging.

Corollary 3 *(i) Suppose that $\alpha^h(t) = \beta^h(t) = \gamma^h(t) = 0$ for all $t \in [0,T]$ (pure mean-variance case). The stock price sensitivity to the fundamental increases over time. The volatility of the stock price, $\sigma_t^S = B(t)\sigma^D$, increases over time. The minimal and maximal volatility values are obtained at the initial and terminal dates,*

$$\sigma_0^S = B(0)\sigma^D = \left(\frac{H(T)}{H(0)}\right)^{\omega}\left(\frac{M(T)}{M(0)}\right)^{1-\omega}\sigma^D, \qquad \lim_{t\to T}\sigma_t^S = B(T)\sigma^D = \sigma^D.$$

The stock's price of risk $\mu_t^S/\sigma_t^S = B(t)\sigma^D/\Gamma - (\alpha(t)Z + \beta(t)D_t + \gamma(t))$ becomes more sensitive to the fundamental over time (i.e., $-\beta(t) > 0$ increases for all $t \in [0,T]$).

(ii) Suppose that $\alpha^h(t), \beta^h(t), \gamma^h(t)$ are given by their equilibrium values. Relative to case (i), the stock price is less sensitive to the fundamental ($\hat{B}(t) < B(t)$), the stock's price of risk is more sensitive ($\hat{\beta}(t) > \beta(t)$) and volatility decreases ($\hat{B}(t)\sigma^D < B(t)\sigma^D$). The coefficient $\hat{\beta}(t)$ can increase or decrease over time. The coefficient $\hat{B}(t)$ and the volatility $\sigma_t^S = \hat{B}(t)\sigma^D$ increase over

time. Volatility converges to the volatility of the fundamental, $\lim_{t\to T}\sigma_t^S = \sigma^D$*, and can reach a minimal value at some intermediate time* $t_* \in (0,T)$*.*

In order to understand the behavior of the price and its characteristics, it is useful to start with the benchmark pure mean-variance case (i). At the initial date, the uninformed extracts the noisy signal Z from the price. In the absence of a hedging motive, this information is most valuable when there is no other source of information, i.e., at the initial date. In the early stages of the economy, the price is heavily influenced by this initial information and, for this reason, does not react significantly to fundamental information. Over time, fundamental information accumulates, reducing the usefulness of the initial piece of information extracted. The impact of fundamental information (resp. the endogenous noisy signal) on the stock price grows (resp. decreases), thereby increasing the stock's volatility. The same phenomenon applies to the price of risk. Its sensitivity to fundamental information increases, implying that the price of risk volatility increases over time.

Intertemporal hedging, in case (ii), has intricate effects on some of these patterns. The positive correlation between the stock's price of risk and the fundamental induces agents to hedge so as to reduce the sensitivities of their demands to the fundamental. The aggregate demand inherits this property. Market clearing implies that the sensitivity of the equilibrium market price of risk is negatively related to the sensitivity of the aggregate hedging demand,

$$\theta_t^m = \frac{\sigma_t^S}{\Gamma} - \Theta_t(Z) - h_t^m(Z)\,\sigma^D \Longrightarrow \frac{\partial\theta_t^m}{\partial D_t} = -\frac{\partial\Theta_t(Z)}{\partial D_t} - \frac{\partial h_t^m(Z)}{\partial D_t}\sigma^D.$$

The sensitivity of the MPR therefore increases. So does the sensitivity of the risk premium. The price, inversely related to the premium, becomes less sensitive. Volatility therefore decreases relative to the mean-variance case. Moreover, hedging incentives have a cumulative effect on volatility, which becomes stronger as the horizon recedes. The volatility discount $B^h(t) = \hat{B}(t)/B(t)$ is therefore smaller at longer horizons ($B^h(t)$ decreases as $T-t$ increases). When the time to dividend payment is sufficiently long, volatility can become a small fraction of the volatility of the fundamental and the volatility of the equilibrium mean-variance price. In such a scenario, private information, which is the ultimate source of all effects, is a dominant stabilizing force. At the opposite end, when the payment date approaches, the hedging motive weakens, to eventually vanish, and volatility behaves as in the pure mean-variance setting. Figure 2 illustrates typical volatility patterns.

Remark 6 *Standard local volatility estimators motivated by weak form informational efficiency, that only condition on the price, are biased in the presence of private information. These estimators converge to the Markovian volatility coefficient* $\sigma_t^{S,M} \equiv \sqrt{\lim_{h\to 0} VAR\left[S_{t+h} - S_t | S_t\right]/h} = \sigma^D\widehat{B}(t)\sqrt{1+2\omega^i\widehat{A}(t)/\widehat{B}(t)}$ *rather than* $\sigma_t^S = \sigma^D\widehat{B}(t)$*. The bias,* $\sigma_t^S\left(\sqrt{1+2\omega^i\widehat{A}(t)/\widehat{B}(t)}-1\right)$*, is positive if and only if* $\hat{A}(t)$ *is positive. A sufficient condition for upward bias is* $\hat{\alpha}(t) > 0$*. This condition is satisfied for sufficiently large noise trading variance or low informed weight. Even if the condition fails,* $\hat{A}(t)$ *tends to be positive due to the positive leading coefficient in (12). See Figure 2.*

6 Welfare Analysis

This section examines the welfare properties of equilibrium when trading based on private information and/or speculation are permitted. In order to carry out a meaningful comparison of welfare across equilibria with and without private information, expected utilities conditional on initial public information, i.e., interim expected utilities, are calculated and compared. The equilibrium with private information is interim PO if and only if the interim welfare of all agents improves.

6.1 Welfare of Informed and Uninformed Agents

The certainty-equivalent (CE) value of private information is an important component of the welfare of the informed. It captures the utility gain from the use of private information for trading. It equals,

$$\begin{aligned}\mathcal{I}^i(G|Z) &= \frac{1}{2}E\left[\xi_T^i \int_0^T \theta_t^{G|m}(G)\left(\theta_t^{G|m}(G)+2\theta_t^m\right)dt\right|\mathcal{F}_0^i\Big] \\ &= \frac{1}{2}E\left[\xi_T^m \int_0^T \theta_t^{G|m}(G)\left(\theta_t^{G|m}(G)+2\theta_t^m\right)dt\right|\mathcal{F}_0^m\Big] \qquad (20)\end{aligned}$$

where the second equality follows because the informed state price density depends on the inverse of the density process of the conditional Wiener measure, i.e., $\xi_T^G = \eta_T^G \xi_T^m$ where $\eta_T^G \equiv (P_T(G \in dx)/P(G \in dx))^{-1}$.[11] In the NREE, the value of private information quantifies the difference between the welfare of the informed and that of the uninformed investor.

In order to compare welfare across economies with and without private information, the equilibrium public information structure matters. In the economy without private information, information is homogeneous and generated by the fundamental. In the NREE, the uninformed extracts the endogenous signal Z from equilibrium. The CE value of this signal for the uninformed stems from the associated trading gains. It is a quadratic function of the hedge-adjusted WAPR $\vartheta(Z|D_t) \equiv \Theta_t(Z;\omega) + h_t^m(Z;\omega)$, given by,

$$\mathcal{I}^u(Z) \equiv \frac{1}{2}E_0\left[\xi_T^m \int_0^T \vartheta_t(Z|D_t)^2\,dt\right] = \frac{1}{2}E\left[\xi_T^m \int_0^T \vartheta_t(Z|D_t)^2\,dt\right|\mathcal{F}_0^u\Big] \qquad (21)$$

where we recall that $E_0[\cdot] \equiv E[\cdot|\mathcal{F}_0^m]$. The weighted average structure of the WAPR implies that it can also be written as a linear function of the value of private information for the informed.

Proposition 6 *In the equilibrium without private information, all agents are uninformed. Initial (interim) utilities and wealth levels are,*

$$\mathcal{U}^{j,ni} = u\left(X_0^j\right)\exp\left(-\frac{1}{2\Gamma^2}\int_0^T\left(\sigma_t^{S,ni}\right)^2dt\right), \qquad j \in \{u,i\} \qquad (22)$$

[11] Note that the expectation of a random variable based on the conditional Wiener measure, $E[\cdot|G=x]$, is the same as the unconditional expectation of the random variable scaled by the density process of the conditional Wiener measure, $E[\cdot|G=x] = E\left[(\eta_T^x)^{-1}\cdot\right]$.

$$X_0^j = N_0^j \left(D_0 + \left(\mu^D - \frac{\left(\sigma^D\right)^2}{\Gamma} \right) T \right), \qquad \sigma_t^{S,ni} = \sigma^D. \tag{23}$$

Optimal share allocations are equal to the initial share endowments, $N_v^i = N_v^u = N_v^n = 1$ *for all* $v \in [0,T]$.[12] *Consider the NREE with initial share endowments* $\overline{N}_0^j = 1$ *for* $j \in \{u,i,n\}$. *Interim expected utilities differ by the interim value of private information,* $E_0\left[\mathcal{U}^i\right] = \mathcal{U}^u E_0\left[\exp\left(-\mathcal{I}^i\left(G|Z\right)\right)\right]$, *where* $\mathcal{I}^i\left(G|Z\right) \geqslant 0$ *is given by (20) and,*

$$\mathcal{U}^u = u\left(X_0^u\right)\exp\left(-\frac{1}{2\Gamma^2}\int_0^T \left(\sigma_t^S\right)^2 dt + \frac{1}{\Gamma}\int_0^T \sigma_t^S \vartheta_t\left(Z|E_0\left[\xi_t^m D_t\right]\right)dt - \mathcal{I}^u\left(Z\right)\right) < 0 \tag{24}$$

$$X_0^u = \overline{N}_0^u\left(\widehat{A}\left(0\right)Z + \widehat{B}\left(0\right)D_0 + \widehat{F}\left(0\right)\right), \qquad \sigma_t^S = \widehat{B}\left(t\right)\sigma^D \tag{25}$$

with $\vartheta\left(Z|D_t\right) \equiv \Theta_t\left(Z;\omega\right) + h_t^m\left(Z;\omega\right)$.

Interim utilities, in the NREE, can be decomposed as,

$$\mathcal{U}^u \equiv E_0\left[u\left(X_T^u\right)\right] = -\Gamma\exp\left(-\frac{X_0^u}{\Gamma}\right)\exp\left(-E_0\left[\xi_T^m\log\xi_T^m\right]\right)$$

$$E_0\left[\mathcal{U}^i\right] \equiv E_0\left[u\left(X_T^i\right)\right] = -\Gamma\exp\left(-\frac{X_0^i}{\Gamma}\right)E_0\left[\exp\left(-E\left[\left.\xi_T^G\log\xi_T^G\right|\mathcal{F}_0^i\right]\right)\right]$$

with $X_0^u = X_0^i = S_0$. The first term, in each decomposition, is the value of initial stock holdings, which is measurable relative to public information. These values are the same because initial endowments correspond to the equilibrium holdings in the economy without private information and are therefore the same across representative agents.[13] The second term, related to the value of the gains from trade, captures the benefits of trading. The value of the gains from trade for the informed is,

$$E\left[\left.\xi_T^G\log\xi_T^G\right|\mathcal{F}_0^i\right] = E\left[\left.\xi_T^G\log\xi_T^m\right|\mathcal{F}_0^i\right] + E\left[\left.\xi_T^G\log\eta_T^G\right|\mathcal{F}_0^i\right] = E\left[\left.\xi_T^m\log\xi_T^m\right|\mathcal{F}_0^m\right] + E\left[\left.\xi_T^m\log\eta_T^x\right|\mathcal{F}_0^m\right]_{|x=G}$$

where $E\left[\left.\xi_t^m\log\xi_T^m\right|\mathcal{F}_0^m\right]$ is the value of the gains from trade for the uninformed. The increment,

$$E\left[\left.\xi_t^m\log\eta_T^x\right|\mathcal{F}_0^m\right]_{|x=G} = \frac{1}{2}E_0\left[\xi_T^m\int_0^T \theta_v^{G|m}\left(x\right)\left(\theta_v^{G|m}\left(x\right) + 2\theta_v^m\right)dv\right]_{|x=G} = \mathcal{I}^i\left(G|Z\right)$$

corresponds to the value of private information (20). It is positive, implying that the informed is always better off than the uninformed.

It is also instructive to examine the constituents of the value of the gains from trade for the uninformed. As shown by (24), this value splits in three parts,

$$-E_0\left[\xi_T^m\log\xi_T^m\right] = -\frac{1}{2}\int_0^T E_0\left[\xi_t^m\left(\theta_t^m\right)^2\right]dt = -\frac{1}{2\Gamma^2}\int_0^T\left(\sigma_t^S\right)^2 dt + \frac{1}{\Gamma}\int_0^T \sigma_t^S\vartheta_t\left(Z|E_0\left[\xi_t^m D_t\right]\right)dt - \mathcal{I}^u\left(Z\right).$$

[12] Optimal demands are identical $N_t^j = \Gamma\theta_t^m/\sigma_t^S$. In equilibrium $\sum_j \omega^j N_t^j = 1$ so that $N_t^j = 1$ for $j \in \{i,u,n\}$.

[13] By assumption, the distribution of initial shares is $\left(\omega^i,\omega^u,\omega^n\right)$. The representative individual of each group therefore owns 1 share at the outset. Aggregate endowment is $\omega^i 1 + \omega^u 1 + \omega^n 1 = 1$, where 1 is the supply of shares.

To shed light on this expression, note that $\theta_t^m = \sigma_t^S/\Gamma - \vartheta_t(Z|D_t)$ so that $(\theta_t^m)^2 = \left(\sigma_t^S/\Gamma\right)^2 - 2\sigma_t^S\vartheta_t(Z|D_t)/\Gamma + \vartheta_t(Z|D_t)^2$. The first term, $\frac{1}{2\Gamma^2}\int_0^T \left(\sigma_t^S\right)^2 dt$, therefore reflects the variance impact on the endogenous market price of risk. This term is positive because a greater riskiness induces an increased equilibrium price of risk. The last term, $\mathcal{I}^u(Z) = \frac{1}{2}\int_0^T E_0\left[\xi_t^m \vartheta_t(Z|D_t)^2\right] dt$, is the value of the endogenous information signal extracted from equilibrium adjusted by the corresponding hedging term. It is also positive, because information improves the efficiency of the pricing of risk and the resulting gains from trade. The middle term, $\frac{1}{\Gamma}\int_0^T \sigma_t^S \vartheta_t(Z|E_0[\xi_t^m D_t])\, dt$, captures the interaction between the risk and information components of the price of risk. This specific form emerges because the stock volatility is deterministic and the hedge-adjusted WAPR is an affine function of the fundamental (implying $E_0[\xi_t^m \vartheta_t(Z|D_t)] = \vartheta_t(Z|E_0[\xi_t^m D_t])$). In the equilibrium without private information, investors are symmetric in all respects. The market price of risk is then entirely determined by the riskiness of the stock. The value of the gains from trade are completely driven by the stock's variance.

The relation between informed and uninformed utilities in the NREE simplifies welfare comparisons across equilibria. Both agents are better off if the welfare of the uninformed improves.

Proposition 7 *The uninformed is better off in the NREE, $\Delta^u \equiv \mathcal{U}^u - \mathcal{U}^{u,ni} \geqslant 0$, if and only if $\frac{\Delta\widehat{P}^u(Z)}{\Gamma} + \Delta\widehat{T}^u(Z) \geqslant 0$, where $\Delta P^u(Z) = S_0 - S_0^{ni}$ is the gain/loss from the valuation of initial share endowments (price impact) and $\Delta T^u(Z)$ is the gain/loss from dynamic trading (trading impact). The price and trading impacts are,*

$$\Delta\widehat{P}^u(Z) = \left(\widehat{B}(0) - 1\right)\left(D_0 + \mu^D T\right) + \widehat{A}(0)\, Z - \frac{\Delta\widehat{V}}{\Gamma} + \sigma^D \int_0^T \widehat{B}(s)\,\widehat{\gamma}(s)\, ds - \omega^n \widehat{I}(0)\,\mu^\phi \tag{26}$$

$$\Delta\widehat{T}^u(Z) = \frac{\Delta\widehat{V}}{2\Gamma^2} - \frac{1}{\Gamma}\int_0^T \sigma_t^S \vartheta(Z|E_0[\xi_t^m D_t])\, dt + \mathcal{I}^u(Z) \tag{27}$$

where $\Delta\widehat{V} = \left(\sigma^D\right)^2 \int_0^T \left(\widehat{B}(t)^2 - 1\right) dt < 0$ is the reduction in the realized variance of the price. The welfare of the uninformed improves, in particular, if risk tolerance is sufficiently large ($\lim_{\Gamma\to+\infty} \Delta^u = +\infty$). For sufficiently small risk tolerance, $\lim_{\Gamma\to 0} \Delta^u = -sgn\left(\Delta\widehat{V} + \Delta\widehat{H}\right) \times \infty$ where $\Delta\widehat{H} \equiv 2\Delta\widehat{H}^S - \Delta\widehat{H}^T$, defined in (89)-(90), captures the net effect of hedging on the price impact ($2\Delta\widehat{H}^S$) and the trading impact ($\Delta\widehat{H}^T$).

Proposition 7 identifies the sources of welfare gains and losses for the uninformed when private information trades are allowed. The first effect, $\Delta\widehat{P}^u(Z)$, captures the price impact on the initial stock holdings of the uninformed. It can be positive or negative depending on parameter values. It is positive if the partial dissemination of private information in the NREE causes a sufficient reduction in risk. The second effect, $\Delta\widehat{T}^u(Z)$, captures the change in the gains from trade. This component splits into three parts, a riskiness effect ($\Delta V/2\Gamma^2$), an informational efficiency effect ($\mathcal{I}^u(Z)$) and a noise trading beliefs effect ($-\int_0^T \sigma_t^S \vartheta(Z|E_0[\xi_t^m D_t])\, dt/\Gamma$). Allowing private information trading reduces the volatility of the stock ($\int_0^T \widehat{B}(t)^2\, dt < T$). As a result, the market price of risk decreases, which also reduces welfare. The first part is then negative. It also disseminates private information and increases the informational efficiency of the market. Better information improves investment allocations and leads to welfare gains. The second part is positive. Finally, permitting the use of private information will prompt the emergence of noise traders, whose activity limits efficiency gains. The bias induced by their activities can be a source of welfare gains or losses. The third part can take either sign. Overall, when risk tolerance is large, the positive informational efficiency effect dominates and the welfare of the uninformed improves. When risk tolerance is small, the riskiness effect dominates. It increases (decreases) the stock price when hedging does not (does) dominate, implying a positive (negative) price impact. It also reduces (increases) the price of risk, generating a negative (positive) trading impact. The price impact dominates leading to an overall welfare gain (loss).

The next corollary provides further insights. To simplify notation, define,

$$J \equiv \left(K_{1,T}^{\Delta\mathcal{U}}\right)^2 - 4K_{0,T}^{\Delta\mathcal{U}}K_{2,T}^{\Delta\mathcal{U}}, \qquad \Gamma_{\pm} = \frac{2K_{0,T}^{\Delta\mathcal{U}}}{-K_{1,T}^{\Delta\mathcal{U}} \pm \sqrt{J}}, \qquad \Gamma_0 = -\frac{K_{1,T}^{\Delta\mathcal{U}}}{K_{2,T}^{\Delta\mathcal{U}}}$$

where the coefficients $K_{j,T}^{\Delta\mathcal{U}}$ for $j \in \{0,1,2\}$ are defined in (91)-(93). The coefficients $K_{0,T}^{\Delta\mathcal{U}}, K_{1,T}^{\Delta\mathcal{U}}$ depend on Z. In contrast, $K_{2,T}^{\Delta\mathcal{U}} = -\frac{1}{2}\left(\Delta\widehat{V} + \Delta\widehat{H}\right)$ is independent of Z. With this notation,

Corollary 4 *The uninformed is as well off in the NREE as in the equilibrium without private information under the following conditions,*

	$K_{0,T}^{\Delta\mathcal{U}} > 0$	$K_{0,T}^{\Delta\mathcal{U}} = 0$	
$J > 0$	$\Gamma \leqslant \Gamma_- \vee 0$ *or* $\Gamma_+ \vee 0 \leqslant \Gamma$	$K_{1,T}^{\Delta\mathcal{U}} > 0$	$\Gamma \geqslant \Gamma_0 \vee 0$
		$K_{1,T}^{\Delta\mathcal{U}} < 0$	$\Gamma \leqslant \Gamma_0 \vee 0$
$J \leqslant 0$	$\Gamma \geqslant 0$	$K_{1,T}^{\Delta\mathcal{U}} = 0$ *and* $K_{2,T}^{\Delta\mathcal{U}} \geqslant 0$	$\Gamma \geqslant 0$

Corollary 4 shows that the welfare of the uninformed depends on hedging. Generically, $K_{0,T}^{\Delta\mathcal{U}} > 0$. Thus, if $\Delta\widehat{V} + \Delta\widehat{H} \leqslant -2\left(K_{1,T}^{\Delta\mathcal{U}}\right)^2 / K_{0,T}^{\Delta\mathcal{U}}$, the uninformed is better off in the NREE as in this case $J \leqslant 0$. If $\Delta\widehat{V} + \Delta\widehat{H} > -2\left(K_{1,T}^{\Delta\mathcal{U}}\right)^2 / K_{0,T}^{\Delta\mathcal{U}}$, the reverse can hold. In this case, an improvement is nevertheless assured for Γ sufficiently large, i.e., $\Gamma \geqslant \Gamma_+$. Moreover, a necessary condition for the dominance of the equilibrium without private information is that the net effect of hedging be strictly positive and sufficiently large, $\Delta\widehat{H} > -\Delta\widehat{V} > 0$. Ultimately, the corollary identifies parameter regions where banning private information reduces the welfare of the uninformed. Figure 3 illustrates possible configurations.

Remark 7 *Utility gains are not monotone in the realized variance change ΔV. The uninformed can therefore be worse off in the presence of private information despite the fact that volatility decreases. This property cannot hold in a static model where maximizing CARA utility is equivalent to mean-variance maximization. In dynamic models, intertemporal hedging alters the monotone relation between volatility and welfare found in static models.*

6.2 Welfare of Noise Trader

Let $f_{\phi|Z}(x|\,Z)$ (resp. $f_{G|Z}(x|\,Z)$ be the Gaussian density of ϕ (resp. G) conditional on Z. The likelihood ratio $\mathcal{L}_{\phi,G}(x|\,Z) \equiv f_{\phi|Z}(x|\,Z)/f_{G|Z}(x|\,Z)$ captures the beliefs divergence between the noise trader and the informed. Explicit formulas are in the proof of Corollary 6 in Appendix B.

Corollary 5 *The noise trader's interim utility is* $E_0[\mathcal{U}^n(\phi)] = \mathcal{U}^u E_0\left[\exp\left(-\mathcal{I}^i(G|Z)\right)\mathcal{L}_{\phi,G}(G|Z)\right]$ *where* $\mathcal{L}_{\phi,G}(G|Z) = f_{\phi|Z}(G)/f_{G|Z}(G)$*. The noise trader is as well off as the uninformed trader if and only if* $E_0\left[\exp\left(-\mathcal{I}^i(G|Z)\right)\mathcal{L}_{\phi,G}(G|Z)\right] \leqslant 1$*. The noise trader is as well of as the informed if* $\omega^i = \omega^n$*,* $VAR[G] = VAR[\phi]$ *and* $E[\phi] = E[G]$*, so that* $\mathcal{L}_{\phi,G}(G|Z) = 1$*.*

The corollary expresses the interim welfare of the noise trader relative to that of the uninformed. The interim utility differential stems from the difference in the gains from trade, reflected in the CE $\mathcal{I}^i(G|Z)$, and the difference in beliefs, reflected in the likelihood ratio $\mathcal{L}_{\phi,G}(G|Z)$. If the noise trader happens to have the same conditional beliefs as the informed, i.e., if the conditional distributions of ϕ and G given public information coincide, then $\mathcal{L}_{\phi,G}(G|\,Z) = 1$. In this case, the interim utilities coincide, $E_0[\mathcal{U}^n(\phi)] = \mathcal{U}^u E_0\left[\exp\left(-\mathcal{I}^i(G|Z)\right)\right] = E_0\left[\mathcal{U}^i(G)\right]$.

6.3 Pareto Optimal NREE

The relations between interim utilities in the NREE simplify welfare comparisons across equilibria. Pareto dominance of the NREE over an equilibrium where investors are symmetric is ensured if the interim welfare of the uninformed and the noise trader improves.

Corollary 6 *The NREE is (weakly) interim PO if and only if* $\Delta^u \geqslant 0$ *and the certainly equivalent gain of the noise trader gain is sufficiently high,* $-\Gamma\log E_0\left[\exp\left(-\mathcal{I}^i(G|Z)\right)\mathcal{L}_{\phi,G}(G|Z)\right] > -\Delta\widehat{P}(Z) + \Gamma\Delta\widehat{T}^u(Z)$*. If* $\mu^\phi = E[G]$*,* $\omega^i = \omega^n$ *and* $VAR[\phi] = VAR[G]$*, then the interim and therefore also the ex-ante utilities of the noise trader and the informed are identical,* $E_0[\mathcal{U}^n(\phi)] = E_0\left[\mathcal{U}^i(G)\right]$ *and* $E[\mathcal{U}^n(\phi)] = E\left[\mathcal{U}^i(G)\right]$*. In this case, the NREE is (weakly) PO under the conditions of Corollary 4.*

The conditions for weak Pareto optimality ensure that all agents are as well off. When risk tolerance converges to zero, the uninformed utility eventually becomes at least as large in the NREE because of the price impact (Proposition 7). At the same time, if beliefs are unbiased, the differential trading impact vanishes, ensuring that the noise trader attains the same ex-ante utility as the uninformed. The NREE becomes PO. If the noise trader happens to have the same beliefs as the informed, he/she reaches the same ex-ante utility. The NREE is then PO under the conditions ensuring that the uninformed agent is as well off.

Corollary 6 has ramifications for market regulation. Permitting private information trades is Pareto efficient when risk tolerance is sufficiently low, or sufficiently large and $0 \leqslant \Delta^u + \mathcal{I}^n + \Delta T^n$ holds. In those cases, either the informational efficiency gains or the decrease in the riskiness of the stock market dominate, leading to a welfare improvement. Scope for regulation exists in intermediate cases. In these cases, factors such as the behavior of the noise trader, the properties of dividends and the weights of the various investor populations matter, and have to be evaluated to determine the relevance of regulatory constraints.

Corollary 7 *Suppose that* $\Delta\widehat{P}(Z) + \Gamma\Delta\widehat{T}^u(Z) > \Gamma \log E_0\left[\exp\left(-\mathcal{I}^i(G|Z)\right)\mathcal{L}_{\phi,G}(G|Z)\right]$ *and that the conditions of Corollary 4 hold. The NREE is then (weakly) PO.*

Under the corollary's conditions, regulation banning usage of private information reduces welfare.

Remark 8 *The results above extend Leland's (1992) analysis to a dynamic competitive setting. In a static framework, the dynamic trading components* $(\Delta T^n, \Delta T^u)$ *are absent. So are intertemporal aspects of the price impacts* $(\Delta P^n, \Delta P^u)$*, such as volatility* (ΔV)*. As conjectured by Leland, some dynamic effects, e.g., hedging effects, can dampen the price impact.*[14] *However, for sufficiently high risk tolerance, the trading impact dominates and ensures Pareto optimality.*

Remark 9 *Suppose that information production costs* $C > 0$*. The assertions in this section remain valid under the additional condition that the certainty equivalent gain for the informed be non-negative,* $GA \equiv CE^{i,wa} - CE^{i,na} \geqslant 0$*. Here* $CE^{i,wa} = -\Gamma \log E\left[\mathcal{U}^i\right] - C$ *with* $\log \mathcal{U}^i = -\frac{S_0}{\Gamma} - E\left[\xi_T^G \log \xi_T^G \middle| \mathcal{F}_0^i\right]$ *is the ex-ante certainty equivalent of the informed in the NREE with information and* $CE^{i,na} = D_0 + \mu^D T - \frac{(\sigma^D)^2}{2\Gamma}$ *is the ex-ante certainty equivalent in the no-trade equilibrium without information.*

6.4 Speculation and Pareto Optimality

In order to pinpoint the role of speculation, suppose that the noise trader behaves as an uninformed agent. The population weights are then $\omega^n = 0$ and $\omega^u = 1 - \omega^i$. Let $\mathcal{E}^{ns}$ denote the resulting equilibrium without speculation. Also assume that the information production cost is $C > 0$.

Corollary 8 *Under the conditions of Corollary 7 and Remark 9, the NREE* $\mathcal{E}^{nree}$ *Pareto dominates the equilibrium without speculation* $\mathcal{E}^{ns}$ *. Dominance holds whether informed agents act independently or coordinate and act as a group.*

In the absence of speculation, the stock price conditional on information acquisition is fully revealing. In this equilibrium, all individuals observe the signal G which becomes public information. The CE value of private information is then null, $\mathcal{I}^i(G|Z) = 0$. The absence of a reward for information acquisition combined with its cost imply that the representative informed is worse off than the uninformed. There are no incentives to acquire.

Recall now that the representative informed is actually composed of a continuum of identical informed agents. If these individuals act independently, they will each choose not to acquire information (free rider problem). The resulting equilibrium is a no information equilibrium in which all agents are identical and common information flows are generated by the fundamental.

[14] "Insider trading "moves up" the resolution of uncertainty. This one time benefit may be relatively more important in a two-period model than in a multiperiod model. If so, my results may overestimate the benefits from insider trading. But we must await the development of multiperiod rational expectations models to answer this question definitively." (Leland (1992), p. 885).

If all agents coordinate and act as a group, they compare their ex-ante utilities with and without acquisition. In both cases equilibrium is fully revealing. The informational content, however, differs across equilibria. With acquisition, the private signal G is disclosed and the common information flow is $\mathcal{F}^i_{(\cdot)} \equiv \mathcal{F}^D_{(\cdot)} \bigvee \sigma(G)$. Without acquisition, the common flow is $\mathcal{F}^D_{(\cdot)}$. Again, information acquisition is not rewarded. The corresponding ex-ante CEs for the informed are,

$$CE^{i,wa} = E\left[S_0^{wa}\right] - \frac{1}{2\Gamma} VAR\left(S_0^{wa}\right) + \frac{1}{2\Gamma}\int_0^T \left(\sigma_v^{S,wa}\right)^2 dv - C$$

$$CE^{i,na} = E\left[S_0^{na}\right] - \frac{1}{2\Gamma} VAR\left(S_0^{na}\right) + \frac{1}{2\Gamma}\int_0^T \left(\sigma_v^{S,na}\right)^2 dv$$

where,

$$E\left[S_0^{wa}\right] = D_0 + \mu^D T - \frac{1}{\Gamma}\int_0^T \left(\sigma_v^{S,wa}\right)^2 dv, \qquad \sigma_t^{S,wa} = \frac{\left(\sigma^{\zeta}\right)^2}{H(t)}\sigma^D$$

$$E\left[S_0^{na}\right] = D_0 + \mu^D T - \frac{1}{\Gamma}\int_0^T \left(\sigma_v^{S,na}\right)^2 dv, \qquad \sigma_t^{S,na} = \sigma^D$$

$$VAR\left[S_0^{wa}\right] = \frac{\left(\sigma^D\right)^2 T}{H(0)} VAR(G) = \left(\sigma^D\right)^2 T, \qquad VAR\left[S_0^{na}\right] = 0.$$

The gain for acquiring $GA \equiv CE^{i,wa} - CE^{i,na}$ is therefore,

$$GA = -\frac{1}{2\Gamma}\left[\int_0^T \left(\left(\sigma_v^{S,wa}\right)^2 - \left(\sigma_v^{S,na}\right)^2\right) dv + VAR\left[S_0^{wa}\right] - VAR\left[S_0^{na}\right]\right] - C = -\int_0^T \frac{\left(\sigma_v^{S,wa}\right)^2}{2\Gamma} dv - C$$

where the right hand side follows from $\int_0^T \left(\sigma_v^{S,na}\right)^2 dv = VAR\left[S_0^{wa}\right]$. This gain is strictly negative. The optimal choice under coordination is again to forego information acquisition.[15,16]

In summary, in the absence of speculation, the equilibrium is non-informative irrespective of the behavior of the informed. With speculation and acquisition, equilibrium is the NREE described in previous sections. The conditions in Corollary 7 ensure that the NREE is weakly PO, hence Pareto dominates the non-informative equilibrium. Under these conditions, regulation banning beliefs-based speculation is welfare reducing (see Figure 4).

7 Regulating Informed Trading: an Implementable Optimal Rule

Informed trading is a major challenge for regulatory agencies. Commonly advocated policies, such as short sales constraints, restrict the choices of all investors and have adverse welfare consequences.

[15] The CE expressions apply to all agents in the economy. The optimal choice under coordination is to forego acquisition even if costs are shared across agents.

[16] The expression for informed utility in Proposition 6 shows that the information acquisition decisions in static and dynamic NREE differ. In a static CARA setting, more precise information decreases volatility, which increases the interim informed utility. Likewise, more speculation (higher σ^{ϕ}) increases the incentive to acquire information. In a dynamic setting, the informed utility is not monotone in volatility and more speculation does not always increase the incentive to acquire information.

Recent regulations, such as disclose-or-abstain, eliminate incentives for information collection and reduce market efficiency. This section studies the properties of an Information Trading Tax (ITT). The ITT is a contingent Tobin tax, so a transaction tax, designed to enforce the PO outcome. It can be implemented by a regulatory agency relying solely on public information.

To pave the way, the next proposition examines the issue of recoverability of structural parameters under public information. Before stating this result, note that the observation of D, S reveals the volatilities σ^D, σ^S.[17] The volatility coefficient $\widehat{B}(t) = \sigma_t^S/\sigma^D$, as well as its growth rate $\Phi(t) = \partial_t \log \widehat{B}(t)$, are therefore known. Likewise, the informational variable $d\chi(t) \equiv \left(\sigma_t^S/\sigma^D\right) dD_t - dS_t$ is observed. With these preliminaries,

Proposition 8 *Consider an agent with information filtration $\mathcal{F}_{(\cdot)}^{D,S}$. The unknown parameters ω, σ^ζ, $\beta, \widehat{\beta}, \beta^h, \omega^u$ can be recovered from the limit growth rate $\Phi_T = \lim_{t\to T} \Phi(t)$ and $\Phi(\tau)$ for some $\tau \in [0, T)$. The remaining unknown parameters $\mu^D, \mu^\phi, \sigma^\phi, \omega^i$, ω^n, Γ and the endogenous signal Z can be inferred by sampling $\chi(t)$ at a finite number of time points. All other equilibrium quantities, with the exception of the private signal G and the noisy signal ϕ, are functions of observables and the recovered parameters.*

The recoverability result in Proposition 8 is strong. It establishes that price and fundamental information are sufficient to identify the unknown parameters of the economy, including the risk tolerance Γ and the endogenous public signal Z. Knowledge of the volatilities σ^D, σ^S ensures that the volatility ratio $\widehat{B}(t) = \sigma_t^S/\sigma^D$ is known and that the parameters $\omega, \sigma^\zeta, \widehat{\beta}, \beta^h$ affecting it can be identified. This follows because $\widehat{B}(t)$ can be written solely as a function of ω, s, σ^D and Φ_T, where $s \equiv \left(\omega^n/\omega^i\right)^2 \left(\sigma^\phi/\sigma^\zeta\right)^2$. Inverting the pair $\widehat{B}(t), \widehat{B}(v)$ at two dates t, v yields ω, s, which also reveals $\left(\sigma^\zeta\right)^2 = \left(\frac{1+\omega s}{1+s}\right)\left(\left(\sigma^D\right)^2/\Phi_T\right)$. The coefficients $\beta, \widehat{\beta}, \beta^h$, that can also be expressed in terms of $\omega, s, \sigma^D, \Phi_T$ are then known.

Information contained in the synthetic informational variable $d\chi(t) \equiv \left(\sigma_t^S/\sigma^D\right) dD_t - dS_t$ is instrumental as well. A simple calculation shows that $\chi(t)$ is locally deterministic with drift $\left(\sigma_t^S/\sigma^D\right)\mu^D - \left(\mu_t^S - \sigma_t^S \theta_t^{D|m}\right)$. This drift is therefore observed and contains information about the fundamental drift μ^D, the stock's risk premium μ_t^S and the price of risk $\theta_t^{D|m}$ associated with the endogenous signal Z. In fact, it can be written as a function of observables and the unknowns $\mu^D, \mu^\phi, \sigma^\phi, \omega^i, \omega^n, \Gamma, Z$. Using a sufficiently large, but discrete sample, and inverting will then reveal the unknown parameters.

Proposition 9 *Let $\Xi \equiv \left\{\omega, \sigma^\zeta, \mu^D, \sigma^D, \mu^\phi, \sigma^\phi, \omega^i, \omega^n, \Gamma\right\}$ be the collection of parameters and $\mathcal{P}$ be the set of parameters in Corollaries 4 and 7 such that the NREE is Pareto dominated. A regulator observing the fundamental and stock price can implement a contingent trading tax, the ITT, that enforces a PO equilibrium. The ITT equals $\left(CE^{i,wa} + C - CE^{i,na}\right) 1_{\sigma_0^S < \sigma^D \cap \Xi \in \mathcal{P}}$, with $CE^{i,wa}$, $CE^{i,na}$ as defined in Remark 9.*

By observing the fundamental and stock price, a regulator can make two key inferences. First, the regulator can determine whether informed trading has occurred or not. The price volatility

[17] Note that $2\left(dS_t/S_t - d\log S_t\right)/dt = \left(\sigma_t^S\right)^2$, respectively, $\left(dD_t^2 - 2D_t dD_t\right)/dt = \left(\sigma^D\right)^2$.

is revealing in that regard. There is conclusive evidence of informed trading if it is less than the volatility of the fundamental. Second, as per Proposition 8, the regulator can determine all the unknown parameters of the economy. This includes the risk tolerance Γ and the endogenous signal Z. Based on that information, the regulator can infer the thresholds $\Gamma_{\pm}, \Gamma_0$ and determine whether trading based on private information is Pareto improving or not. If it is not, a contingent trading tax equal to the informed gross CE gain, i.e., gross of production cost, can be imposed on all traders. Such a tax will deter informed trading irrespective of the cost of information production and ensure that the PO equilibrium without private information prevails. The revenues from this contingent tax are therefore null.

8 Optimal Information Production Infrastructure

This section endogeneizes the frequency at which private information is produced.

Let $\Im(n) \equiv \left\{t_j^n : j = 0, ..., n\right\}$ be an equidistant partition of the time interval $[0, T]$, with $t_0^n = 0$ and $t_n^n = T$. The last date t_n^n is the dividend payment date. Each other date t_j^n represents an information arrival (production) time. The interarrival time is $t_j^n - t_{j-1}^n = \Delta^n$. It represents the time required to produce the next signal. The partition $\Im(n)$ is an information production infrastructure with n arrival dates. Such an infrastructure has an upfront cost $C(n)$, an increasing function of the number of arrival dates. Once the upfront cost has been paid, the infrastructure produces a proprietary signal $G_j = D_T + \zeta_j$ at time t_j^n, $j = 0, ..., n-1$, where $\zeta_j \sim \mathcal{N}\left(0, \left(\sigma_j^{\zeta}\right)^2\right)$ and the noise process $\zeta = \{\zeta_j : j = 0, ..., n\}$ is independent of all other uncertainty (but ζ_j, ζ_k could be correlated for $j \neq k$). The ex-ante utility associated with the infrastructure $\Im(n)$ is $V^i(n)$. The informed selects n so as to maximize ex-ante utility subject to his/her budget constraint.[18]

Once the information infrastructure is set up, it enters production and generates the sequence of private signals described above. Let N_t be a deterministic counting process that tallies the number of signals received up to time t. The vector of private signals received by time t is $G_t' = [G_1, \ldots, G_{N_t}]$. This vector increases in size every time a new signal arrives.

The noise trader emulates the behavior of the informed, but based on the sequence of irrelevant forecasts $\phi_t' = [\phi_1, \ldots, \phi_{N_t}]$ instead of G_t. Noise trading beliefs are identical to informed beliefs, except that they are parameterized by ϕ instead of G. The rest of the economy remains unchanged.

The certainty equivalent of the information infrastructure $\Im(n)$ is,

$$CE^i(n) = -\Gamma \log E\left[\exp\left(-\frac{S_0^n}{\Gamma} - K_0^n - E\left[\sum_{k=0}^{n-1}\prod_{i=0}^{k}\xi^m_{t_i^n, t_{i+1}^n}\Delta D^{KL}_{t_k^n}\middle|\mathcal{F}_0^{m,n}\right]\right)\right] - C(n)$$

where $S_0^n \equiv E\left[\prod_{i=0}^{n-1}\xi^m_{t_i^n, t_{i+1}^n} D_T \middle| Z_1\right]$ and $\Delta D^{KL}_{t_k^n}$ is the change in the Kullback-Leibler divergence

[18] The cost of modifying the infrastructure once it is built is assumed to be prohibitive.

measure between the signal density conditional on initial information $\mathcal{F}_0^{m,n}$ and information $\mathcal{F}_t^{m,n}$,

$$\Delta D_{t_k^n}^{KL} = \frac{1}{2}\log\frac{VAR\left[G_{k+1}|\,\mathcal{F}_{t_{k+1}^n}^{m,n}\right]}{VAR\left[G_{k+1}|\,\mathcal{F}_{t_k^n}^{m,n}\right]} + \frac{1}{2}\left(\frac{VAR\left[G_{k+1}|\,\mathcal{F}_0^{m,n}\right]}{VAR\left[G_{k+1}|\,\mathcal{F}_{t_{k+1}^n}^{m,n}\right]} - \frac{VAR\left[G_{k+1}|\,\mathcal{F}_0^{m,n}\right]}{VAR\left[G_{k+1}|\,\mathcal{F}_{t_k^n}^{m,n}\right]}\right)$$
$$+\frac{1}{2}\left(\left(\frac{E\left[G_{k+1}|\,\mathcal{F}_{t_{k+1}^n}^{m,n}\right]-E\left[G_{k+1}|\,\mathcal{F}_0^{m,n}\right]}{\sqrt{VAR\left[G_{k+1}|\,\mathcal{F}_{t_{k+1}^n}^{m,n}\right]}}\right)^2 - \left(\frac{E\left[G_{k+1}|\,\mathcal{F}_{t_k^n}^{m,n}\right]-E\left[G_{k+1}|\,\mathcal{F}_0^{m,n}\right]}{\sqrt{VAR\left[G_{k+1}|\,\mathcal{F}_{t_k^n}^{m,n}\right]}}\right)^2\right)$$

Finally, denote by ζ_{N_t} the $N_t \times 1$ vector of signals before time t.

Proposition 10 *The optimal information structure has n^* information production dates, where $n^* \geqslant 1$ solves the inequality,*

$$-\Gamma\log\left(\frac{E\left[\exp\left(-\frac{S_0^{m,n^*}}{\Gamma}-K_0^{n^*}-E\left[\sum_{k=0}^{n^*}\prod_{i=0}^{k}\xi_{t_i^{n^*},t_{i+1}^{n^*}}^m \Delta D_{t_k^{n^*+1}}^{KL}\middle|\,\mathcal{F}_0^{m,n^*}\right]\right)\right]}{E\left[\exp\left(-\frac{S_0^{m,n^*}}{\Gamma}-K_0^{n^*}-E\left[\sum_{k=0}^{n^*-1}\prod_{i=0}^{k}\xi_{t_i^{n^*},t_{i+1}^{n^*}}^m \Delta D_{t_k^{n^*}}^{KL}\middle|\,\mathcal{F}_0^{m,n^*}\right]\right)\right]}\right) \leqslant C\left(n^*+1\right)-C\left(n^*\right).$$

Furthermore, if $1/1_{N_t}'VAR\left[\zeta_{N_t}\right]^{-1}1_{N_t} \downarrow \left(\sigma^\zeta\right)^2$ for some constant $\left(\sigma^\zeta\right)^2 \in (0,+\infty)$ (no-singularity condition) and the cost increment $\Delta k_n = C(n+1) - C(n)$ is bounded away from zero (boundedness condition), then $n^* < \infty$.

The choice of an optimal information production infrastructure trades off the marginal utility benefit with the marginal cost. The marginal utility benefit depends on the incremental CE value of the information structure. When the latter falls below the marginal cost, raising the frequency at which information is produced becomes suboptimal. The optimal frequency is the solution of the equality that attains the highest expected utility.

The no-singularity (NS) condition prevents an asymptotic arbitrage. As shown in Proposition 11, an increase in the number of signals is isomorph to a reduction of the variance of the signal error. If the signal error disappears, there is an asymptotic arbitrage opportunity because the limiting risk neutral probability measure is not absolutely continuous. Condition NS guarantees that information remains noisy in the limit.

Remark 10 *Condition NS is a weak restriction on the information infrastructure. For instance, it holds when the noise process is $\zeta_{t_k^n} = \zeta + \bar{\zeta}_{t_k^n}$ where ζ is a common component. In this case,*

$$VAR\left[\zeta_{N_t}\right] = \left(\sigma^\zeta\right)^2 1_{N_t}1_{N_t}' + VAR\left[\bar{\zeta}_{N_t}\right]$$

$$VAR\left[\widetilde{\zeta}_t\right] = \frac{1}{1_{N_t}'VAR\left[\zeta_{N_t}\right]^{-1}1_{N_t}} = \left(\sigma^\zeta\right)^2 + \frac{1}{1_{N_t}'VAR\left[\bar{\zeta}_{N_t}\right]^{-1}1_{N_t}} \downarrow \left(\sigma^\zeta\right)^2$$

as $N_t \to \infty$. Information is asymptotically incomplete and it is therefore optimal to acquire a finite number of signals when the cost increment is bounded away from zero.

The boundedness (B) condition on the cost increment also represents a mild restriction. In practice, it becomes prohibitively costly to increase the private information frequency beyond some level. Information production methods, which are typically time intensive, are simply unable to generate arbitrarily large numbers of non-redundant signals.

Under conditions NS and B, the optimal information structure produces signals at a finite frequency. The endogenous arrival frequency of private news is therefore lower than the arrival frequency of public news. The asynchrony of information flows is an endogenous phenomenon.

Remark 11 *The optimal infrastructure consists of a single signal at the initial date if the maximal solution of the inequality in Proposition 10 is $n^* = 1$. This outcome arises, in particular, if the cost of producing multiple signals is sufficiently large.*

The next result describes equilibrium in this multi-signal environment.

Proposition 11 *The information flow generated by the vector signal process $G_t' = [G_1, \ldots, G_{N_t}]$ is equivalent to the information flow generated by the univariate process $\widetilde{G}_t = D_T + \widetilde{\zeta}_t$ where $\widetilde{\zeta}_t \equiv \frac{\zeta_{N_t}' VAR[\zeta_{N_t}]^{-1} 1_{N_t}}{1_{N_t}' VAR[\zeta_{N_t}]^{-1} 1_{N_t}}$. The noise trader using the conditional beliefs of the informed evaluated at the vector randomization process $\phi_t' = [\phi_1, \ldots, \phi_{N_t}]$ is identical to a noise trader using conditional beliefs based on the distribution of $\widetilde{G}_t$ and evaluated at the univariate randomization $\tilde{\phi}_t = \frac{\phi_{N_t}' VAR[\phi_{N_t}]^{-1} 1_{N_t}}{1_{N_t}' VAR[\zeta_{N_t}]^{-1} 1_{N_t}}$. The resulting NREE has the same structure as in Proposition 5, but with $\left(\sigma^{\zeta}\right)^2 = VAR\left[\zeta\right]$ replaced by the time-varying variance $\left(\sigma_t^{\tilde{\zeta}}\right)^2 = VAR\left[\widetilde{\zeta}_t\right]$, $\left(\sigma^{\phi}\right)^2 = VAR\left[\phi\right]$ replaced by $\left(\sigma_t^{\tilde{\phi}}\right)^2 = VAR\left[\widetilde{\phi}_t\right]$ and Z replaced by $\widetilde{Z}_t = \omega^i \widetilde{G}_t + \omega^n \widetilde{\phi}_t$.*

Proposition 11 is an information aggregation result. It shows that the sequential observation of the collection of noisy signals pertaining to the terminal dividend payment, G_t, is informationally equivalent to the reception of a univariate signal with time-varying noise, $\widetilde{G}_t$. The filtration generated by the two signal structures is effectively the same. Simple adjustments to the formulas in Section 5, as described in the proposition, lead to the NREE in this multi-signal setting.

Corollary 9 *The volatility of the stock price $\sigma_t^S = \widehat{B}\left(t\right) \sigma^D$ jumps down at information arrival times $t = t_j$, $j = 1, ..., J$ and increases in between private information arrival times $t \neq t_j, j = 1, ..., J$. Private information has a stabilizing effect on price volatility.*

Each time a new private signal arrives, it partially disseminates through the market. Dissemination is partial, because the activity of the noise trader hides the true value of the signal. Nevertheless, instantaneously, all market participants acquire new information about the future dividend. They immediately become less responsive to public news, such as information carried by the fundamental. This instantaneous reaction lowers volatility at arrival times. Between arrival times, the information related to past private signals does not change, whereas new fundamental information keeps materializing. Investors therefore become progressively more responsive to public news. The sensitivity of demands to public news increases, ultimately leading to an increase in the stock price volatility.

In the limit, as the dividend date approaches, past private information becomes irrelevant and the stock price volatility converges to the volatility of the fundamental. Overall, private information has a stabilizing effect. It lowers the price volatility relative to an economy without private signals. With multiple signals, it lowers volatility at each arrival time. As described in earlier sections, this recurring reduction in volatility is a source of welfare benefits for all agents involved.

9 Extensions

Various extensions of the model are now considered. Section 9.1 studies heterogeneous risk tolerances. Section 9.2 deals with differential information. Section 9.3 allows for a dividend surprise. It is shown that these extensions can all be mapped into the structure analyzed in Sections 2-5. Section 9.4 presents the NREE in a setting with a stochastic speculation process.

9.1 Heterogeneous Risk Tolerances

Suppose that risk tolerances are heterogenous, $\Gamma^u \neq \Gamma^i \neq \Gamma^n$. The NREE is described next.

Proposition 12 *Consider the economy with heterogeneous risk tolerances $\Gamma^u \neq \Gamma^i \neq \Gamma^n$. A competitive NREE exists. It is given by the formulas in Proposition 5, where risk tolerance Γ is replaced by the aggregate risk tolerance $\Gamma^a \equiv \sum_{j\in\{u,i,n\}} \omega^j \Gamma^j$ and where population weights $\left(\omega^i, \omega^n, \omega\right)$ are replaced by risk-tolerance-adjusted weights $(\widetilde{\omega}^\iota, \widetilde{\omega}^n, \widetilde{\omega})$ with $\widetilde{\omega}^\iota \equiv \omega^\iota \Gamma^\iota / \Gamma^a$ for $\iota \in \{i, n\}$ and $\widetilde{\omega} \equiv \widetilde{\omega}^i + \widetilde{\omega}^n$. The endogenous public signal becomes $\widetilde{Z} = \widetilde{\omega}^i G + \widetilde{\omega}^n \phi$, thus depends on risk tolerance. The WAPR and the stock price volatility also depend on risk tolerance.*

The heterogeneity in risk tolerances has an effect on the informational content of equilibrium. When the ratio Γ^n/Γ^i increases, the endogenous signal becomes less precise and informational efficiency decreases. The noise trader invests more aggressively, relatively speaking, which clouds the information conveyed by the residual demand. In the limit, when the noise trader becomes infinitely more risk tolerant than the uninformed so that $\Gamma^n/\Gamma^i \to \infty$, the endogenous signal is overwhelmed by noise and the private signal of the informed remains concealed. At the opposite end of the spectrum, when $\Gamma^n/\Gamma^i \to 0$, equilibrium fully reveals the private information available.

The diversity in risk attitudes has multiple effects on volatility. The next corollary describes the impact of an increase in the ratio of risk tolerances between the noise trader and the informed.

Corollary 10 *Suppose that Γ^n/Γ^i increases, but Γ^a stays constant. Then, $\partial B\left(t\right) > 0$, $\partial \beta^h\left(t\right) < 0$ and $\partial \hat{B}(t) < 0$ for all $t \in [0, T]$. Thus, $\partial \sigma_t^S < 0$ for all $t \in [0, T]$. Intertemporal hedging has a taming effect on the reaction of volatility to the increase in the tolerance ratio Γ^n/Γ^i.*

Under the conditions of the corollary, the endogenous information revealed becomes less precise. In the absence of hedging, investors increase their reliance on fundamental information, which increases the price volatility ($\partial B\left(t\right) > 0$). Intertemporal hedging has a taming effect. It causes agents to reduce the sensitivities of their individual demands to the fundamental, which reduces the sensitivity of the aggregate demand ($\partial \beta^h\left(t\right) < 0$). To ensure market clearing, the sensitivity, hence the volatility, of the equilibrium market price of risk must increase. The negative relation between the stock price and its risk premium explains the taming effect on the stock price volatility.

9.2 Differential Information

Suppose that there is a continuum of informed investors, $\iota \in [0,1]$, distributed on the unit interval according to the measure $\mu(d\iota)$. Informed agent ι receives the private signal $G^{\iota} = D_T + \zeta + \zeta^{\iota}$, where ζ^i, ζ are mutually independent Gaussian random variable with mean zero. Assume further that there is a continuum of noise traders, so that each informed agent has a follower. Noise trader ι has the same beliefs as informed agent ι , but evaluated at $\phi^{\iota} = \phi + \varepsilon^{\iota}$. The random forecast ϕ^{ι} consists of a common component ϕ and an independent Gaussian random variable ε^{ι} with mean zero. All other components of the economy remain the same as before.

Let $\varsigma_t^{\iota} \equiv VAR[\zeta^{\iota}]/VAR_t[G^{\iota}]$ be the ratio of the idiosyncratic signal noise relative to the public signal, i.e., the inverse signal-to-noise ratio.

The next proposition shows that the NREE is given by the formulas in Proposition 5 with adjusted coefficients. The adjustments required depend on population moments involving the signal-to-noise ratios, namely $\overline{\varrho}_t^{[1]} \equiv \int_0^1 \frac{\mu(d\iota)}{1+\varsigma_t^{\iota}}$ and $\overline{\varrho}_t^{[2]} \equiv \int_0^1 \left(\frac{1}{1+\varsigma_t^{\iota}}\right)^2 \mu(d\iota)$. They modify the coefficients of the ODEs that determine the parameters $\breve{\beta}^h, \breve{\alpha}^h, \breve{\gamma}^h$ of the aggregate hedging demand.

Proposition 13 *Suppose that,*
(i) The signal noise has a systematic component ζ and an idiosyncratic component ζ^{ι}: $\int_0^1 \zeta^{\iota}\mu(d\iota) = \lim_{N\to\infty} N^{-1}\sum_{i=1}^N \zeta^i = 0$.
(ii) The speculative signal forecasts of noise traders have a systematic component ϕ and an idiosyncratic component ε^{ι}: $\int_0^1 \varepsilon^{\iota}\mu(d\iota) = \lim_{N\to\infty} N^{-1}\sum_{i=1}^N \varepsilon^i = 0$.

Under conditions (i) and (ii), the equilibrium with differentially informed investors is identical to the NREE with a representative informed investor described in Proposition 5, but with coefficients $\alpha^h, \beta^h, \gamma^h$ replaced by $\breve{\alpha}^h, \breve{\beta}^h, \breve{\gamma}^h$ where:
(a) $\breve{\beta}^h$ solves the same Riccati ODE as β^h in (37) but with κ_0, κ_1 in (38) replaced by $\breve{\kappa}_0, \breve{\kappa}_1$ in (94),
(b) $\breve{\alpha}^h, \breve{\gamma}^h$ are as α^h, γ^h in (39), but with κ_j, $j = 2, ..., 5$ in (40)-(43) replaced by $\breve{\kappa}_j$, $j = 2, ..., 5$ in (95)-(98).

A competitive NREE for an economy with purely differential information (non-hierarchical information structure) is obtained by setting $\omega^u \equiv 1 - \omega^i - \omega^n = 0$. The NREE in Proposition 5 without differential private information is recovered by setting $\varsigma_t^i = 0$.

Coefficients in the NREE with differential information are modified by the inverse signal-to-noise ratio ς_t^{ι} of the idiosyncratic components ζ^{ι}. The assumption that the private information signals differ by idiosyncratic noise components implies that the infinite regress problem, typically associated with non-hierarchical information structures (see Townsend (1983)), does not arise. This assumption ensures that only the systematic information component matters in the aggregate. It implies that the equilibrium information flow is generated by D, Z, as in the equilibrium without differential information. A NREE exists and is given by the formulas in Proposition 5, with suitably adjusted parameters. All properties documented in Sections 5 and 6 continue to hold. In particular, welfare implications and Pareto rankings of equilibria are as in the model without differential information. The price impact and gains from trade impact are obtained by using the adjusted coefficients.

If $\zeta = \omega^u = 0$, the model specializes to a setting with differential information similar to He and Wang (1995), but with endogenous noise trading demand instead of exogenous stochastic supply.

9.3 Dividend Surprise

Suppose that $\widetilde{D}_T = D_{T-} + \epsilon$ where ϵ represents a surprise, i.e., a jump. The jump size ϵ is assumed to be normally distributed and independent of the signal noise ζ and of $\mathcal{F}_T^D$. Furthermore, suppose that private information pertains to the continuous part of the dividend, $G = D_{T-} + \zeta$.[19]

Proposition 14 *The equilibrium stock price $\widetilde{S}_t$ is,*

$$\widetilde{S}_t = \begin{cases} S_t + E[\epsilon] - \frac{VAR[\epsilon]}{\Gamma} & \text{for } t < T \\ S_t + \epsilon & \text{for } t = T \end{cases} \tag{28}$$

where S_t is the price in the absence of a surprise, described in Proposition 5. The price jump at T is $\Delta\widetilde{S}_T = \epsilon - E[\epsilon] + \frac{VAR[\epsilon]}{\Gamma}$. The volatility and risk premium are unaffected by the surprise. The risk premium converges to $E_{T-}\left[\Delta\widetilde{S}_T\right] = \frac{VAR[\epsilon]}{\Gamma}$ as T approaches. The conditional variance of the terminal price jump converges to $VAR_{T-}\left[\Delta\widetilde{S}_T\right] = VAR[\epsilon]$. The equilibrium at T_- is a symmetric no-trade equilibrium where $N_{T-}^{\iota} = 1$ for $\iota \in \{u, i, n\}$.

The dividend surprise at the terminal date does not affect the informational content of equilibrium, prior to the payment date. It also leaves the structure and properties of equilibrium unchanged. The reason for this irrelevance result is the absence of information regarding the eventual surprise.

The stock price volatility in this setting is still driven by the volatility of the fundamental. Volatility, in the NREE, has the same properties relative to the equilibrium without private information. The welfare properties of equilibrium and the Pareto dominance of the NREE under the conditions outlined in Section 6 continue to apply.

9.4 Equilibrium with stochastic speculation process

Consider now a speculative noise trader who uses the signal process $\phi_t = \phi_0 + V_t$ where ϕ is a Gaussian random variable and $V_t = \int_0^t \sigma^{\phi}(s,t)\, dW_v^{\phi}$ with deterministic function $\sigma^{\phi}(s,t)$ is a Gaussian process independent of ϕ and $\mathcal{F}_t^D$. The random variable ϕ represents speculation about the private signal G. The process V is extraneous speculative noise. The NREE is described next.

Proposition 15 *A NREE exists if the system (104)-(111) has a solution. The endogenous signal is $Z_0 = \omega^i G + \omega^n \phi_0$ and the public information flow is $\mathcal{F}_{(\cdot)}^m = \mathcal{F}_{(\cdot)}^{D,W^{\phi}} \bigvee \sigma(Z_0)$. In the NREE, the*

[19] The dividend paid, D_T, differs from the terminal value of the fundamental D_{T-} by the independent random variable ϵ. Private information pertains to the fundamental value D_{T-}.

stock price and the market price of risk are,

$$S_t = \Upsilon^Z(t) Z_0 + \Upsilon^D(t) D_t + \Upsilon^V(t) V_t + \Upsilon^0(t)$$

$$\theta_t^S \equiv \frac{\mu_t^S}{\sigma_t^S} = \varphi^Z(t) Z_0 + \varphi^D(t) D_t + \varphi^V(t) V_t + \varphi^0(t)$$

where $\Upsilon^Z, \Upsilon^D, \Upsilon^V, \Upsilon^0$ *are deterministic coefficients defined in (112)-(113) and* φ^Z, φ^D, φ^Z, φ^0 *are deterministic coefficients satisfying (104)-(107). The stock price volatility is* $\sigma_t^S \equiv \|\sigma_t\|$ *where* $\sigma_t' \equiv \left[\sigma_t^{S,D}, \sigma_t^{S,W^\phi}\right]$ *solves (111). The stock price evolves as* $dS_t = \mu_t^S dt + \sigma_t^S dW_t^S$ *where*

$$dW_t^S = \rho_t^{S,D}\left(dW_t^D - \theta_t^{Z|D} dt\right) + \sqrt{1 - \left(\rho_t^{S,D}\right)^2} dW_t^\phi$$

is a Brownian motion, $\rho_t^{S,D} \equiv \sigma_t^{S,D}/\sigma_t^S$ *is the correlation coefficient between the stock price and the fundamental and* $\theta_t^{Z_0|D}$*, the information price of risk of the endogenous signal, is as in the equilibrium with non time-varying belief-based speculation* ($V_t = 0$)*.*

The equilibrium filtration is now generated by the independent Brownian motions W^D and W^ϕ, enlarged by the initial endogenous signal Z_0. The financial market is therefore incomplete. Given CARA utility and constant interest rate, the shadow price of incompleteness is null, i.e., the minimal martingale measure solves the dual problem. The structure of equilibrium is then as before except for the additional noise factor V_t in the price. This stochastic factor is a source of excess volatility.

Corollary 11 *There is excess volatility,* $\sigma_t^S > \sigma^D$*, at* $t < T$ *if and only if* $\|\varrho(t)\| > 1$*, where,*

$$\varrho(t) \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \int_t^T \begin{bmatrix} \sigma^D\left(\varphi^D(v)\rho_v^{S,D} - \beta^D(v)\right) \\ \sigma^\phi(t,t)\varphi^V(v)\rho_v^{S,D} \end{bmatrix} dv.$$

A necessary condition for excess volatility is,

$$\dot{\sigma}_t^S = \left(\rho_t^{S,D}\right)^2 \varphi^D(t)\sigma^D - \sigma^D\beta^D(t)\rho_t^{S,D} + \rho_t^{S,D}\sqrt{1 - \left(\rho_t^{S,D}\right)^2}\varphi^V(t)\sigma^V < 0 \quad \text{for some } t \in [0,T].$$

If $t \to T$*, then* $\rho_t^{S,D} \to 1$ *and* $\varphi^D(t) \to -\beta(T)$*, so that* $\lim_{t\uparrow T}\dot{\sigma}_t^S = -\left(\beta(T) + \beta^D(T)\right)\sigma^D > 0$*. As* T *approaches, the price volatility converges to the volatility of the fundamental* σ^D *from below.*

As there are two Brownian motions, the stock price is imperfectly correlated with the fundamental. This can give rise to excess volatility in periods where correlation is low. As the dividend payment date approaches, the correlation converges to one and the stock price volatility falls below the fundamental volatility. As discussed next, the presence of private information reduces volatility.

Remark 12 *Consider the economy without private information and where the noise trader speculates based solely on the extraneous noise* V*. The equilibrium filtration is* $\mathcal{F}_{(\cdot)}^m = \mathcal{F}_{(\cdot)}^{D,W^\phi}$*. The equilibrium stock price and market price of risk are* $S_t = D_t + \Upsilon^{V,ni}(t) V_t + \Upsilon^{0,ni}(t)$ *and* $\theta^{S,ni}(t) = \varphi^{V,ni}(t) V_t +$

$\varphi^{0,ni}(t)$ where coefficients satisfy (116)-(117) and (121). The volatility $\sigma_t^{S,ni}$ in (115) is larger than the fundamental volatility σ^D before T and converges to σ^D from above as T approaches. There is excess volatility. Consider next the economy with private information and noise trading based on both an irrelevant forecast ϕ_0 of G and the extraneous noise process V. In this NREE, volatility is lower than in the benchmark economy. Private information again stabilizes the market.

10 Conclusion

This paper studied the effects of information production and speculation on the structure and properties of a non-stationary noisy rational expectations equilibrium. Private information dissemination was found to have a stabilizing effect, as it reduces the volatility of the stock price. Costly information production, resulting in asynchrony between private and public information flows, leads to an increase in volatility in between information arrival dates. Private information, although fundamentally detrimental to non-informed agents, has nevertheless welfare benefits for all market participants. By lowering risk, it increases the value of the stock and therefore of initial allocations. By disseminating through the market, it improves the decisions of investors and the resulting gains from trade. For economies with sufficiently high or low risk tolerances, the welfare benefits offset the costs for all agents involved. Under these conditions, the NREE with private information trades Pareto dominates the equilibrium without private information collection. Speculation plays a central role. In its absence, incentives for information collection vanish and the potential welfare benefits of information are lost.

The dynamic model developed in the paper is tractable and produces closed form solutions for the NREE, with the exception of a single coefficient. It therefore offers a useful platform to examine complex issues related to information asymmetry in financial markets. For instance, it provides a natural setting to further study policy questions. Should trades based on private information be banned? The results in this paper suggest that an outright ban may not be best for society and that a contingent trading tax may be a useful regulation to consider. Admittedly, such a regulation requires some degree of distributional knowledge as well as monitoring and inference skills by the regulator. Nevertheless, in the context of the model, it implements the PO equilibrium. Likewise, should speculation be restricted and if so in what ways? These questions are fundamental for the smooth functioning of financial markets and the welfare of market participants. Further analysis requires extensions of the model incorporating more general specifications of the economic environment and is therefore beyond the scope of the present study. Issues such as these could prove interesting avenues for future research.

Appendix

Appendix A: Conditional Moment Formulas

The next lemma provides formulas for the conditional moments of the fundamental under a change of measure.

Lemma 1 *Define* $d\widetilde{P}^z = \exp\left(-\frac{1}{2}\int_0^T f(z)^2\,dv - \int_0^T f(z)\,dW_v^D\right)dP$ *where* $f(z) = -\left(\widetilde{\alpha}(t)\,z + \widetilde{\beta}(t)\,D_t + \overline{\gamma}(t)\right)$ *and* $\widetilde{\alpha}, \widetilde{\beta}, \overline{\gamma}$ *are functions of time. The first two conditional moment of* D *under* $\widetilde{P}^z$ *for* $s \geqslant t$ *are,*

$$\widetilde{E}^z\left[\left.D_s\right|\mathcal{F}_t^D\right] = a_1(t,s)\,z + b_1(t,s)\,D_t + g_1(t,s) \tag{29}$$

$$\widetilde{E}^z\left[\left.D_s^2\right|\mathcal{F}_t^D\right] = a_2(t,s)\,z + b_2(t,s)\,D_t + c_2(t,s)\,D_t z + d_2(t,s)\,z^2 + b_1(t,s)^2 D_t^2 + g_2(t,s) \tag{30}$$

where,

$$a_1(t,s) \equiv \sigma^D \int_t^s b_1(v,s)\,\widetilde{\alpha}(v)\,dv, \qquad b_1(t,s) \equiv \exp\left(\sigma^D \int_t^s \widetilde{\beta}(u)\,du\right), \qquad g_1(t,s) \equiv \int_t^s b_1(v,s)\,k(v)\,dv$$

$$a_2(t,s) \equiv 2\int_t^s b_1(v,s)^2\left(k(v)\,a_1(t,v) + \sigma^D\widetilde{\alpha}(v)\,g_1(t,v)\right)dv, \qquad k(v) \equiv \mu^D + \sigma^D\overline{\gamma}(v)$$

$$b_2(t,s) \equiv 2\int_t^s b_1(v,s)^2\,k(v)\,b_1(t,v)\,dv, \quad c_2(t,s) \equiv 2\sigma^D \int_t^s b_1(v,s)^2\,\widetilde{\alpha}(v)\,b_1(t,v)\,dv$$

$$d_2(t,s) \equiv 2\sigma^D \int_t^s b_1(v,s)^2\,\widetilde{\alpha}(v)\,a_1(t,v)\,dv, \qquad g_2(t,s) \equiv 2\int_t^s b_1(v,s)^2\left(\frac{\left(\sigma^D\right)^2}{2} + k(v)\,g_1(t,v)\right)dv.$$

Moreover, $\partial_{D_t}\widetilde{E}^z\left[\left.D_s\right|\mathcal{F}_t^D\right] = b_1(t,s)$ *and* $\partial_{D_t}\widetilde{E}^z\left[\left.D_v^2\right|\mathcal{F}_t^D\right] = b_2(t,v) + c_2(t,v)\,z + 2b_1(t,v)^2 D_t$.

Proof of Lemma 1. Using $\widetilde{E}^z\left[\left.D_s\right|\mathcal{F}_t^D\right] = D_t + \mu^D(s-t) + \sigma^D\widetilde{E}^z\left[\left.W_s^D - W_t^D\right|\mathcal{F}_t^D\right]$, $\widetilde{E}^z\left[\left.W_s^D - W_t^D\right|\mathcal{F}_t^D\right] = -\widetilde{E}^z\left[\left.\int_t^s\left(\theta_v^m - \theta_v^{Z|D}(z)\right)dv\right|\mathcal{F}_t^D\right]$ and $\widetilde{E}^z\left[\left.\theta_v^m - \theta_v^{Z|D}(z)\right|\mathcal{F}_t^D\right] = -\left(\overline{\gamma}(v) + \widetilde{\beta}(v)\,\widetilde{E}^z\left[\left.D_v\right|\mathcal{F}_t^D\right] + \widetilde{\alpha}(v)\,z\right)$ gives,

$$\widetilde{E}^z\left[\left.D_s\right|\mathcal{F}_t^D\right] = D_t + \mu^D(s-t) + \sigma^D\int_t^s\left(\overline{\gamma}(v) + \widetilde{\alpha}(v)\,z\right)dv + \sigma^D\int_t^s\widetilde{\beta}(v)\,\widetilde{E}^z\left[\left.D_v\right|\mathcal{F}_t^D\right]dv.$$

Defining $k(v) \equiv \mu^D + \sigma^D\overline{\gamma}(v)$ and solving gives (29). Also, as $\widetilde{E}^z\left[\left.dW_v^D\right|\mathcal{F}_t^D\right] = -\widetilde{E}^z\left[\left.\left(\theta_v^m - \theta_v^{Z|D}(z)\right)\right|\mathcal{F}_t^D\right]dv$,

$$\widetilde{E}^z\left[\left.\int_t^s D_v dD_v\right|\mathcal{F}_t^D\right] = \int_t^s\widetilde{E}^z\left[\left.D_v\right|\mathcal{F}_t^D\right]dv\mu^D - \sigma^D\widetilde{E}^z\left[\left.\int_t^v D_v\left(\theta_v^m - \theta_v^{Z|D}(z)\right)dv\right|\mathcal{F}_t^D\right]$$

$$\widetilde{E}^z\left[\left.D_v\left(\theta_v^m - \theta_v^{Z|D}(z)\right)\right|\mathcal{F}_t^D\right] = -\widetilde{E}^z\left[\left.D_v\left(\overline{\gamma}(v) + \widetilde{\beta}(v)\,D_v + \widetilde{\alpha}z\right)\right|\mathcal{F}_t^D\right]$$

it follows that,

$$\begin{aligned}
\widetilde{E}^z\left[\left.D_s^2\right|\mathcal{F}_t^D\right] &= D_t^2 + 2\widetilde{E}^z\left[\left.\int_t^s D_v dD_v\right|\mathcal{F}_t^D\right] + \left(\sigma^D\right)^2(s-t) = D_t^2 + 2\mu^D\int_t^s\widetilde{E}^z\left[\left.D_v\right|\mathcal{F}_t^D\right]dv \\
&\quad + 2\sigma^D\int_t^s\widetilde{E}^z\left[\left.D_v\left(\overline{\gamma}(v) + \widetilde{\beta}(v)\,D_v + \widetilde{\alpha}(v)\,z\right)\right|\mathcal{F}_t^D\right]dv + \left(\sigma^D\right)^2(s-t) \\
&= D_t^2 + 2\int_t^s\left(k(v) + \sigma^D\widetilde{\alpha}(v)\,z\right)\widetilde{E}^z\left[\left.D_v\right|\mathcal{F}_t^D\right]dv + 2\sigma^D\int_t^s\widetilde{\beta}(v)\,\widetilde{E}^z\left[\left.D_v^2\right|\mathcal{F}_t^D\right]dv + \left(\sigma^D\right)^2(s-t).
\end{aligned}$$

Solving this linear equation gives

$$\begin{aligned}
\widetilde{E}^z\left[\left.D_s^2\right|\mathcal{F}_t^D\right] &= b_1(t,s)^2 D_t^2 + \left(\sigma^D\right)^2\int_t^s b_1(v,s)^2\,dv + 2\int_t^s b_1(v,s)^2\left(k(v) + \sigma^D\widetilde{\alpha}(v)\,z\right)\widetilde{E}^z\left[\left.D_v\right|\mathcal{F}_t^D\right]dv \\
&= b_1(t,s)^2 D_t^2 + \left(\sigma^D\right)^2\int_t^s b_1(v,s)^2\,dv \\
&\quad + 2\int_t^s b_1(v,s)^2\left(k(v) + \sigma^D\widetilde{\alpha}(v)\,z\right)\left(g_1(t,v) + b_1(t,v)\,D_t + a_1(t,v)\,z\right)dv \\
&= b_1(t,s)^2 D_t^2 + \left(\sigma^D\right)^2\int_t^s b_1(v,s)^2\,dv + 2\int_t^s b_1(v,s)^2\,k(v)\,g_1(t,v)\,dv \\
&\quad + 2\int_t^s b_1(v,s)^2\,k(v)\,b_1(t,v)\,dvD_t + 2\left(\int_t^s b_1(v,s)^2\,k(v)\,a_1(t,v)\,dv + \sigma^D\int_t^s b_1(v,s)^2\,\widetilde{\alpha}(v)\,g_1(t,v)\,dv\right)z \\
&\quad + 2\sigma^D\int_t^s b_1(v,s)^2\,\widetilde{\alpha}(v)\,b_1(t,v)\,dvzD_t + 2\sigma^D\int_t^s b_1(v,s)^2\,\widetilde{\alpha}(v)\,a_1(t,v)\,dvz^2
\end{aligned}$$

Defining the coefficients as in the lemma leads to the expressions stated. ∎

Appendix B: Proofs

Proof of Proposition 1. The state price density (SPD) for the uninformed is,

$$\xi_t^m = \exp\left(-\frac{1}{2}\int_0^t (\theta_v^m)^2\, dv - \int_0^t \theta_v^m dW_v^S\right) \equiv \mathcal{E}\left(-\int_0^{\cdot} \theta_v^m dW_v^S\right)_t.$$

Optimal terminal wealth equals $X_T^u = -\Gamma \log y^u - \Gamma \log \xi_T^m$ where the shadow price of initial wealth y^u solves $X_0^u = -\Gamma log y^u - \Gamma E\left[\xi_T^m \log \xi_T^m\right]$. Intermediate wealth is $X_t^u = -\Gamma \log y^u - \Gamma \log \xi_t^m - \Gamma E_t\left[\xi_{t,T}^m \log \xi_{t,T}^m\right]$. Using $\log \xi_{t,T}^m = -\int_t^T \theta_v^m \left(dW_v^S + \theta_v^m dv\right) + \frac{1}{2}\int_t^T (\theta_v^m)^2\, dv$ gives $E_t\left[\xi_{t,T}^m \log \xi_{t,T}^m\right] = \frac{1}{2}E_t\left[\xi_{t,T}^m \int_t^T (\theta_v^m)^2\, dv\right]$. If the conditional expectation is Markovian in D_t, which will be shown to hold in equilibrium, the optimal demand is $N_t^u \sigma_t^S = \Gamma\theta_t^m - \frac{\Gamma}{2}\partial_{D_t} E_t\left[\xi_{t,T}^m \int_t^T (\theta_v^m)^2\, dv\right]\sigma^D$. The formulas for the mean-variance and hedging components follow.

For the informed $\xi_t^i = \xi_t^m \eta_t^G$ where $\eta_{t,T}^x \equiv P\left(G \in dx \middle| \mathcal{F}_T^m\right)/P\left(G \in dx \middle| \mathcal{F}_t^m\right)^{-1} = \mathcal{E}\left(\int_0^t \theta_v^{G|m}(x)\, dW_v^S\right)_t^{-1}$. Optimal terminal wealth is $X_T^i = -\Gamma \log y^i - \Gamma \log \xi_T^i$, where y^i solves $X_0^i = -\Gamma log y^i - \Gamma E\left[\xi_T^i \log \xi_T^i \middle| \mathcal{F}_0^i\right]$. Thus,

$$X_t^i = -\Gamma \log y^i - \Gamma \log \xi_t^i - \Gamma E\left[\xi_{t,T}^i \log \xi_{t,T}^i \middle| \mathcal{F}_t^i\right]$$

because $\log \xi_t^i = \log \xi_t^m + \log \eta_t^G$ and $E\left[\xi_{t,T}^i \log \xi_{t,T}^i \middle| \mathcal{F}_t^i\right] = \frac{1}{2}\int_t^T E\left[\xi_{t,v}^i \left(\theta_v^m + \theta_v^{G|m}(G)\right)^2 \middle| \mathcal{F}_t^i\right] dv$, where,

$$\begin{aligned} E\left[\xi_{t,v}^i \left(\theta_v^m + \theta_v^{G|m}(G)\right)^2 \middle| \mathcal{F}_t^i\right] &= E\left[\xi_{t,v}^m \left(\theta_v^m + \theta_v^{G|m}(x)\right)^2 \middle| \mathcal{F}_t^m\right]_{|x=G} \\ &= E_t\left[\xi_{t,v}^m (\theta_v^m)^2\right] + E_t\left[\xi_{t,v}^m \theta_v^{G|m}(x)^2\right]_{|x=G} + 2E_t\left[\xi_{t,v}^m \theta_v^m \theta_v^{G|m}(x)\right]_{|x=G}. \end{aligned}$$

Thus, $X_t^i = X_t^u + \Gamma \log\left(y^u/y^i\right) - \Gamma \log \eta_t^G - \frac{\Gamma}{2}\int_t^T E_t\left[\xi_{t,v}^m \theta_v^{G|m}(x)\left(\theta_v^{G|m}(x) + 2\theta_v^m\right)\right] dv_{|x=G}$. If the conditional expectation is Markovian in D_t, as will be shown to hold in equilibrium, the portfolio of the informed follows. ∎

The proofs of Theorems 3 and 4 are provided in Appendix C. They use Proposition 5. The proof of Proposition 5 relies on auxiliary Lemmas 2-10.

Lemma 2 *Suppose that public information is carried by the filtration $\mathcal{F}_{(\cdot)}^m = \mathcal{F}_{(\cdot)}^{D,Z}$. The PIPR is the affine function,*

$$\theta_t^{G|m}(x) = \frac{x - \mu_t^{G|D,Z}}{\left(\sigma_t^{G|D,Z}\right)^2}\left(1 - \kappa_t \omega^i\right)\sigma^D = \frac{x - \mu_t^{G|D,Z}}{H(t)}\sigma^D$$

$$\mu_t^{G|D,Z} = D_t + \mu^D (T-t) + \kappa_t\left[Z - \omega^i\left(D_t + \mu^D (T-t)\right) - \omega^n \mu^\phi\right]$$

$$\left(\sigma_t^{G|D,Z}\right)^2 = \left(\left(\sigma^D\right)^2 (T-t) + \left(\sigma^\zeta\right)^2\right)\left(1 - \kappa_t\omega^i\right) \equiv H(t)\left(1 - \kappa_t\omega^i\right)$$

$$\kappa_t = \frac{\omega^i H(t)}{M(t)}, \quad H(t) = \left(\sigma^D\right)^2 (T-t) + \left(\sigma^\zeta\right)^2, \quad M(t) = \left(\omega^i\right)^2 H(t) + (\omega^n)^2\left(\sigma^\phi\right)^2.$$

Proof of Lemma 2. The PIPR satisfies $d\left[\log p^G(x), W^S\right] = \theta_t^{G|m}(x)\, dt$, where $p^G(x)$ is the conditional density at time t of the signal G, given public information $\mathcal{F}_{(\cdot)}^m$. To determine the conditional density, note that,

$$G = D_T + \zeta = D_t + \mu^D (T-t) + \int_t^T \sigma^D dW_s^D + \zeta$$

$$Z = \omega^i G + \omega^n \phi = \omega^i\left(D_t + \mu^D (T-t)\right) + \omega^i\left(\int_t^T \sigma^D dW_s^D + \zeta\right) + \omega^n \phi.$$

Under the assumption $\mathcal{F}_{(\cdot)}^m = \mathcal{F}_{(\cdot)}^{D,Z}$, the conditional density is $p_t^G(x) = \frac{1}{\sigma_t^{G|D,Z}} n\left(\frac{x - \mu_t^{G|D,Z}}{\sigma_t^{G|D,Z}}\right)$ with parameters

$\mu_t^{G|D,Z}, \sigma_t^{G|D,Z}$ as defined in the lemma ($M(t)$ is the variance of $Z - \omega^i \left(D_t + \mu^D (T-t)\right) = \omega^i \left(\int_t^T \sigma^D dW_s^D + \zeta\right) + \omega^n \phi$). An application of Ito's lemma establishes the result announced. ■

Lemma 3 *Suppose that $\mathcal{F}_{(\cdot)}^m = \mathcal{F}_{(\cdot)}^{D,Z}$. The WAPR is an affine function of Z, $\Theta_t(Z;\omega) \equiv \omega^i \theta_t^{G|m}(G) + \omega^n \theta_t^{G|m}(\phi) = \alpha(t) Z + \beta(t) D_t + \gamma(t)$ where,*

$$\alpha(t) = \frac{1-\kappa_t \omega}{H(t)} \sigma^D, \quad \beta(t) = -\omega \frac{1-\kappa_t \omega^i}{H(t)} \sigma^D, \quad \gamma(t) = -\omega \frac{\left(1-\kappa_t \omega^i\right) \mu^D (T-t) - \omega^n \kappa_t \mu^\phi}{H(t)} \sigma^D, \quad \omega = \omega^i + \omega^n.$$

Proof of Lemma 3. From Lemma 2 and the definition of the WAPR,

$$\begin{aligned} \Theta_t(Z;\omega) &\equiv \omega^i \theta_t^{G|m}(G) + \omega^n \theta_t^{G|m}(\phi) = \frac{Z - \left(\omega^i + \omega^n\right) \mu_t^{G|D,Z}}{H(t)} \sigma^D \equiv \frac{Z - \omega \mu_t^{G|D,Z}}{H(t)} \sigma^D \\ &= \frac{Z - \omega \left(\left(1-\kappa_t \omega^i\right)\left(D_t + \mu^D (T-t)\right) + \kappa_t \left(Z - \omega^n \mu^\phi\right)\right)}{H(t)} \sigma^D \\ &= \left(\frac{1-\kappa_t \omega}{H(t)} Z - \omega \frac{1-\kappa_t \omega^i}{H(t)} D_t - \omega \frac{\left(1-\kappa_t \omega^i\right) \mu^D (T-t) - \omega^n \kappa_t \mu^\phi}{H(t)}\right) \sigma^D \equiv \alpha(t) Z + \beta(t) D_t + \gamma(t). \end{aligned}$$

This establishes the claim. ■

Lemma 4 *Suppose that $\mathcal{F}_{(\cdot)}^m = \mathcal{F}_{(\cdot)}^{D,Z}$. The optimal hedging demand of the uninformed is $N_t^{h,u} = \Gamma h_t^u(Z) \sigma^D / \sigma_t^S$ where $h_t^u(Z) = \psi_{1t}^u Z + \psi_{2t}^u D_t + \psi_{3t}^u$ with,*

$$\psi_{1t}^u = -\int_t^T \left(\widehat{\alpha}(v) b_1(t,v) + \frac{1}{2} \widehat{\beta}(v)^2 c_2(t,v)\right) dv, \quad \psi_{2t}^u = -\int_t^T \widehat{\beta}(v)^2 b_1(t,v)^2 dv$$
$$\psi_{3t}^u = -\int_t^T \left(\widehat{\beta}(v) \widehat{\gamma}(v) b_1(t,v) + \frac{1}{2} \widehat{\beta}(v)^2 b_2(t,v)\right) dv$$

and the functions b_1, b_2, c_2 are defined in Lemma 1 with $\overline{\gamma}(t) \equiv \widetilde{\gamma}(t) - \frac{\sigma_t^S}{\Gamma}$. The remaining coefficients are $\widehat{\alpha}(v) \equiv \alpha(v) + \alpha^h(v)$, $\widehat{\beta}(v) \equiv \beta(v) + \beta^h(v)$ and $\widehat{\gamma}(v) \equiv \gamma(v) + \gamma^h(v)$.

Proof of Lemma 4. Uninformed hedging demands are determined by $H_{t,T}^m \equiv -\frac{1}{2} E\left[\xi_{t,T}^m \int_t^T (\theta_v^m)^2 dv \middle| \mathcal{F}_t^m\right]$. As $E\left[K | \mathcal{F}_t^m\right] = E\left[\mathcal{E}\left(\int_t^\cdot \theta_v^{Z|D}(z) dW_v^D\right)_T K \middle| \mathcal{F}_t^D\right]_{|z=Z}$ for arbitrary K (recall that $\mathcal{F}_{(\cdot)}^m \equiv \mathcal{F}_{(\cdot)}^{D,Z}$) and $dW_v^m = dW_v^D - \theta_v^{Z|D} dv$ it follows that,

$$\begin{aligned} H_{t,T}^m &= -\frac{1}{2} E\left[\mathcal{E}\left(\int_t^\cdot \theta_v^{Z|D}(z) dW_v^D\right)_T \mathcal{E}\left(-\int_t^\cdot \theta_v^m(z) dW_v^m\right)_T \int_t^T \left(\theta_v^m(z)\right)^2 dv \middle| \mathcal{F}_t^D\right]_{|z=Z} \\ &= -\frac{1}{2} E\left[\mathcal{E}\left(-\int_t^\cdot \left(\theta_v^m(z) - \theta_v^{Z|D}(z)\right) dW_v^D\right)_T \int_t^T \theta_v^m(z)^2 ds \middle| \mathcal{F}_t^D\right]_{|z=Z} \equiv -\frac{1}{2} \widetilde{E}^z \left[\int_t^T \theta_v^m(z)^2 dv \middle| \mathcal{F}_t^D\right] \end{aligned}$$

where the last equality uses the definition $d\widetilde{P}^z/dP \equiv \mathcal{E}\left(-\int_0^\cdot \left(\theta_v^m(z) - \theta_v^{Z|D}(z)\right) dW_v^D\right)_T$, the next to last $\mathcal{E}(M)\mathcal{E}(N) = \mathcal{E}(M + N + [M,N])$ and $\theta_t^{Z|D}(z)$ is the volatility of the conditional density of D_T given Z,

$$\theta_t^{Z|D}(z) = \alpha^D(t) z + \beta^D(t) D_t + \gamma^D(t)$$

$$\alpha^D(t) \equiv \frac{\omega^i \sigma^D}{M(t)}, \quad \beta^D(t) \equiv -\omega^i \alpha^D(t), \quad \gamma^D(t) \equiv -\alpha^D(t) \left(\omega^i \mu^D (T-t) + \omega^n \mu^\phi\right). \tag{31}$$

It follows from Propositions 1 and 2, and from market clearing that $\theta_t^m = \sigma_t^S/\Gamma - h_t^u(z) \sigma^D - \Theta_t(Z;\omega) - h_t(Z;\omega) \sigma^D$, where $h_t(z;\omega) \equiv \omega^i h_t^{G|m}(G) + \omega^n h_t^{G|m}(\phi)$ is the weighted sum of the informational hedge of the informed and the corresponding hedge of the noise trader. The aggregate hedging demand depends on $h_t^m(z) \equiv h_t^u(z) + h_t(z;\omega)$.

To proceed, let us conjecture an affine aggregate hedging demand and a time-dependent equilibrium volatility,

$$h_t^m(z) \sigma^D \equiv \left(h_t^u(z) + h_t(z;\omega)\right) \sigma^D \equiv \alpha^h(t) z + \beta^h(t) D_t + \gamma^h(t) \quad \text{and} \quad \sigma_t^S \text{ non-stochastic.} \tag{32}$$

Under this double conjecture,

$$\theta_t^m = \frac{\sigma_t^S}{\Gamma} - \left(\alpha(t) + \alpha^h(t)\right) z - \left(\beta(t) + \beta^h(t)\right) D_t - \left(\gamma(t) + \gamma^h(t)\right) \equiv \frac{\sigma_t^S}{\Gamma} - \widehat{\alpha}(t) Z - \widehat{\beta}(t) D_t - \widehat{\gamma}(t)$$

$$\begin{aligned}\theta_t^m(z) - \theta_t^{Z|D}(z) &= \frac{\sigma_t^S}{\Gamma} - \left(\widehat{\alpha}(t) + \alpha^D(t)\right) z - \left(\widehat{\beta}(t) + \beta^D(t)\right) D_t - \left(\widehat{\gamma}(t) + \gamma^D(t)\right) \\ &\equiv \frac{\sigma_t^S}{\Gamma} - \left(\widetilde{\alpha}(t) z + \widetilde{\beta}(t) D_t + \widetilde{\gamma}(t)\right) \equiv -\left(\widetilde{\alpha}(t) z + \widetilde{\beta}(t) D_t + \overline{\gamma}(t)\right)\end{aligned}$$

where $\overline{\gamma}(t) \equiv \widetilde{\gamma}(t) - \sigma_t^S/\Gamma$. Substituting in the expression for $h_t^u(Z) = \partial_{D_t} H_{t,T}^m$ gives,

$$\begin{aligned}h_t^u(z) &= -\frac{1}{2}\int_t^T \partial_{D_t} \widetilde{E}^z\left[\theta_v^m(z)^2 \middle| \mathcal{F}_t^D\right] dv = -\frac{1}{2}\int_t^T \partial_{D_t} \widetilde{E}^z\left[\left(\widehat{\gamma}(v) + \widehat{\beta}(v) D_v + \widehat{\alpha}(v) z\right)^2 \middle| \mathcal{F}_t^D\right] dv \\ &= -\int_t^T \left(\widehat{\beta}(v)\left(\widehat{\gamma}(v) + \widehat{\alpha}(v) z\right) \partial_{D_t} \widetilde{E}^z\left[D_v \middle| \mathcal{F}_t^D\right] + \frac{1}{2}\widehat{\beta}(v)^2 \partial_{D_t} \widetilde{E}^z\left[D_v^2 \middle| \mathcal{F}_t^D\right]\right) dv \\ &= -\int_t^T \left(\widehat{\beta}(v)\left(\widehat{\gamma}(v) + \widehat{\alpha}(v) z\right) b_1(t,v) + \frac{1}{2}\widehat{\beta}(v)^2\left(b_2(t,v) + c_2(t,v) z + 2 b_1(t,v)^2 D_t\right)\right) dv.\end{aligned}$$

Collecting terms and defining coefficients as indicated leads to the formulas stated. ■

Lemma 5 *Suppose that* $\mathcal{F}_{(\cdot)}^m = \mathcal{F}_{(\cdot)}^{D,Z}$. *If the conjectures in (32) are satisfied, the optimal hedging demand of the informed is* $N_t^{h,i} = N_t^{h,u} + \Gamma h_t^{G|m}(G|Z)\sigma^D/\sigma_t^S$, *where* $h_t^{G|m}(G|Z) = h_{1t}^{G|m}(G|Z) + h_{2t}^{G|m}(G|Z)$ *with,*

$$h_{1t}^{G|m}(G|Z) = \psi_{11}^{i,G}(t) G + \psi_{11}^{i,Z}(t) Z + \psi_{21}^i(t) D_t + \psi_{31}^i(t)$$

$$h_{2t}^{G|m}(G|Z) = \psi_{12}^{i,G}(t) G + \psi_{12}^{i,Z}(t) Z + \psi_{22}^i(t) D_t + \psi_{32}^i(t)$$

$$\psi_{11}^{i,G}(t) = -\int_t^T \alpha_1(v)\beta_0(v) b_1(t,v)\, dv, \quad \psi_{11}^{i,Z}(t) = -\int_t^T \left(\alpha_2(v)\beta_0(v) b_1(t,v) + \frac{1}{2}\beta_0(v)^2 c_2(t,v)\right) dv$$

$$\psi_{21}^i(t) = -\int_t^T \beta_0(v)^2 b_1(t,v)^2\, dv, \quad \psi_{31}^i(t) = -\int_t^T \left(\gamma_0(v)\beta_0(v) b_1(t,v) + \frac{1}{2}\beta_0(v)^2 b_2(t,v)\right) dv$$

$$\psi_{12}^{i,G}(t) = \int_t^T \widehat{\beta}(v)\alpha_1(v) b_1(t,v)\, dv, \quad \psi_{12}^{i,Z}(t) = \int_t^T \left(\left(\widehat{\beta}(v)\alpha_2(v) + \beta_0(v)\widehat{\alpha}(v)\right) b_1(t,v) + \widehat{\beta}(v)\beta_0(v) c_2(t,v)\right) dv$$

$$\psi_{22}^i(t) = 2\int_t^T \beta_0(v)\widehat{\beta}(v) b_1(t,v)^2\, dv, \quad \psi_{32}^i(t) = \int_t^T \left(\left(\gamma_0(v)\widehat{\beta}(v) + \check{\gamma}(v)\beta_0(v)\right) b_1(t,v) + \beta_0(v)\widehat{\beta}(v) b_2(t,v)\right) dv$$

where $\check{\gamma}(t) \equiv \widehat{\gamma}(t) - \sigma_t^S/\Gamma$ *and* b_1, b_2, c_2 *are defined in Lemma 1. The optimal hedging demand of the noise trader has the same structure, but is evaluated at* ϕ *instead of* G.

Proof of Lemma 5. The proof proceeds as for the uninformed. Optimal wealth is,

$$\begin{aligned}X_t^i &= c_0(Z,G) + X_t^u - \Gamma\log\left(\eta_t^G\right) - \frac{\Gamma}{2}\int_t^T E_t\left[\xi_{t,v}^m \theta_v^{G|m}(x|z)\left(\theta_v^{G|m}(x|Z) + 2\theta_v^m(z)\right)\right]_{|x=G} dv \\ &= c_0(Z,G) + X_t^u - \Gamma\log\left(\eta_t^G\right) - \frac{\Gamma}{2}\int_t^T \widetilde{E}^z\left[\theta_v^{G|m}(x|z)\left(\theta_v^{G|m}(x|z) + 2\theta_v^m(z)\right) \middle| \mathcal{F}_t^D\right]_{|z=Z,x=G} dv \\ &\equiv c_0(Z,G) + X_t^u - \Gamma\log\left(\eta_t^G\right) + \Gamma H_{1t,T}^{G|m}(G,Z) + \Gamma H_{2t,T}^{G|m}(G|Z)\end{aligned}$$

where $c_0(Z,G) \equiv \Gamma\log\left(y^u/y^i\right)$ and

$$H_{1t,T}^{G|m}(x|z) = -\frac{1}{2}\int_t^T \widetilde{E}^z\left[\theta_v^{G|m}(x)^2 \middle| \mathcal{F}_t^D\right] dv, \qquad H_{2t,T}^{G|m}(x|z) = -\int_t^T \widetilde{E}^z\left[\theta_v^{G|m}(x)\theta_v^m(z) \middle| \mathcal{F}_t^D\right] dv.$$

The hedging demand of the informed can be decomposed as $h_t^u + h_t^{G|m}(G|Z)$ where $h_t^{G|m}(G|Z) = h_{1t}^{G|m}(G|Z) + h_{2t}^{G|m}(G|Z)$ with $h_{1t}^{G|m}(G|Z) \equiv \partial_{D_t} H_{1t,T}^{G|m}(G|Z)$ and $h_{2t}^{G|m}(G|Z) \equiv \partial_{D_t} H_{2t,T}^{G|m}(G|Z)$. Given the affine structure of the

PIPR, $\theta_t^{G|m}(x) = \alpha_1(t)x + \alpha_2(t)z + \beta_0(t)D_t + \gamma_0(t)$,

$$\begin{aligned}
2h_{1t}^{G|m}(x|z) &= -\int_t^T \left(2(\alpha_1(v)x + \alpha_2(v)z + \gamma_0(v))\beta_0(v)\partial_{D_t}\widetilde{E}^z\left[D_v|\mathcal{F}_t^D\right] + \beta_0(v)^2\partial_{D_t}\widetilde{E}^z\left[D_v^2|\mathcal{F}_t^D\right]\right)dv \\
&= -\int_t^T \left(2(\alpha_1(v)x + \alpha_2(v)z + \gamma_0(v))\beta_0(v)b_1(t,v) + \beta_0(v)^2\left(b_2(t,v) + c_2(t,v)z + 2b_1(t,v)^2D_t\right)\right)dv \\
&= -2\left(\int_t^T \alpha_1(v)\beta_0(v)b_1(t,v)dv\right)x + \left(\int_t^T \left(2\alpha_2(v)\beta_0(v)b_1(t,v) + \beta_0(v)^2c_2(t,v)\right)dv\right)z \\
&\quad -2\left(\int_t^T \beta_0(v)^2b_1(t,v)^2dv\right)D_t + \int_t^T \left(2\gamma_0(v)\beta_0(v)b_1(t,v) + \beta_0(v)^2b_2(t,v)\right)dv
\end{aligned}$$

and using $\theta_t^m = \frac{\sigma_t^S}{\Gamma} - \widehat{\alpha}(t)Z - \widehat{\beta}(t)D_t - \widehat{\gamma}(t) \equiv -\left(\widehat{\alpha}(t)Z + \widehat{\beta}(t)D_t + \check{\gamma}(t)\right)$ where $\check{\gamma}(t) \equiv \widehat{\gamma}(t) - \sigma_t^S/\Gamma$,

$$\begin{aligned}
h_{2t}^{G|m}(x|z) &= \int_t^T \left((\alpha_1(v)x + \alpha_2(v)z + \gamma_0(v))\widehat{\beta}(v) + (\check{\gamma}(v) + \widehat{\alpha}(v)z)\beta_0(v)\right)\partial_{D_t}\widetilde{E}^z\left[D_v|\mathcal{F}_t^D\right]dv \\
&\quad + \int_t^T \beta_0(v)\widehat{\beta}(v)\partial_{D_t}\widetilde{E}^z\left[D_v^2|\mathcal{F}_t^D\right]dv \\
&= \int_t^T \left((\alpha_1(v)x + \alpha_2(v)z + \gamma_0(v))\widehat{\beta}(v) + (\check{\gamma}(v) + \widehat{\alpha}(v)z)\beta_0(v)\right)b_1(t,v)dv \\
&\quad + \int_t^T \beta_0(v)\widehat{\beta}(v)\left(b_2(t,v) + c_2(t,v)z + 2b_1(t,v)^2D_t\right)dv \\
&= \int_t^T \widehat{\beta}(v)\alpha_1(v)b_1(t,v)dvx + \int_t^T \left(\left(\widehat{\beta}(v)\alpha_2(v) + \beta_0(v)\widehat{\alpha}(v)\right)b_1(t,v) + \widehat{\beta}(v)\beta_0(v)c_2(t,v)\right)dvz \\
&\quad +2\int_t^T \beta_0(v)\widehat{\beta}(v)b_1(t,v)^2dvD_t + \int_t^T \left(\left(\gamma_0(v)\widehat{\beta}(v) + \check{\gamma}(v)\beta_0(v)\right)b_1(t,v) + \beta_0(v)\widehat{\beta}(v)b_2(t,v)\right)dv.
\end{aligned}$$

Defining the coefficients as indicated in the lemma establishes the result. ■

Lemma 6 *Suppose that $\mathcal{F}_{(\cdot)}^m = \mathcal{F}_{(\cdot)}^{D,Z}$. If the conjectures in (32) are satisfied, the information-related component of the residual hedging demand depends on,*

$$h_t(Z;\omega) \equiv \omega^i h_t^{G|m}(G|Z) + \omega^n h_t^{G|m}(\phi|Z) = h_{1t}(Z;\omega) + h_{2t}(Z;\omega)$$

$$h_{1t}(Z;\omega) = \left(\psi_{11}^{i,G}(t) + \omega\psi_{11}^{i,Z}(t)\right)Z + \omega\psi_{21}^i(t)D_t + \omega\psi_{31}^i(t)$$

$$h_{2t}(Z;\omega) = \left(\psi_{12}^{i,G}(t) + \omega\psi_{12}^{i,Z}(t)\right)Z + \omega\psi_{22}^i(t)D_t + \omega\psi_{32}^i(t).$$

The aggregate hedging demand depends on,

$$h_t^m(Z) \equiv h_t^u(Z) + h_t(Z;\omega) = \psi_{1t}^m Z + \psi_{2t}^m D_t + \psi_{3t}^m$$

$$\psi_{1t}^m = \psi_{1t}^u + \psi_{11}^{i,G}(t) + \psi_{12}^{i,G}(t) + \omega\left(\psi_{11}^{i,Z}(t) + \psi_{12}^{i,Z}(t)\right)$$

$$\psi_{2t}^m = \psi_{2t}^u + \omega\left(\psi_{21}^i(t) + \psi_{22}^i(t)\right), \quad \psi_{3t}^m = \psi_{3t}^u + \omega\left(\psi_{31}^i(t) + \psi_{32}^i(t)\right).$$

The aggregate hedging demand is an affine function of Z.

Proof of Lemma 6. Substituting the formulas from Lemmas 3 and 4 in $h_t^{G|m}(Z;\omega) \equiv \omega^i h_t^{G|m}(G|Z) + \omega^n h_t^{G|m}(\phi|Z)$ and $h_t^m(Z) \equiv h_t^u(Z) + h_t(Z;\omega)$ leads to the expressions stated. ■

The coefficients of the aggregate hedging demand depend on the coefficients of the conjectures in (32). That is, $h_t^m(Z) = \alpha^h(t)Z + \beta^h(t)D_t + \gamma^h(t) = \psi_{1t}^m Z + \psi_{2t}^m D_t + \psi_{3t}^m$. For consistency, the following integral equations must be satisfied, $\alpha^h(t) = \psi_{1t}^m\sigma^D$, $\beta^h(t) = \psi_{2t}^m\sigma^D$, $\gamma^h(t) = \psi_{3t}^m\sigma^D$. Simplifying the expressions for $(\psi_{1t}^m, \psi_{2t}^m, \psi_{3t}^m)$ gives,

$$\psi_{1t}^m = -\int_t^T \left(\widehat{\alpha}(v)b_1(t,v) + \frac{1}{2}\widehat{\beta}(v)^2c_2(t,v)\right)dv - \int_t^T \alpha_1(v)\beta_0(v)b_1(t,v)dv + \int_t^T \widehat{\beta}(v)\alpha_1(v)b_1(t,v)dv$$

$$
\begin{aligned}
&-\omega \int_t^T \left(\alpha_2(v)\,\beta_0(v)\,b_1(t,v)+\frac{1}{2}\beta_0(v)^2\,c_2(t,v)\right)dv \\
&+\omega \int_t^T \left(\left(\widehat{\beta}(v)\,\alpha_2(v)+\beta_0(v)\,\widehat{\alpha}(v)\right)b_1(t,v)+\widehat{\beta}(v)\,\beta_0(v)\,c_2(t,v)\right)dv \\
= &-\int_t^T \left(\widehat{\alpha}(v)+\alpha_1(v)\,\beta_0(v)-\widehat{\beta}(v)\,\alpha_1(v)+\omega\left(\alpha_2(v)\,\beta_0(v)-\left(\widehat{\beta}(v)\,\alpha_2(v)+\beta_0(v)\,\widehat{\alpha}(v)\right)\right)\right)b_1(t,v)\,dv \\
&-\int_t^T \left(\frac{1}{2}\widehat{\beta}(v)^2+\omega\left(\frac{1}{2}\beta_0(v)^2-\widehat{\beta}(v)\,\beta_0(v)\right)\right)c_2(t,v)\,dv \\
= &-\int_t^T \left((1-\beta(v))\,\widehat{\alpha}(v)+\alpha(v)\left(\beta_0(v)-\widehat{\beta}(v)\right)\right)b_1(t,v)\,dv \\
&-\frac{1}{2}\int_t^T \left(\widehat{\beta}(v)^2+\omega\beta_0(v)\left(\beta_0(v)-2\widehat{\beta}(v)\right)\right)c_2(t,v)\,dv
\end{aligned}
$$

$$
\begin{aligned}
\psi_{2t}^m &= -\int_t^T \widehat{\beta}(v)^2\,b_1(t,v)^2\,dv+\omega\left(-\int_t^T \beta_0(v)^2\,b_1(t,v)^2\,dv+2\int_t^T \beta_0(v)\,\widehat{\beta}(v)\,b_1(t,v)^2\,dv\right) \\
&= -\int_t^T \left(\widehat{\beta}(v)^2-2\omega\beta_0(v)\left(\widehat{\beta}(v)-\frac{1}{2}\beta_0(v)\right)\right)b_1(t,v)^2\,dv
\end{aligned}
$$

$$
\begin{aligned}
\psi_{3t}^m = &-\int_t^T \left(\widehat{\beta}(v)\,\widehat{\gamma}(v)\,b_1(t,v)+\frac{1}{2}\widehat{\beta}(v)^2\,b_2(t,v)\right)dv-\omega\int_t^T \left(\gamma_0(v)\,\beta_0(v)\,b_1(t,v)+\frac{1}{2}\beta_0(v)^2\,b_2(t,v)\right)dv \\
&+\omega\int_t^T \left(\left(\gamma_0(v)\,\widehat{\beta}(v)+\check{\gamma}(v)\,\beta_0(v)\right)b_1(t,v)+\beta_0(v)\,\widehat{\beta}(v)\,b_2(t,v)\right)dv \\
= &-\int_t^T \left(\widehat{\beta}(v)\,\widehat{\gamma}(v)-\omega\left(\widehat{\beta}(v)-\beta_0(v)\right)\gamma_0(v)-\omega\check{\gamma}(v)\,\beta_0(v)\right)b_1(t,v)\,dv \\
&-\int_t^T \left(\frac{1}{2}\widehat{\beta}(v)^2-\omega\beta_0(v)\left(\widehat{\beta}(v)-\frac{1}{2}\beta_0(v)\right)\right)b_2(t,v)\,dv
\end{aligned}
$$

and substituting in the equations for α^h , β^h , γ^h gives,

$$
\begin{aligned}
\frac{\alpha^h(t)}{\sigma^D} = &-\int_t^T \left((1-\beta(v))\,\widehat{\alpha}(v)+\alpha(v)\left(\beta_0(v)-\widehat{\beta}(v)\right)\right)b_1(t,v)\,dv \\
&-\frac{1}{2}\int_t^T \left(\widehat{\beta}(v)^2+\omega\beta_0(v)\left(\beta_0(v)-2\widehat{\beta}(v)\right)\right)c_2(t,v)\,dv \qquad (33)
\end{aligned}
$$

$$
\frac{\beta^h(t)}{\sigma^D} = -\int_t^T \left(\widehat{\beta}(v)^2-2\omega\beta_0(v)\left(\widehat{\beta}(v)-\frac{1}{2}\beta_0(v)\right)\right)b_1(t,v)^2\,dv \qquad (34)
$$

$$
\begin{aligned}
\frac{\gamma^h(t)}{\sigma^D} = &-\int_t^T \left(\widehat{\beta}(v)\,\widehat{\gamma}(v)-\omega\left(\widehat{\beta}(v)-\beta_0(v)\right)\gamma_0(v)-\omega\check{\gamma}(v)\,\beta_0(v)\right)b_1(t,v)\,dv \\
&-\int_t^T \left(\frac{1}{2}\widehat{\beta}(v)^2-\omega\beta_0(v)\left(\widehat{\beta}(v)-\frac{1}{2}\beta_0(v)\right)\right)b_2(t,v)\,dv. \qquad (35)
\end{aligned}
$$

where $\alpha(v)=\alpha_1(v)+\omega\alpha_2(v)$. The system (33)-(35) can be solved sequentially. As will be shown next, σ_t^S depends only on β^h. Hence, (34) is autonomous and determines β^h. The coefficients $\left(\alpha^h,\gamma^h\right)$ follow from (33) and (35).

Lemma 7 *Suppose that $\mathcal{F}_{(\cdot)}^m=\mathcal{F}_{(\cdot)}^{D,Z}$. If the conjectures in (32) are satisfied, the equilibrium stock price is $S_t=\widehat{A}(t)\,Z+\ \widehat{B}(t)\,D_t\ +\widehat{F}(t)$ where,*

$$
\widehat{B}(t)=B(t)\,B^h(t)\,,\quad B(t)=\left(\frac{H(T)}{H(t)}\right)^{\omega}\left(\frac{M(T)}{M(t)}\right)^{1-\omega},\quad B^h(t)=e^{\sigma^D\int_t^T \beta^h(v)dv}
$$

$$
\widehat{A}(t)=\lambda(t,T)+\sigma^D\left(\int_t^T \widehat{B}(s)\left(\widehat{\alpha}(s)+\widehat{\beta}(s)\lambda(t,s)\right)ds\right)
$$

$$\widehat{F}(t)=\widehat{B}\left(t\right)\mu^{D}\left(T-t\right)-\frac{\left(\sigma^{D}\right)^{2}}{\Gamma}\int_{t}^{T}\widehat{B}\left(s\right)^{2}ds+\sigma^{D}\int_{t}^{T}\widehat{B}\left(s\right)\widehat{\gamma}\left(s\right)ds-\omega^{n}\widehat{I}\left(t\right)\mu^{\phi}$$

$$\widehat{I}\left(t\right)=\lambda\left(t,T\right)+\sigma^{D}\int_{t}^{T}\widehat{B}\left(s\right)\widehat{\beta}\left(s\right)\lambda\left(t,s\right)ds,\qquad\lambda\left(t,s\right)=\frac{\omega^{i}\left(\sigma^{D}\right)^{2}\left(s-t\right)}{M\left(t\right)}.$$

The stock price volatility equals $\sigma_t^S=\widehat{B}\left(t\right)\sigma^D$, *a function of time.*

Proof of Lemma 7. Straightforward calculations give,

$$\begin{aligned}
S_t &= \mathbb{E}\left[D_T-\int_t^T\mu_s^S ds\,\middle|\,\mathcal{F}_t^m\right]=D_t+\mu^D(T-t)+\sigma^D\mathbb{E}\left[W_T^D-W_t^D\,\middle|\,\mathcal{F}_t^{D,Z}\right]\\
&\quad-\frac{1}{\Gamma}\int_t^T\left(\sigma_s^S\right)^2ds+\int_t^T\mathbb{E}\left[\sigma_s^S\left(\widehat{\alpha}(s)Z+\widehat{\beta}(s)D_s+\widehat{\gamma}(s)\right)\middle|\,\mathcal{F}_t^{D,Z}\right]\\
&= D_t+\mu^D(T-t)-\frac{1}{\Gamma}\int_t^T\left(\sigma_s^S\right)^2ds+\int_t^T\sigma_s^S\widehat{\gamma}(s)ds+\left(\int_t^T\sigma_s^S\widehat{\alpha}(s)ds\right)Z\\
&\quad+\sigma^D\mathbb{E}\left[W_T^D-W_t^D\,\middle|\,\mathcal{F}_t^{Z,D}\right]+\int_t^T\sigma_s^S\widehat{\beta}(s)\mathbb{E}\left[D_s|\,\mathcal{F}_t^{D,Z}\right]ds\\
&\equiv D_t+\widehat{G}_0\left(t,T\right)Z+\overline{F}\left(\sigma^S,t\right)+\sigma^D\mathbb{E}\left[W_T^D-W_t^D\,\middle|\,\mathcal{F}_t^{D,Z}\right]+\int_t^T\sigma_s^S\widehat{\beta}\left(s\right)\mathbb{E}\left[D_s|\mathcal{F}_t^{D,Z}\right]ds
\end{aligned}$$

where $\widehat{G}_0\left(t,T\right)=\int_t^T\sigma_s^S\widehat{\alpha}\left(s\right)ds$ and $\overline{F}\left(\sigma^S,t\right)=\mu^D\left(T-t\right)-(1/\Gamma)\int_t^T\left(\sigma_s^S\right)^2ds+\int_t^T\sigma_s^S\widehat{\gamma}\left(s\right)ds$. Moreover,

$$\begin{aligned}
\mathbb{E}\left[\int_t^T\sigma^D dW_s^D\,\middle|\,\mathcal{F}_t^{D,Z}\right] &= \lambda\left(t,T\right)\left(Z-\omega^i\left(D_t+\mu^D\left(T-t\right)\right)-\omega^n\mu^\phi\right)\\
&= \lambda\left(t,T\right)Z-\omega^i\lambda\left(t,T\right)D_t-\lambda\left(t,T\right)\left(\omega^i\mu^D\left(T-t\right)+\omega^n\mu^\phi\right)
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\left[D_s|\mathcal{F}_t^{D,Z}\right] &= D_t+\mu^D\left(T-t\right)+\lambda\left(t,s\right)\left(Z-\omega^i\left(D_t+\mu^D\left(T-t\right)\right)-\omega^n\mu^\phi\right)\\
&= \left(D_t+\mu^D\left(T-t\right)\right)\left(1-\omega^i\lambda\left(t,s\right)\right)+\lambda\left(t,s\right)Z-\omega^n\lambda\left(t,s\right)\mu^\phi
\end{aligned}$$

where $\lambda\left(t,s\right)=\frac{\omega^i\left(\sigma^D\right)^2(s-t)}{M(t)}$, so that,

$$\int_t^T\sigma_s^S\widehat{\beta}(s)\mathbb{E}\left[D_s|\mathcal{F}_t^{D,Z}\right]ds=\widehat{G}_1\left(t,T\right)\left(D_t+\mu^D\left(T-t\right)\right)+\widehat{G}_2\left(t,T\right)\left(Z-\omega^n\mu^\phi\right)$$

$$\widehat{G}_1\left(t,T\right)=\int_t^T\sigma_s^S\widehat{\beta}(s)\left(1-\omega^i\lambda(t,s)\right)ds,\qquad G_2\left(t,T\right)=\int_t^T\sigma_s^S\widehat{\beta}(s)\lambda(t,s)ds.$$

Hence,

$$\begin{aligned}
S_t &= D_t+\widehat{G}_0\left(t,T\right)Z+\overline{F}\left(\sigma^S,t\right)+\lambda\left(t,T\right)Z-\omega^i\lambda\left(t,T\right)D_t\\
&\quad-\lambda\left(t,T\right)\left(\omega^i\mu^D\left(T-t\right)+\omega^n\mu^\phi\right)+G_1\left(t,T\right)\left(D_t+\mu^D\left(T-t\right)\right)+G_2\left(t,T\right)\left(Z-\omega^n\mu^\phi\right)\\
&= \left(1-\omega^i\lambda\left(t,T\right)+\widehat{G}_1\left(t,T\right)\right)D_t+\left(\widehat{G}_0\left(t,T\right)+\lambda\left(t,T\right)+\widehat{G}_2\left(t,T\right)\right)Z+\overline{F}\left(\sigma^S,t\right)\\
&\quad+\left(-\omega^i\lambda\left(t,T\right)+\widehat{G}_1\left(t,T\right)\right)\mu^D\left(T-t\right)-\left(\lambda\left(t,T\right)+\widehat{G}_2\left(t,T\right)\right)\omega^n\mu^\phi\equiv\widehat{A}(t)Z+\widehat{B}(t)D_t+\widehat{F}(t)
\end{aligned}$$

where,

$$\widehat{A}(t)=\widehat{G}_0\left(t,T\right)+\lambda\left(t,T\right)+\widehat{G}_2\left(t,T\right)=\lambda\left(t,T\right)+\int_t^T\sigma_s^S\left(\widehat{\alpha}\left(s\right)+\widehat{\beta}(s)\lambda(t,s)\right)ds$$

$$\widehat{B}(t)=1-\omega^i\lambda\left(t,T\right)+\widehat{G}_1\left(t,T\right)=1-\omega^i\lambda\left(t,T\right)+\int_t^T\sigma_s^S\widehat{\beta}(s)\left(1-\omega^i\lambda(t,s)\right)ds$$

$$\widehat{F}(t)=\overline{F}\left(\sigma^S,t\right)+\left(\widehat{B}\left(t\right)-1\right)\mu^D\left(T-t\right)-\left(\widehat{B}\left(t\right)-\int_t^T\sigma_s^S\widehat{\alpha}\left(s\right)ds\right)\omega^n\mu^\phi.$$

An application of Ito's lemma gives $\sigma_t^S = \widehat{B}(t)\,\sigma^D$. The volatility coefficient is deterministic as conjectured. This validates the construction of the equilibrium stock price to this stage. Substituting in the coefficients above yields,

$$\widehat{A}(t) = \lambda(t,T) + \sigma^D \int_t^T \widehat{B}(s)\left(\widehat{\alpha}(s) + \widehat{\beta}(s)\lambda(t,s)\right) ds, \qquad \widehat{B}(t) = 1 - \omega^i \lambda(t,T) + \sigma^D \int_t^T \widehat{B}(s)\,\widehat{\beta}(s)\left(1 - \omega^i \lambda(t,s)\right) ds$$

$$\widehat{F}(t) = \overline{F}\left(\widehat{B}(t)\,\sigma^D, t\right) + \left(\widehat{B}(t) - 1\right)\mu^D (T-t) - \widehat{I}(t)\,\omega^n \mu^\phi, \qquad \widehat{I}(t) = \widehat{B}(t) - \sigma^D \int_t^T \widehat{B}(s)\,\widehat{\alpha}(s)\, ds.$$

Inserting $\overline{F}\left(B(t)\,\sigma^D, t\right) = \mu^D (T-t) - \frac{1}{\Gamma}\left(\sigma^D\right)^2 \int_t^T \widehat{B}(s)^2\, ds + \sigma^D \int_t^T \widehat{B}(s)\,\widehat{\gamma}(s)\, ds$ in the last coefficient and collecting terms leads to,

$$\widehat{F}(t) = \widehat{B}(t)\,\mu^D (T-t) - \frac{\left(\sigma^D\right)^2}{\Gamma} \int_t^T \widehat{B}(s)^2\, ds + \sigma^D \int_t^T \widehat{B}(s)\,\widehat{\gamma}(s)\, ds - \widehat{I}(t)\,\omega^n \mu^\phi$$

$$\widehat{I}(t) = \lambda(t,T) + \sigma^D \int_t^T \widehat{B}(s)\,\widehat{\beta}(s)\,\lambda(t,s)\, ds.$$

In these expressions, with $\omega = \omega^i + \omega^n$,

$$\widehat{\alpha}(t) = \alpha(t) + \alpha^h(t), \quad \widehat{\beta}(t) = \beta(t) + \beta^h(t), \quad \widehat{\gamma}(t) = \gamma(t) + \gamma^h(t), \quad \alpha(t) = \frac{1-\kappa_t \omega}{H(t)}\sigma^D, \quad \beta(t) = -\omega \frac{1-\kappa_t \omega^i}{H(t)}\sigma^D$$

$$\gamma(t) = -\omega \frac{\left(1-\kappa_t \omega^i\right)\mu^D (T-t) - \omega^n \kappa_t \mu^\phi}{H(t)}\sigma^D, \quad \lambda(t,s) = \frac{\omega^i \left(\sigma^D\right)^2 (s-t)}{M(t)}.$$

Equilibrium exists if the backward Volterra equation,

$$\widehat{B}(t) = 1 - \omega^i \lambda(t,T) + \sigma^D \left(\int_t^T \widehat{B}(s)\,\widehat{\beta}(s)\left(1 - \omega^i \lambda(t,s)\right) ds\right), \quad \widehat{B}(T) = 1 \tag{36}$$

for the coefficient $\widehat{B}(\cdot)$ has a solution. This issue is addressed in the next lemma. ■

Lemma 8 *The unique solution of (36) is* $\widehat{B}(t) = B(t)\,B^h(t)$ *where* $B(t) = \left(\frac{H(T)}{H(t)}\right)^\omega \left(\frac{M(T)}{M(t)}\right)^{1-\omega}$ *with* $M(t) = \left(\omega^i\right)^2 H(t) + (\omega^n)^2 \left(\sigma^\phi\right)^2$, $\omega = \omega^i + \omega^n$ *and* $B^h(t) = \exp\left(\sigma^D \int_t^T \beta^h(v)\, dv\right)$. *Moreover,* $\widehat{B}(t) > 0$ *for* $t \in [0,T]$.

Proof of Lemma 8. With $M(t) = \left(\omega^i\right)^2 H(t) + (\omega^n)^2 \left(\sigma^\phi\right)^2$, note that,

$$1 - \omega^i \lambda(t,T) = 1 - \frac{\left(\omega^i\right)^2 \left(\sigma^D\right)^2 (T-t)}{M(t)} = \frac{\left(\omega^i\right)^2 \left(\sigma^\zeta\right)^2 + (\omega^n)^2 \left(\sigma^\phi\right)^2}{M(t)} \equiv \frac{M(T)}{M(t)}$$

$$1 - \omega^i \lambda(t,s) = \frac{\left(\omega^i\right)^2 \left(\left(\sigma^D\right)^2 (T-s) + \left(\sigma^\zeta\right)^2\right) + (\omega^n)^2 \left(\sigma^\phi\right)^2}{M(t)} = \frac{M(s)}{M(t)}.$$

Substituting in (36) and using the change of variables $C(t) = \widehat{B}(t)M(t)$ leads to,

$$\begin{aligned}
\widehat{B}(t) \quad &= \quad 1 - \omega^i \lambda(t,T) + \sigma^D \left(\int_t^T \widehat{B}(s)\,\widehat{\beta}(s)\left(1 - \omega^i \lambda(t,s)\right) ds\right) = \frac{M(T)}{M(t)} + \sigma^D \left(\int_t^T \widehat{B}(s)\,\widehat{\beta}(s)\frac{M(s)}{M(t)} ds\right) \\
\iff \quad & \widehat{B}(t)M(t) = M(T) + \sigma^D \left(\int_t^T \widehat{B}(s)\,\widehat{\beta}(s) M(s)\, ds\right) \iff C(t) = M(T) + \sigma^D \left(\int_t^T C(s)\,\widehat{\beta}(s) ds\right)
\end{aligned}$$

subject to the boundary condition $C(T) = M(T)$. Equivalently, $dC(t) = -\sigma^D C(t)\,\widehat{\beta}(t)dt$. The solution is $C(t) = M(T)\exp\left(\sigma^D \int_t^T \widehat{\beta}(s)ds\right)$. Substituting, $\widehat{\beta}(t) = \beta(t) + \beta^h(t)$,

$$\beta(t) = -\frac{\omega}{H(t)}\left(1 - \kappa_t \omega^i\right)\sigma^D = -\frac{\omega}{H(t)}\left(1 - \frac{\left(\omega^i\right)^2 H(t)}{M(t)}\right)\sigma^D = -\omega \sigma^D \left(\frac{1}{H(t)} - \frac{\left(\omega^i\right)^2}{M(t)}\right)$$

and performing the integration,

$$C(t) = M(T)\exp\left(\omega\left(\log\left(\frac{H(T)}{H(t)}\right) - \log\left(\frac{M(T)}{M(t)}\right)\right)\right)B^h(t) = M(T)\left(\frac{H(T)}{H(t)}\right)^{\omega}\left(\frac{M(T)}{M(t)}\right)^{-\omega}B^h(t).$$

Substituting $C(t) = B(t)M(t)$ and rearranging leads to the formula stated. ■

Under the conjectures in (32), $\widehat{B}$ depends on $\widehat{\beta}$. To validate the construction of hedging demands and equilibrium, conditional on the information structure, it remains to show the existence of $\widehat{\beta}$. This is done next.

Lemma 9 *The coefficient β^h solves the Riccati equation,*

$$\dot{\beta}^h(t) = \kappa_0(t) + \kappa_1(t)\beta^h(t) - \sigma^D\beta^h(t)^2; \quad \beta^h(T) = 0 \tag{37}$$

$$\kappa_0(t) \equiv \left(\frac{1-\omega}{\omega}\right)\beta(t)^2\sigma^D, \qquad \kappa_1(t) \equiv -2\sigma^D\left(\beta(t) + \beta^D(t)\right) \tag{38}$$

where $\beta(t)$ is defined in (6). A unique solution exists.

Proof of Lemma 9. Differentiating (34) relative to time and using $\partial_t b_1(t,v)^2 = -2\sigma^D b_1(t,v)^2\widetilde{\beta}(t)$ yields,

$$\frac{\dot{\beta}^h(t)}{\sigma^D} = \widehat{\beta}(t)^2 - 2\omega\beta_0(t)\left(\widehat{\beta}(t) - \frac{1}{2}\beta_0(t)\right) - 2\sigma^D\frac{\beta^h(t)}{\sigma^D}\widetilde{\beta}(t)$$

with boundary condition $\beta^h(T) = 0$. Using the definitions $\widetilde{\beta} = \beta + \beta^h + \beta^D$, $\widehat{\beta} = \beta + \beta^h$ and $\beta(t) = \omega\beta_0(t)$ gives, after simplifications, $\dot{\beta}^h(t) = \left(\frac{1-\omega}{\omega}\right)\beta(t)^2\sigma^D - 2\sigma^D\left(\beta(t) + \beta^D(t)\right)\beta^h(t) - \sigma^D\beta^h(t)^2$. Defining κ_0, κ_1 as in (38) proves (37). Existence of a unique solution follows from continuity of the ODE and the fact (see Technical Appendix) that $\beta^h(t) \in \left(\underline{\beta}(t), 0\right]$ where $\underline{\beta}^h(t) \equiv -\int_t^T \exp\left(-\int_t^s k_1(u)\,du\right)k_0(s)\,ds < 0$. ■

The next lemma gives closed form expressions for $\left(\alpha^h, \gamma^h\right)$.

Lemma 10 *The coefficients $\left(\alpha^h, \gamma^h\right)$ are given by,*

$$\alpha^h(t) = -\int_t^T \exp\left(-\int_t^s \kappa_2(v)\,dv\right)\kappa_3(s)\,ds, \qquad \gamma^h(t) = -\int_t^T \exp\left(-\int_t^s \kappa_4(v)\,dv\right)\kappa_5(s)\,ds \tag{39}$$

where, with $K(t,T) \equiv \int_t^T \left(\beta^h(v)^2 + \frac{\kappa_0(v)}{\sigma^D}\right)b_1(t,v)^2\,dv$,

$$\kappa_2(t) \equiv \left(1 - \beta(t) - \widetilde{\beta}(t)\right)\sigma^D + \left(\sigma^D\right)^2 K(t,T) \tag{40}$$

$$\kappa_3(t) \equiv \alpha(t)\left(1 + \beta(t)\left(\frac{1-\omega}{\omega}\right) - \widehat{\beta}(t)\right)\sigma^D + \left(\sigma^D\right)^2 K(t,T)\left(\alpha(t) + \alpha^D(t)\right) \tag{41}$$

$$\kappa_4(t) \equiv -\left(\beta(t) + \beta^D(t)\right)\sigma^D + \sigma^D K(t,T) \tag{42}$$

$$\kappa_5(t) \equiv \beta(t)\left(\frac{1-\omega}{\omega}\gamma(t) + \frac{\sigma_t^S}{\Gamma}\right)\sigma^D + \sigma^D K(t,T)\left(\mu^D + \sigma^D\left(\gamma(t) + \gamma^D(t) - \frac{\sigma_t^S}{\Gamma}\right)\right). \tag{43}$$

Proof of Lemma 10. As $\alpha(t) = \alpha_1(t) + \omega\alpha_2(t)$, $b_1(t,t) = 1$, $c_2(t,t) = 0$, $\partial_t b_1(t,v) = -\sigma^D b_1(t,v)\widetilde{\beta}(t)$ and $\partial_t c_2(t,v) = -2\sigma^D b_1(t,v)^2\widetilde{\alpha}(t) - \sigma^D c_2(t,v)\widetilde{\beta}(t)$, it follows from (33) that,

$$\begin{aligned}
\frac{\dot{\alpha}^h(t)}{\sigma^D} &= (1-\beta(t))\widehat{\alpha}(t) + \alpha(t)\left(\beta_0(t) - \widehat{\beta}(t)\right) - \sigma^D\frac{\alpha^h(t)}{\sigma^D}\widetilde{\beta}(t) \\
&\quad +2\sigma^D\int_t^T\left(\frac{1}{2}\widehat{\beta}(v)^2 + \omega\beta_0(v)\left(\frac{1}{2}\beta_0(v) - \widehat{\beta}(v)\right)\right)b_1(t,v)^2\,dv\widetilde{\alpha}(t) \\
&= \left(1 - \beta(t) - \widetilde{\beta}(t) + 2\sigma^D\int_t^T\left(\frac{1}{2}\widehat{\beta}(v)^2 + \omega\beta_0(v)\left(\frac{1}{2}\beta_0(v) - \widehat{\beta}(v)\right)\right)b_1(t,v)^2\,dv\right)\alpha^h(t) \\
&\quad +\alpha(t)\left(1 - \beta(t) + \beta_0(t) - \widehat{\beta}(t)\right) \\
&\quad +2\sigma^D\int_t^T\left(\frac{1}{2}\widehat{\beta}(v)^2 + \omega\beta_0(v)\left(\frac{1}{2}\beta_0(v) - \widehat{\beta}(v)\right)\right)b_1(t,v)^2\,dv\left(\alpha(t) + \alpha^D(t)\right)
\end{aligned}$$

with boundary condition $\alpha^h(T) = 0$. Expression (39) follows because,

$$\begin{aligned} 2\left(\frac{1}{2}\widehat{\beta}(v)^2 + \omega\beta_0(v)\left(\frac{1}{2}\beta_0(v) - \widehat{\beta}(v)\right)\right) &= \left(\beta(v) + \beta^h(v)\right)^2 + \beta(v)\left(\frac{\beta(v)}{\omega} - 2\left(\beta(v) + \beta^h(v)\right)\right) \\ &= \beta^h(v)^2 + \beta(v)^2\left(\frac{1}{\omega} - 1\right) = \beta^h(v)^2 + \frac{\kappa_0(v)}{\sigma^D} \end{aligned}$$

where $\kappa_0(v)$ is given in (38).

Using $\partial_t b_2(t,v) = -\sigma^D b_2(t,v)\tilde{\beta}(t) - 2b_1(t,v)^2\left(\mu^D + \sigma^D\left(\gamma(t) + \gamma^D(t) - \sigma_t^S/\Gamma\right)\right) - 2\sigma^D b_1(t,v)^2\gamma^h(t)$ and $b_1(t,t) = b_2(t,t) = 0$, $\check{\gamma}(t) = \widehat{\gamma}(t) - \sigma_t^S/\Gamma$, $\omega\beta_0(t) = \beta(t)$, $\omega\gamma_0(t) = \gamma(t)$ and differentiating the integral equation gives,

$$\begin{aligned} \frac{\dot{\gamma}^h(t)}{\sigma^D} &= \left(\beta^h(t)\widehat{\gamma}(t) - \omega\left(\widehat{\beta}(t) - \beta_0(t)\right)\gamma_0(t) + \frac{\sigma_t^S}{\Gamma}\beta(t)\right) - \sigma^D\frac{\gamma^h(t)}{\sigma^D}\widetilde{\beta}(t) \\ &\quad + 2\int_t^T\left(\frac{1}{2}\widehat{\beta}(v)^2 - \omega\beta_0(v)\left(\widehat{\beta}(v) - \frac{1}{2}\beta_0(v)\right)\right)b_1(t,v)^2\,dv\left(\mu^D + \sigma^D\left(\gamma(t) + \gamma^D(t) - \sigma_t^S/\Gamma + \gamma^h(t)\right)\right) \\ &= \left(\beta^h(t) - \widetilde{\beta}(t) + 2\int_t^T\left(\frac{1}{2}\widehat{\beta}(v)^2 - \omega\beta_0(v)\left(\widehat{\beta}(v) - \frac{1}{2}\beta_0(v)\right)\right)b_1(t,v)^2\,dv\right)\gamma^h(t) \\ &\quad + \left(\beta^h(t)\gamma(t) - \omega\left(\widehat{\beta}(t) - \beta_0(t)\right)\gamma_0(t) + \frac{\sigma_t^S}{\Gamma}\beta(t)\right) \\ &\quad + 2\int_t^T\left(\frac{1}{2}\widehat{\beta}(v)^2 - \omega\beta_0(v)\left(\widehat{\beta}(v) - \frac{1}{2}\beta_0(v)\right)\right)b_1(t,v)^2\,dv\left(\mu^D + \sigma^D\left(\gamma(t) + \gamma^D(t) - \sigma_t^S/\Gamma\right)\right). \end{aligned}$$

The expressions for the coefficients in the Lemma and the representation of γ^h in (39) follow. ■

Proof of Proposition 5. Lemmas 2-10 establish that an equilibrium exists under the assumption $\mathcal{F}_{(\cdot)}^m = \mathcal{F}_{(\cdot)}^{D,Z}$. To complete the proof of existence, it remains to verify that the endogenous public filtration is indeed $\mathcal{F}_{(\cdot)}^m = \mathcal{F}_{(\cdot)}^{D,Z}$. To this end, use the relations,

$$h_t^m = \psi_1^m(t)Z + \psi_2^m(t)D_t + \psi_3^m(t), \qquad h_t^m = h_t^u + h_t(Z;\omega)$$

$$\Theta_t(Z;\omega) = \alpha(t)Z + \beta(t)D_t + \gamma(t), \qquad \theta_t^m = \frac{\sigma_t^S}{\Gamma} - \Theta_t(Z;\omega) - h_t^m(Z;\omega)\sigma^D$$

to write the residual demand as,

$$\begin{aligned} \frac{N_t^a\sigma_t^S}{\Gamma} &= \omega^i\left(\theta_t^m + \theta_t^{G|m}(G) + (h_t^u + h_t(G;\omega))\sigma^D\right) + \omega^n\left(\theta_t^m + \theta_t^{G|m}(\phi) + (h_t^u + h_t(\phi;\omega))\sigma^D\right) \\ &= \omega\left(\frac{\sigma_t^S}{\Gamma} - \Theta_t(Z;\omega) - h_t^m(Z;\omega)\sigma^D\right) + \Theta_t(Z;\omega) + (\omega h_t^u + h_t(Z;\omega))\sigma^D \\ &= \omega\frac{\sigma_t^S}{\Gamma} + (1-\omega)\left(\Theta_t(Z;\omega) + h_t(Z;\omega)\sigma^D\right) \\ &= \omega\frac{\sigma_t^S}{\Gamma} + (1-\omega)\left(\left(\alpha(t) + \psi_1^i(t)\sigma^D\right)Z + \left(\beta(t) + \psi_2^i(t)\sigma^D\right)D_t + \gamma(t) + \psi_3^i(t)\sigma^D\right). \end{aligned}$$

As N_t^a, σ_t^S and D_t are observed, the product $\left(\alpha(t) + \psi_1^i(t)\sigma^D\right)Z$ is known. If $\alpha(0) + \psi_1^i(0)\sigma^D \neq 0$, the endogenous signal Z is revealed. If $\alpha(0) + \psi_1^i(0)\sigma^D = 0$, then $\alpha(t) + \psi_1^i(t)\sigma^D \neq 0$ for $t = 0^+$. In both cases $\mathcal{F}_{(\cdot)}^m = \mathcal{F}_{(\cdot)}^{D,Z}$ and the candidate equilibrium constructed is a rational expectations equilibrium. Moreover, as $\mathcal{F}_{(\cdot)}^m = \mathcal{F}_{(\cdot)}^{D,Z} \subset \mathcal{F}_{(\cdot)}^m = \mathcal{F}_{(\cdot)}^i = \mathcal{F}_{(\cdot)}^{D,G}$, the equilibrium is a NREE. ■

Proof of Remark 3. Fix ω^u and let $\omega^i \to 0$ and $\omega^n \to 1 - \omega^u = \omega$. Then,

$$\alpha^{si}(t) = \frac{\sigma^D}{H(t)}, \quad \beta^{si}(t) = -\omega\frac{\sigma^D}{H(t)}, \quad \kappa_t^{si} = 0, \quad \gamma^{si}(t) = -\omega\frac{\mu^D(T-t)}{H(t)}\sigma^D, \quad \lambda^{si}(t,s) = 0, \; s \in [t,T] \tag{44}$$

$$H(t) = \left(\sigma^D\right)^2(T-t) + \left(\sigma^\zeta\right)^2, \qquad M^{si}(t) = \omega^2\left(\sigma^\phi\right)^2 \tag{45}$$

$$\alpha^{D,si}(t) = \beta^{D,si}(t) = \gamma^{D,si}(t) = 0 \Longrightarrow \widetilde{\alpha}^{si}(t) = \widehat{\alpha}^{si}(t), \quad \widetilde{\beta}^{si}(t) = \widehat{\beta}^{si}(t), \quad \widetilde{\gamma}^{si}(t) = \widehat{\gamma}^{si}(t) \tag{46}$$

$$\widehat{\alpha}^{si}(t) = \alpha^{si}(t) + \alpha^{h,si}(t), \quad \widehat{\beta}^{si}(t) = \beta^{si}(t) + \beta^{h,si}(t), \quad \widehat{\gamma}^{si}(t) = \gamma^{si}(t) + \gamma^{h,si}(t) \tag{47}$$

$$\dot{\beta}^{h,si}(t) = \kappa_0^{si}(t) + \kappa_1^{si}(t)\beta^{h,si}(t) - \sigma^D\beta^{h,si}(t)^2; \quad \beta^{h,si}(T) = 0 \tag{48}$$

$$\alpha^{h,si}(t) = -\int_t^T \exp\left(-\int_t^s \kappa_2^{si}(v)\,dv\right)\kappa_3^{si}(s)\,ds, \qquad \gamma^{h,si}(t) = -\int_t^T \exp\left(-\int_t^s \kappa_4^{si}(v)\,dv\right)\kappa_5^{si}(s)\,ds \tag{49}$$

$$\kappa_0^{si}(t) = \left(\frac{1-\omega}{\omega}\right)\beta^{si}(t)^2\sigma^D, \quad \kappa_1^{si}(t) = -2\sigma^D\beta^{si}(t), \quad \kappa_2^{si}(t) = \left(1-\beta^{si}(t)-\widehat{\beta}^{si}(t)\right)\sigma^D + \left(\sigma^D\right)^2 K^{si}(t,T) \tag{50}$$

$$\kappa_3^{si}(t) = \alpha^{si}(t)\left(1+\beta^{si}(t)\left(\frac{1-\omega}{\omega}\right)-\widehat{\beta}^{si}(t)\right)\sigma^D + \left(\sigma^D\right)^2 K^{si}(t,T)\,\alpha^{si}(t) \tag{51}$$

$$\kappa_4^{si}(t) = -\beta^{si}(t)\,\sigma^D + \sigma^D K^{si}(t,T)$$

$$\kappa_5^{si}(t) = \beta^{si}(t)\left(\frac{1-\omega}{\omega}\gamma^{si}(t) + \frac{\sigma_t^{S,si}}{\Gamma}\right)\sigma^D + \sigma^D K^{si}(t,T)\left(\mu^D + \sigma^D\left(\gamma^{si}(t) - \frac{\sigma_t^{S,si}}{\Gamma}\right)\right) \tag{52}$$

where $K^{si}(t,T) \equiv \int_t^T \left(\beta^{h,si}(v)^2 + \frac{\kappa_0^{si}(v)}{\sigma^D}\right) b_1^{si}(t,v)^2\,dv$ with $b_1^{si}(t,v)$ as in Lemma 1, but with $\widetilde{\beta}^{si}(t)$ instead of $\widetilde{\beta}(t)$. The stock price becomes $S_t^{si} = \hat{A}^{si}(t)\,Z^{si} + \hat{B}^{si}(t)\,D_t + \hat{F}^{si}(t)$ where $Z^{si} = \omega\phi$ and,

$$\hat{B}^{si}(t) = B^{si}(t)\,B^{h,si}(t), \quad B^{si}(t) = \left(\frac{H(T)}{H(t)}\right)^{\omega}, \quad B^{h,si}(t) = e^{\sigma^D\int_t^T \beta^{h,si}(v)dv} \tag{53}$$

$$\hat{A}^{si}(t) = \sigma^D\left(\int_t^T \hat{B}^{si}(s)\,\hat{\alpha}^{si}(s)\,ds\right), \qquad \hat{I}^{si}(t) = 0 \tag{54}$$

$$\hat{F}^{si}(t) = \hat{B}^{si}(t)\,\mu^D(T-t) - \frac{\left(\sigma^D\right)^2}{\Gamma}\int_t^T \hat{B}^{si}(s)^2\,ds + \sigma^D\int_t^T \hat{B}^{si}(s)\,\hat{\gamma}^{si}(s)\,ds. \tag{55}$$

The pair $\left(S^{si}, D\right)$ in the limit economy is uninformative.

If in addition $\omega^n \to 0$ (i.e., $\omega^u \to 1$), then $M^{si}(t) = Z^{si} = 0$ and,

$$\alpha^{si}(t) = \frac{\sigma^D}{H(t)}, \quad \beta^{si}(t) = \gamma^{si}(t) = \kappa_t^{si} = \lambda^{si}(t,s) = 0, \quad \frac{\beta^{si}(t)}{\omega} = -\frac{\sigma^D}{H(t)}, \quad \frac{\gamma^{si}(t)}{\omega} = -\frac{\mu^D(T-t)}{H(t)}\sigma^D$$

$$\kappa_0^{si}(t) = \kappa_1^{si}(t) = 0, \quad \kappa_2^{si}(t) = \left(1-\widehat{\beta}^{si}(t)\right)\sigma^D + \left(\sigma^D\right)^2 K^{si}(t,T)$$

$$\kappa_3^{si}(t) = \alpha^{si}(t)\left(1+\frac{\beta^{si}(t)}{\omega}-\widehat{\beta}^{si}(t)\right)\sigma^D + \left(\sigma^D\right)^2 K^{si}(t,T)\,\alpha^{si}(t)$$

$$\kappa_4^{si}(t) = \sigma^D K^{si}(t,T), \quad \kappa_5^{si}(t) = \sigma^D K^{si}(t,T)\left(\mu^D - \sigma^D\frac{\sigma_t^{S,si}}{\Gamma}\right)$$

$$\alpha^{D,si}(t) = \beta^{D,si}(t) = \gamma^{D,si}(t) = 0 \Longrightarrow \widetilde{\alpha}^{si}(t) = \widehat{\alpha}^{si}(t), \quad \widetilde{\beta}^{si}(t) = \widehat{\beta}^{si}(t), \quad \widetilde{\gamma}^{si}(t) = \widehat{\gamma}^{si}(t)$$

$$\widehat{\alpha}^{si}(t) = \alpha^{si}(t) + \alpha^{h,si}(t), \quad \widehat{\beta}^{si}(t) = \beta^{h,si}(t), \quad \widehat{\gamma}^{si}(t) = \gamma^{h,si}(t)$$

$$\alpha^{h,si}(t) = -\int_t^T \exp\left(-\int_t^s \kappa_2^{si}(v)\,dv\right)\kappa_3^{si}(s)\,ds, \qquad \gamma^{h,si}(t) = -\int_t^T \exp\left(-\int_t^s \kappa_4^{si}(v)\,dv\right)\kappa_5^{si}(s)\,ds.$$

Thus, $\dot{\beta}^{h,si}(t) = -\sigma^D\beta^{h,si}(t)^2$, $\beta^{h,si}(T) = 0$, which has solution $\beta^{h,si} = 0$. It follows that $\widetilde{\beta}^{si}(t) = \widehat{\beta}^{si}(t) = \beta^{h,si}(t) = K^{si}(t,T) = \kappa_4^{si}(t) = \kappa_5^{si}(t) = \gamma^{h,si}(t) = 0$ and $\kappa_2^{si}(t) = \sigma^D$, $\kappa_3^{si}(t) = \alpha^{si}(t)\left(1+\frac{\beta^{si}(t)}{\omega}\right)\sigma^D$. Finally, $\hat{B}^{si}(t) = B^{si}(t) = B^{h,si}(t) = 1$ and $Z^{si} = 0$ so that $S_t^{si,0} = D_t + \hat{F}^{si,0}(t)$ with $\hat{F}^{si,0}(t) = \left(\mu^D - \frac{\left(\sigma^D\right)^2}{\Gamma}\right)(T-t)$ and $\sigma_t^{S,si,0} = \sigma^D$ for all $t \in [0,T]$.

Note that $0 < B(t) < B^{si}(t) < 1$ for $t < T$ and $B(T) = B^{si}(T) = 1$. Moreover,

$$\dot{\beta}^{h}(t) = \left(\frac{1-\omega}{\omega}\right)\beta(t)^2\sigma^D - 2\sigma^D\left(\beta(t)+\beta^D(t)\right)\beta^h(t) - \sigma^D\beta^h(t)^2; \qquad \beta^h(T) = 0$$

$$\dot{\beta}^{h,si}(t) = \left(\frac{1-\omega}{\omega}\right)\beta^{si}(t)^2\sigma^D - 2\sigma^D\beta^{si}(t)\,\beta^{h,si}(t) - \sigma^D\beta^{h,si}(t)^2; \qquad \beta^{h,si}(T) = 0$$

$$\beta(t) + \beta^D(t) = -\omega\frac{1-\kappa_t\omega^i}{H(t)}\sigma^D - \left(\omega^i\right)^2\frac{1}{M(t)}\sigma^D = -\omega\left(\frac{\left(\omega^n\right)^2\left(\sigma^\phi\right)^2}{H(t)}\right)\frac{\sigma^D}{M(t)} - \left(\omega^i\right)^2\frac{\sigma^D}{M(t)}$$

$$\beta^{si}(t) - \left(\beta(t) + \beta^{D}(t)\right) = -\omega\left(\omega^{i}\right)^{2}\frac{\sigma^{D}}{M(t)} + \left(\omega^{i}\right)^{2}\frac{\sigma^{D}}{M(t)} = \left(\omega^{i}\right)^{2}\frac{\sigma^{D}}{M(t)}(1-\omega) > 0$$

$$U \equiv \left(\frac{1-\omega}{\omega}\right)\beta(t)^{2} = \left(\frac{1-\omega}{\omega}\right)\left(-\omega\frac{1-\kappa_{t}\omega^{i}}{H(t)}\sigma^{D}\right)^{2} = \omega(1-\omega)\left(\frac{(\omega^{n})^{2}(\sigma^{\phi})^{2}}{M(t)}\right)^{2}\left(\frac{\sigma^{D}}{H(t)}\right)^{2}$$

$$U^{si} \equiv \left(\frac{1-\omega}{\omega}\right)\beta^{si}(t)^{2} = \left(\frac{1-\omega}{\omega}\right)\left(-\omega\frac{\sigma^{D}}{H(t)}\right)^{2} = \omega(1-\omega)\left(\frac{\sigma^{D}}{H(t)}\right)^{2}$$

so that $\Delta \equiv U^{si} - U > 0$. Thus, $\dot{\beta}^{h,si}(T) - \dot{\beta}^{h}(T) = \Delta > 0$. Moreover, if $\beta^{h,si}(t) = \beta^{h}(t)$ at any $t < T$, then

$$\dot{\beta}^{h,si}(t) - \dot{\beta}^{h}(t) = \Delta\sigma^{D} - 2\sigma^{D}\left(\beta^{si}(t) - \left(\beta(t) + \beta^{D}(t)\right)\right)\beta^{h,si}(t) > 0.$$

Thus, $\beta^{h,si}(t) < \beta^{h}(t) < 0$ for all $t \in [0,T]$ and $\beta^{h,si}(T) - \beta^{h}(T) = 0$. Therefore $B^{h,si}(t) < B^{h}(t)$. As $0 < B(t) < B^{si}(t) < 1$, the overall effect on price volatility is ambiguous: $\sigma_{t}^{S,si} \gtrless \sigma_{t}^{S}$ for $t < T$. However, $\max\left(\sigma_{t}^{S,si}, \sigma_{t}^{S}\right) < \sigma_{t}^{S,si,0} = \sigma^{D}$ for $t < T$ and $lim_{t\to T}\sigma_{t}^{S} = lim_{t\to T}\sigma_{t}^{S,si} = lim_{t\to T}\sigma_{t}^{S,si,0} = \sigma^{D}$. ■

Proof of Remark 4. Fix ω^{n} and let $\omega^{u} \to 0$ and $\omega^{i} \to 1 - \omega^{n}$. Then, $\omega \to 1$ and,

$$\alpha^{su}(t) = \frac{1-\kappa_{t}^{su}}{H(t)}\sigma^{D}, \qquad \beta^{su}(t) = -\frac{1-\kappa_{t}^{su}\omega^{i}}{H(t)}\sigma^{D}, \qquad \kappa_{t}^{su} = \frac{\omega^{i}H(t)}{M^{su}(t)} \tag{56}$$

$$\gamma^{su}(t) = -\frac{\left(1-\kappa_{t}^{su}\omega^{i}\right)\mu^{D}(T-t) - \omega^{n}\kappa_{t}^{su}\mu^{\phi}}{H(t)}\sigma^{D}, \qquad \lambda^{su}(t,s) = \frac{\omega^{i}\left(\sigma^{D}\right)^{2}(s-t)}{M^{su}(t)}, \; s \in [t,T] \tag{57}$$

$$H(t) = \left(\sigma^{D}\right)^{2}(T-t) + \left(\sigma^{\zeta}\right)^{2}, \qquad M^{su}(t) = \left(\omega^{i}\right)^{2}H(t) + (\omega^{n})^{2}\left(\sigma^{\phi}\right)^{2} \tag{58}$$

$$\widetilde{\alpha}^{su}(t) = \widehat{\alpha}^{su}(t) + \alpha^{D,su}(t), \qquad \widetilde{\beta}^{su}(t) = \widehat{\beta}^{su}(t) + \beta^{D,su}(t), \qquad \widetilde{\gamma}^{su}(t) = \widehat{\gamma}^{su}(t) + \gamma^{D,su}(t)$$

$$\widehat{\alpha}^{su}(t) = \alpha^{su}(t) + \alpha^{h,su}(t), \qquad \widehat{\beta}^{su}(t) = \beta^{su}(t) + \beta^{h,su}(t), \qquad \widehat{\gamma}^{su}(t) = \gamma^{su}(t) + \gamma^{h,su}(t)$$

$$\alpha^{D,su}(t) \equiv \frac{\omega^{i}\sigma^{D}}{M^{su}(t)}, \qquad \beta^{D,su}(t) \equiv -\omega^{i}\alpha^{D,su}(t), \qquad \gamma^{D,su}(t) \equiv -\alpha^{D,su}(t)\left(\omega^{i}\mu^{D}(T-t) + \omega^{n}\mu^{\phi}\right) \tag{59}$$

$$\dot{\beta}^{h,su}(t) = \kappa_{1}^{su}(t)\beta^{h,su}(t) - \sigma^{D}\beta^{h,su}(t)^{2}; \qquad \beta^{h,su}(T) = 0 \tag{60}$$

$$\alpha^{h,su}(t) = -\int_{t}^{T}\exp\left(-\int_{t}^{s}\kappa_{2}^{su}(v)\,dv\right)\kappa_{3}^{su}(s)\,ds, \qquad \gamma^{h,su}(t) = -\int_{t}^{T}\exp\left(-\int_{t}^{s}\kappa_{4}^{su}(v)\,dv\right)\kappa_{5}^{su}(s)\,ds \tag{61}$$

$$\kappa_{0}^{su}(t) = 0, \qquad \kappa_{1}^{su}(t) = -2\sigma^{D}\left(\beta^{su}(t) + \beta^{D,su}(t)\right), \qquad \kappa_{2}^{su}(t) = \left(1 - \beta^{su}(t) - \widetilde{\beta}^{su}(t)\right)\sigma^{D} + \left(\sigma^{D}\right)^{2}K^{su}(t,T)$$

$$\kappa_{3}^{su}(t) = \alpha^{su}(t)\left(1 - \widehat{\beta}^{su}(t)\right)\sigma^{D} + \left(\sigma^{D}\right)^{2}K^{su}(t,T)\left(\alpha^{su}(t) + \alpha^{D,su}(t)\right)$$

$$\kappa_{4}^{su}(t) = -\left(\beta^{su}(t) + \beta^{D,su}(t)\right)\sigma^{D} + \sigma^{D}K^{su}(t,T)$$

$$\kappa_{5}^{su}(t) = \beta^{su}(t)\frac{\sigma_{t}^{S,su}}{\Gamma}\sigma^{D} + \sigma^{D}K^{su}(t,T)\left(\mu^{D} + \sigma^{D}\left(\gamma^{su}(t) + \gamma^{D,su}(t) - \frac{\sigma_{t}^{S,su}}{\Gamma}\right)\right)$$

where $K^{su}(t,T) \equiv \int_{t}^{T}\beta^{h,su}(v)^{2}b_{1}^{su}(t,v)^{2}\,dv$ with $b_{1}^{su}(t,v)$ as in Lemma 1, but with $\widetilde{\beta}^{su}(t)$ instead of $\widetilde{\beta}(t)$. As the solution of (60) is $\beta^{h,su}(t) = 0$, it follows that $K^{su}(t,T) = 0$ and,

$$\kappa_{0}^{su}(t) = 0, \qquad \kappa_{1}^{su}(t) = -2\sigma^{D}\left(\beta^{su}(t) + \beta^{D,su}(t)\right), \qquad \kappa_{2}^{su}(t) = \left(1 - \beta^{su}(t) - \widetilde{\beta}^{su}(t)\right)\sigma^{D} \tag{62}$$

$$\kappa_{3}^{su}(t) = \alpha^{su}(t)\left(1 - \widehat{\beta}^{su}(t)\right)\sigma^{D}, \qquad \kappa_{4}^{su}(t) = -\left(\beta^{su}(t) + \beta^{D,su}(t)\right)\sigma^{D}, \qquad \kappa_{5}^{su}(t) = \beta^{su}(t)\frac{\sigma_{t}^{S,su}}{\Gamma}\sigma^{D} \tag{63}$$

$$\widetilde{\alpha}^{su}(t) = \widehat{\alpha}^{su}(t) + \alpha^{D,su}(t), \qquad \widetilde{\beta}^{su}(t) = \widehat{\beta}^{su}(t) + \beta^{D,su}(t), \qquad \widetilde{\gamma}^{su}(t) = \widehat{\gamma}^{su}(t) + \gamma^{D,su}(t) \tag{64}$$

$$\widehat{\alpha}^{su}(t) = \alpha^{su}(t) + \alpha^{h,su}(t), \qquad \widehat{\beta}^{su}(t) = \beta^{su}(t), \qquad \widehat{\gamma}^{su}(t) = \gamma^{su}(t) + \gamma^{h,su}(t) \tag{65}$$

$$\hat{B}^{su}(t) = B^{su}(t), \qquad B^{su}(t) = \frac{H(T)}{H(t)}, \qquad B^{h,su}(t) = 1 \tag{66}$$

$$\hat{A}^{su}(t) = \lambda^{su}(t,T) + \sigma^D \left(\int_t^T B^{su}(s) \left(\hat{\alpha}^{su}(s) + \hat{\beta}^{su}(s) \lambda^{su}(t,s) \right) ds \right) \tag{67}$$

$$\hat{F}^{su}(t) = B^{su}(t) \mu^D (T-t) - \frac{\left(\sigma^D\right)^2}{\Gamma} \int_t^T B^{su}(s)^2 ds + \sigma^D \int_t^T B^{su}(s) \hat{\gamma}^{su}(s) ds - \omega^n \hat{I}^{su}(t) \mu^{\phi} \tag{68}$$

$$\hat{I}^{su}(t) = \lambda^{su}(t,T) + \sigma^D \int_t^T B^{su}(s) \hat{\beta}^{su}(s) \lambda^{su}(t,s) ds. \tag{69}$$

If, in addition, $\omega^n \to 0$, then $\omega^i \to 1$, $M^{su}(t) = H(t)$, $\kappa_t^{su} = 1$ and,

$$\alpha^{su}(t) = \beta^{su}(t) = \gamma^{su}(t) = 0, \qquad \lambda^{su}(t,s) = \frac{\left(\sigma^D\right)^2 (s-t)}{H(t)}, \ s \in [t,T] \tag{70}$$

$$\kappa_0^{su}(t) = \kappa_3^{su}(t) = \kappa_5^{su}(t) = 0, \ \ \kappa_1^{su}(t) = -2\sigma^D \beta^{D,su}(t), \ \ \kappa_2^{su}(t) = \left(1 - \widetilde{\beta}^{su}(t)\right) \sigma^D, \ \ \kappa_4^{su}(t) = -\beta^{D,su}(t) \sigma^D \tag{71}$$

$$\alpha^{h,su}(t) = \gamma^{h,su}(t) = 0, \qquad \widehat{\alpha}^{su}(t) = \widehat{\beta}^{su}(t) = \widehat{\gamma}^{su}(t) = 0 \tag{72}$$

$$\widetilde{\alpha}^{su}(t) = \alpha^{D,su}(t), \qquad \widetilde{\beta}^{su}(t) = \beta^{D,su}(t), \qquad \widetilde{\gamma}^{su}(t) = \gamma^{D,su}(t) \tag{73}$$

$$\hat{B}^{su}(t) = B^{su}(t), \quad B^{su}(t) = \frac{H(T)}{H(t)}, \quad B^{h,su}(t) = 1, \quad \hat{A}^{su}(t) = \hat{I}^{su}(t) = \lambda^{su}(t,T) \tag{74}$$

$$\hat{F}^{su}(t) = B^{su}(t) \mu^D (T-t) - \frac{\left(\sigma^D\right)^2}{\Gamma} \int_t^T B^{su}(s)^2 ds + \sigma^D \int_t^T B^{su}(s) \hat{\gamma}^{su}(s) ds. \tag{75}$$

As $Z^{su,0} = G$, the pair $\left(D, S^{su,0}\right)$ is fully revealing. Price volatilities in the different economies are $\sigma_t^{S,su,0} = \sigma_t^{S,su} = \hat{B}^{su}(t) \sigma^D = B^{su}(t) \sigma^D < B(t) \sigma^D = \sigma_t^S < \sigma^D$ for $t < T$. As $t \to T$, $\sigma_t^{S,su,0} = \sigma_t^{S,su} \to \sigma^D$ and $\sigma_t^S \to \sigma^D$. ■

Proof of Remark 5. Fix ω^u and let $\omega^i \to 1 - \omega^u$ and $\omega^n \to 0$. Then, $\omega \to \omega^i \to 1 - \omega^u$ and,

$$M^{sn}(t) = \left(\omega^i\right)^2 H(t), \quad \kappa_t^{sn} = \frac{\omega^i H(t)}{M^{sn}(t)}, \quad \kappa_t^{sn} \omega^i = 1 \tag{76}$$

$$\alpha^{sn}(t) = \beta^{sn}(t) = \gamma^{sn}(t) = 0, \quad \lambda^{sn}(t,s) = \frac{\omega^i \left(\sigma^D\right)^2 (s-t)}{M^{sn}(t)} \tag{77}$$

$$\widetilde{\alpha}^{sn}(t) = \widehat{\alpha}^{sn}(t) + \alpha^{D,sn}(t), \quad \widetilde{\beta}^{sn}(t) = \widehat{\beta}^{sn}(t) + \beta^{D,sn}(t), \quad \widetilde{\gamma}^{sn}(t) = \widehat{\gamma}^{sn}(t) + \gamma^{D,sn}(t) \tag{78}$$

$$\widehat{\alpha}^{sn}(t) = \alpha^{h,sn}(t), \qquad \widehat{\beta}^{sn}(t) = \beta^{h,sn}(t), \qquad \widehat{\gamma}^{sn}(t) = \gamma^{h,sn}(t) \tag{79}$$

$$\alpha^{D,sn}(t) \equiv \frac{\omega^i \sigma^D}{M^{sn}(t)}, \quad \beta^{D,sn}(t) \equiv -\omega^i \alpha^{D,sn}(t), \quad \gamma^{D,sn}(t) \equiv -\alpha^{D,sn}(t) \omega^i \mu^D (T-t) \tag{80}$$

$$\dot{\beta}^{h,sn}(t) = \kappa_1^{sn}(t) \beta^{h,sn}(t) - \sigma^D \beta^{h,sn}(t)^2; \quad \beta^{h,sn}(T) = 0 \tag{81}$$

$$\alpha^{h,sn}(t) = -\int_t^T \exp\left(-\int_t^s \kappa_2^{sn}(v) dv\right) \kappa_3^{sn}(s) ds, \qquad \gamma^{h,sn}(t) = -\int_t^T \exp\left(-\int_t^s \kappa_4^{sn}(v) dv\right) \kappa_5^{sn}(s) ds \tag{82}$$

$$\kappa_0^{sn}(t) = 0, \qquad \kappa_1^{sn}(t) \equiv -2\sigma^D \beta^{D,sn}(t), \qquad \kappa_2^{sn}(t) \equiv \left(1 - \widetilde{\beta}^{sn}(t)\right) \sigma^D + \left(\sigma^D\right)^2 K^{sn}(t,T)$$

$$\kappa_3^{sn}(t) \equiv \left(\sigma^D\right)^2 K^{sn}(t,T) \alpha^D(t), \qquad \kappa_4^{sn}(t) \equiv -\beta^{D,sn}(t) \sigma^D + \sigma^D K^{sn}(t,T)$$

$$\kappa_5^{sn}(t) \equiv \sigma^D K^{sn}(t,T) \left(\mu^D + \sigma^D \left(\gamma(t) + \gamma^D(t) - \frac{\sigma_t^{S,sn}}{\Gamma}\right)\right)$$

where $K^{sn}(t,T) \equiv \int_t^T \beta^{h,sn}(v)^2 b_1^{sn}(t,v)^2 dv$ with $b_1^{sn}(t,v)$ as in Lemma 1, but with $\widetilde{\beta}^{sn}(t)$ instead of $\widetilde{\beta}(t)$. As the solution of (81) is $\beta^{h,sn}(t) = 0$, it follows that $\widehat{\beta}^{sn}(t) = 0$, $\widetilde{\beta}^{sn}(t) = \beta^{D,sn}(t)$ and $K^{sn}(t,T) = 0$. Then,

$$\kappa_0^{sn}(t) = \kappa_3^{sn}(t) = \kappa_5^{sn}(t) = 0, \quad \kappa_1^{sn}(t) = -2\sigma^D \beta^{D,sn}(t), \quad \kappa_2^{sn}(t) = \left(1 - \beta^{D,sn}(t)\right) \sigma^D, \quad \kappa_4^{sn}(t) = -\beta^{D,sn}(t) \sigma^D \tag{83}$$

$$\alpha^{h,sn}(t) = \gamma^{h,sn}(t) = 0, \qquad \widehat{\alpha}^{sn}(t) = \widehat{\beta}^{sn}(t) = \widehat{\gamma}^{sn}(t) = 0 \tag{84}$$

$$\hat{B}^{sn}(t) = B^{sn}(t), \quad B^{sn}(t) = \frac{H(T)}{H(t)}, \quad B^{h,sn}(t) = 1 \tag{85}$$

$$\hat{A}^{sn}(t) = \hat{I}^{sn}(t) = \lambda^{sn}(t,T), \quad \hat{F}^{sn}(t) = B^{sn}(t)\mu^D(T-t) - \frac{(\sigma^D)^2}{\Gamma}\int_t^T B^{sn}(s)^2 ds. \tag{86}$$

The pair (D, S^{sn}), in the limit economy, is fully revealing because $Z = \omega^i G$. The stock price volatility is $\sigma_t^{S,sn} = \hat{B}^{sn}(t)\sigma^D = B^{sn}(t)\sigma^D$ where $B^{sn}(t) < \hat{B}(t)$. Volatilities rank as $\sigma_t^{S,sn} < \sigma_t^S < \sigma_t^D$ for $t < T$. ■

The proofs of Corollaries 1-3 follows from Lemma 11, given next. The proofs for Lemma 11 are straightforward, but long and tedious. They are in a companion Technical Appendix.

Lemma 11 *The following holds,*

$$\frac{\partial H(t)}{\partial t} = -\left(\sigma^D\right)^2 < 0, \quad \frac{\partial M(t)}{\partial t} = \left(\omega^i\right)^2 \frac{\partial H(t)}{\partial t} < 0, \quad \frac{\partial \kappa_t}{\partial t} = \frac{\omega^i (\omega^n)^2 (\sigma^\phi)^2}{M(t)^2}\frac{\partial H(t)}{\partial t} < 0$$

$$\frac{\partial \lambda(t,s)}{\partial t} = -\omega^i\left(\sigma^D\right)^2 \frac{M(s)}{M(t)^2} < 0, \quad \frac{\partial B(t)}{\partial t} = -B(t)\left(\frac{\omega}{H(t)} + \frac{(1-\omega)(\omega^i)^2}{M(t)}\right)\frac{\partial H(t)}{\partial t} > 0,$$

$$\frac{\partial \alpha(t)}{\partial t} = -\frac{(\omega^n)^2(\sigma^\phi)^2\omega\kappa_t + (1-\kappa_t\omega)M(t)}{M(t)H(t)^2}\frac{\partial H(t)}{\partial t}\sigma^D \gtrless 0 \Longleftrightarrow \kappa_t^2 \lessgtr \frac{1}{\omega^i\omega}$$

$$\frac{\partial \beta(t)}{\partial t} = \omega\frac{(\omega^n)^4(\sigma^\phi)^4 + 2(\omega^i)^2 H(t)(\omega^n)^2(\sigma^\phi)^2}{M(t)^2 H(t)^2}\frac{\partial H(t)}{\partial t}\sigma^D < 0$$

$$\frac{\partial \gamma(t)}{\partial t} = \frac{\omega\sigma^D}{H(t)^2}\left(\frac{\partial \kappa_t}{\partial t}\left(\omega^i\mu^D(T-t) - \omega^n\mu^\phi\right)H(t) - \omega^n\kappa_t\mu^\phi\left(\sigma^D\right)^2 + \left(1-\kappa_t\omega^i\right)\mu^D\left(\sigma^\zeta\right)^2\right)$$

$$\begin{cases} \partial\gamma(t)/\partial t > 0 \Longleftrightarrow 0 \leqslant H(t) < H(t)^+ \\ \partial\gamma(t)/\partial t < 0 \Longleftrightarrow H^+ < H(t) \end{cases}, \quad H^+ = \frac{-b+\sqrt{b^2-4ac}}{2a} \tag{87}$$

$$a = s^2\left(s\left(\sigma^D\right)^2\mu^\phi + \left(\sigma^\phi\right)^2\mu^D\right), \quad b = -2s^2\left(\sigma^\phi\right)^2\mu^D\left(\sigma^\zeta\right)^2, \quad c = -\left(\sigma^\phi\right)^2\left(\sigma^\phi\right)^2\mu^D\left(\sigma^\zeta\right)^2, \quad s = \frac{\omega^i}{\omega^n} \tag{88}$$

Proof of Proposition 6. Marginal utility is $\exp(-X_T^u/\Gamma) = y^u\xi_T^m$ where the shadow price of initial wealth y^u satisfies $X_0^u = -\Gamma\log(y^u) - \Gamma E_0[\xi_T^m \log\xi_T^m]$, and therefore, $y^u = \exp(-X_0^u/\Gamma - E_0[\xi_T^m\log\xi_T^m])$. Interim expected utility of the public investor is $\mathcal{U}^u \equiv E_0[u(X_T^u)] = -\Gamma\exp\left(-\frac{X_0^u}{\Gamma} - E_0[\xi_T^m\log\xi_T^m]\right)$. In the presence of private information, $\theta_v^m = \sigma_v^S/\Gamma - \vartheta_v(Z|D_v)$. It follows that $-E_0[\xi_T^m\log\xi_T^m] = \frac{1}{2\Gamma^2}\int_0^T\left(\sigma_v^S\right)^2 dv - \frac{1}{\Gamma}\int_0^T \sigma_v^S\vartheta_v(Z|E_0[\xi_v^m D_v]) + \mathcal{I}^u$. Initial wealth is $X_0^u = N_0^u S_0 = N_0^u\left(\widehat{A}(0)Z + \widehat{B}(0)D_0 + \widehat{F}(0)\right)$. In the absence of private information, $\theta_v^{m,ni} = \sigma^D/\Gamma$ such that $-E[\xi_T^m\log\xi_T^m|\mathcal{F}_0^m] = -\frac{1}{2\Gamma^2}\left(\sigma^D\right)^2 T$. The corresponding expression for the insider is $E\left[u\left(X_T^i\right)\middle|\mathcal{F}_0^i\right] = -\Gamma$ $\exp\left(-\frac{X_0^i}{\Gamma} - E\left[\xi_T^G\log\xi_T^G\middle|\mathcal{F}_0^i\right]\right)$. As $E\left[\xi_T^G\log\xi_T^G\middle|\mathcal{G}_0\right] = E_0\left[\xi_{t,T}^m\log\left(\xi_T^m\eta_T^x\right)\right]_{|x=G} = E_0[\xi_T^m\log\xi_T^m] + E_0\left[\xi_T^m\log\eta_T^x\right]_{|x=G}$ and $N_0^u = N_0^j = 1$, we have $X_0^i = N_0^i S_0 = N_0^u S_0 = X_0^u$ and $E\left[u\left(X_T^i\right)\middle|\mathcal{F}_0^i\right] = \mathcal{U}^u\exp\left(-\mathcal{I}^i(G|Z)\right)$. The informed is always better off, i.e. $\mathcal{U}^u < 0$ and $\mathcal{I}^i(G|Z) \geqslant 0$. This inequality follows as by optimality of the informed investor's choice and $\mathcal{F}_{(\cdot)}^m \subset \mathcal{F}_{(\cdot)}^i$, necessarily, $\mathcal{U}^i \geqslant \mathcal{U}^u\left(\exp\left(-\mathcal{I}^i(G|Z)\right) - 1\right)$. This establishes the results announced. ■

Proof of Proposition 7. Proposition 6 shows that $\mathcal{U}^u = \mathcal{U}^{u,ni}\exp\left(-\left(\frac{\Delta\widehat{P}^u(Z)}{\Gamma} + \Delta\widehat{T}^u(Z)\right)\right)$ with

$$\Delta\widehat{P}^u(Z) \equiv S_0 - S_0^{ni} = \left(\widehat{B}(0) - 1\right)\left(D_0 + \mu^D T\right) + \widehat{A}(0)Z - \frac{\Delta\widehat{V}}{\Gamma} + \sigma^D\int_0^T \widehat{B}(s)\widehat{\gamma}(s)ds - \omega^n\widehat{I}(0)\mu^\phi$$

$$\Delta\widehat{T}^u(Z) = \frac{\Delta\widehat{V}}{2\Gamma^2} - \frac{1}{\Gamma}\int_0^T \sigma_t^S\vartheta(Z|E_0[\xi_t^m D_t])dt + \mathcal{I}^u(Z) \text{ and } \Delta\widehat{V} \equiv \left(\sigma^D\right)^2\int_0^T\left(\widehat{B}(v)^2 - 1\right)dv.$$

Hence,

$$\Delta\mathcal{U} \equiv \mathcal{U}^u - \mathcal{U}^{u,ni} = \left(\exp\left(-\left(\frac{\Delta\widehat{P}^u(Z)}{\Gamma} + \Delta\widehat{T}^u(Z)\right)\right) - 1\right)\mathcal{U}^{u,ni}$$

and, as $\mathcal{U}^{u,ni} \leqslant 0$, $\Delta\mathcal{U} \geqslant 0$ if and only if $\Delta\widehat{P}^u(Z)/\Gamma + \Delta\widehat{T}^u(Z) \geqslant 0$.

To study the limit behavior, note that $\lim_{\Gamma\to\infty}\widehat{\gamma}(s) = \gamma(t) + \gamma^h(t;\infty) \equiv \widehat{\gamma}(t;\infty)$ where

$$\gamma^h(s;\infty) \equiv -\int_s^T \exp\left(-\int_s^v \kappa_4(u)du\right)\kappa_5(v;\infty)dv$$

$$\kappa_5(t;\infty) \equiv \left(\beta(t)\left(\frac{1-\omega}{\omega}\right)\gamma(t) + \int_t^T \left(\beta^h(v)^2 + \frac{\kappa_0(v)}{\sigma^D}\right) b_1(t,v)^2\, dv \left(\mu^D + \sigma^D\left(\gamma(t) + \gamma^D(t)\right)\right)\right)\sigma^D.$$

Moreover, as $\gamma^h(s) \leqslant \gamma^h(s;\infty)$ uniformly in Γ for all $s \in [0,T]$, Lebesgue's dominated convergence theorem gives,

$$\lim_{\Gamma\to\infty} \Delta\widehat{P}^u(Z) = \left(\widehat{B}(0) - 1\right)\left(D_0 + \mu^D T\right) + \widehat{A}(0)\, Z + \sigma^D \int_0^T \widehat{B}(s)\,\widehat{\gamma}(s;+\infty)\, ds - \omega^n \widehat{I}(0)\,\mu^\phi.$$

Hence, $\lim_{\Gamma\to\infty}\left(-\frac{\Delta\widehat{P}^u(z)}{\Gamma}\right) = 0$. As $\theta_t^m = \sigma_t^S/\Gamma - \vartheta_t(Z|D_t)$,

$$\mathcal{I}^u(Z) = \frac{1}{2}E\left[\xi_T^m \int_0^T \vartheta_t(Z|D_t)^2\, dt \middle| \mathcal{F}_0^u\right] = \frac{1}{2}E\left[\xi_T^m \int_0^T \left(\theta_t^m - \frac{\sigma_t^S}{\Gamma}\right)^2 dt \middle| \mathcal{F}_0^u\right]$$

$$\lim_{\Gamma\to\infty}\left(-\Delta\widehat{T}^u(Z)\right) = \lim_{\Gamma\to\infty}\left(-\mathcal{I}^u(Z)\right) = \lim_{\Gamma\to\infty}\left(-\frac{1}{2}E\left[\xi_T^m \int_0^T (\theta_t^m)^2\, dt \middle| \mathcal{F}_0^u\right]\right) = E_0\left[\lim_{\Gamma\to\infty}\left(-\xi_T^m \int_0^T (\theta_v^m)^2\, dv\right)\right] \leqslant 0.$$

The third equality above follows from the Lebesgue dominated convergence theorem and $-\xi_T^m \int_0^T (\theta_v^m)^2\, dv \leqslant 0$ uniformly in Γ. But $\lim_{\Gamma\to\infty} \theta_v^m = -\widehat{\alpha}(v)\, Z - \widehat{\beta}(v)\, D_v - \gamma(v) - \gamma^h(v;+\infty) \equiv \overline{\theta}_v^m$ and $\lim_{\Gamma\to\infty} \xi_v^m = \mathcal{E}\left(-\int_0^\cdot \overline{\theta}_s^m\, dW_s^m\right)_v \equiv \overline{\xi}_v^m$ (indeed, as θ_t^m has finite variance uniformly in Γ for all $t \in [0,T]$, the sequence θ_t^m indexed by Γ is tight, so that $\lim_{\Gamma\to\infty} \int_0^T \theta_t^m dW_t^m = \int_0^T \overline{\theta}_t dW_t^m$ when $\lim_{\Gamma\to\infty} \theta_t^m = \overline{\theta}_t^m$). Thus, $\lim_{\Gamma\to\infty} \xi_T^m \int_0^T (\theta_v^m)^2\, dv = \overline{\xi}_T^m \int_0^T \left(\overline{\theta}_v^m\right)^2 dv$ and $\lim_{\Gamma\to\infty} -\left(\Delta\widehat{T}^u(Z) + \Delta\widehat{P}(Z)/\Gamma\right) \leqslant 0$. Because $\lim_{\Gamma\to\infty} \mathcal{U}^{u,ni} = -\infty$, it follows that $\lim_{\Gamma\to\infty} \Delta\mathcal{U} = +\infty$.

For small risk tolerance,

$$d_{\gamma^h}(v) \equiv \lim_{\Gamma\to 0} \Gamma\widetilde{\gamma}(v) = \lim_{\Gamma\to 0} \Gamma\widehat{\gamma}(v) = \lim_{\Gamma\to 0} \Gamma\gamma^h(v) = -\int_v^T \exp\left(-\int_v^s \kappa_4(u)\, du\right) d_{\kappa_5}(s)\, ds$$

$$d_{\kappa_5}(s) \equiv \lim_{\Gamma\to 0} \Gamma\kappa_5(s) = \left(\beta(s) - \sigma^D \int_s^T \left(\beta^h(v)^2 + \frac{\kappa_0(v)}{\sigma^D}\right) b_1(t,v)^2\, dv\right)\sigma_s^S \sigma^D < 0.$$

The price effect is,

$$\Gamma\Delta P(Z) = \Gamma\left(\widehat{B}(0) - 1\right)\left(D_0 + \mu^D T\right) + \Gamma\widehat{A}(0)\, Z - \Delta\widehat{V} + \Gamma\sigma^D \int_0^T \widehat{B}(s)\,\widehat{\gamma}(s)\, ds - \Gamma\omega^n \widehat{I}(0)\,\mu^\phi$$

$$\lim_{\Gamma\to 0} \Gamma\Delta P(Z) = -\Delta\widehat{V} + \sigma^D \int_0^T \widehat{B}(v)\, d_{\gamma^h}(v;0)\, dv \equiv -\left(\Delta\widehat{V} + \Delta\widehat{H}^S\right)$$

where $d_{\gamma^h}(v;0) = \lim_{\Gamma\to 0} \Gamma\widehat{\gamma}(s) = \lim_{\Gamma\to 0} \Gamma\gamma^h(s) > 0$ and

$$\Delta\widehat{H}^S \equiv -\sigma^D \int_0^T \widehat{B}(v)\, d_{\gamma^h}(v;0)\, dv < 0 \tag{89}$$

measures the impact of hedging on the price effect. To find the trading impact for small risk tolerance, note that

$$\Delta\widehat{T}^u(Z) = \frac{\Delta\widehat{V}}{2\Gamma^2} - \frac{1}{\Gamma}\int_0^T \sigma_t^S \vartheta\left(Z|E_0\left[\xi_t^m D_t\right]\right) dt + \mathcal{I}^u(Z) = -\left(H_{0,T}^m - \frac{\left(\sigma^D\right)^2 T}{2\Gamma^2}\right) \equiv -\left(H_{0,T}^m - H_{0,T}^{m,ni}\right)$$

where $H^m_{0,T} \equiv -\frac{1}{2}\int_0^T \widetilde{E}^z\left[\theta^m_v(z)^2\right]dv$ from Lemma 4 . Using (29)-(30),

$$\begin{aligned}
\widetilde{E}^z\left[\theta^m_v(z)^2\right] &= \tilde{E}^z\left[\left(\frac{\sigma^S_v}{\Gamma} - \widehat{\gamma}(v) - \widehat{\alpha}(v)z - \widehat{\beta}(v)D_v\right)^2\right] \\
&= \left(\frac{\sigma^S_v}{\Gamma} - \widehat{\gamma}(v) - \widehat{\alpha}(v)z\right)^2 - 2\left(\frac{\sigma^S_v}{\Gamma} - \widehat{\gamma}(v) - \widehat{\alpha}(v)z\right)\widehat{\beta}(v)\widetilde{E}^z[D_v] + \widehat{\beta}(v)^2\widetilde{E}^z\left[D_v^2\right] \\
&= \left(\frac{\sigma^S_v}{\Gamma} - \widehat{\gamma}(v)\right)^2 - 2\left(\frac{\sigma^S_v}{\Gamma} - \widehat{\gamma}(v)\right)\widehat{\alpha}(v)z + \widehat{\alpha}(v)^2 z^2 \\
&\quad -2\left(\frac{\sigma^S_v}{\Gamma} - \widehat{\gamma}(v) - \widehat{\alpha}(v)z\right)\widehat{\beta}(v)\left(a_1(0,v)z + b_1(0,v)D_0 + g_1(0,v)\right) \\
&\quad +\widehat{\beta}(v)^2\left(a_2(0,v)z + b_2(0,v)D_0 + c_2(0,v)D_0 z + d_2(0,v)z^2 + b_1(0,v)^2 D_0^2 + g_2(0,v)\right) \\
&= \left(\frac{\sigma^S_v}{\Gamma} - \widehat{\gamma}(v)\right)^2 - 2\left(\frac{\sigma^S_v}{\Gamma} - \widehat{\gamma}(v)\right)\widehat{\beta}(v)g_1(0,v) + \widehat{\beta}(v)^2 g_2(0,v) \\
&\quad -\left[2\left(\frac{\sigma^S_v}{\Gamma} - \widehat{\gamma}(v)\right)\left(\widehat{\alpha}(v) + \widehat{\beta}(v)a_1(0,v)\right) - 2\widehat{\alpha}(v)\widehat{\beta}(v)g_1(0,v) - \widehat{\beta}(v)^2 a_2(0,v)\right]z \\
&\quad +\left(\widehat{\alpha}(v)^2 + 2\widehat{\alpha}(v)\widehat{\beta}(v)a_1(0,v) + \widehat{\beta}(v)^2 d_2(0,v)\right)z^2 \\
&\quad -\left[2\left(\frac{\sigma^S_v}{\Gamma} - \widehat{\gamma}(v)\right)\widehat{\beta}(v)b_1(0,v) - \widehat{\beta}(v)^2 b_2(0,v)\right]D_0 \\
&\quad +\left[2\widehat{\alpha}(v)\widehat{\beta}(v)b_1(0,v) + \widehat{\beta}(v)^2 c_2(0,v)\right]D_0 z + \widehat{\beta}(v)^2 b_1(0,v)^2 D_0^2.
\end{aligned}$$

Using $\Gamma k(s) \equiv \sigma^D\Gamma\overline{\gamma}(s) = \sigma^D\left(\Gamma\widetilde{\gamma}(s) - \sigma^S_s\right)$ and $\widetilde{\gamma}(s) = \gamma^D(s) + \widehat{\gamma}(s)$ gives,

$$\begin{aligned}
d_{\gamma^h}(s) &= \lim_{\Gamma\to 0}\Gamma\widehat{\gamma}(s) = \lim_{\Gamma\to 0}\Gamma\gamma^h(s), \qquad \lim_{\Gamma\to 0}\Gamma k(s) = \sigma^D\left(\lim_{\Gamma\to 0}\Gamma\widehat{\gamma}(s) - \sigma^S_s\right) = \sigma^D\left(d_{\gamma^h}(s) - \sigma^S_s\right) \\
d_{g_1}(s) &\equiv \lim_{\Gamma\to 0}\Gamma g_1(0,s) = -\sigma^D\int_0^s \exp\left(\sigma^D\int_v^s \widetilde{\beta}(u)\,du\right)\left(d_{\gamma^h}(v) + \sigma^S_v\right)dv \\
d_{g_2}(s) &\equiv \lim_{\Gamma\to 0}\Gamma^2 g_2(0,s) = 2\sigma^D\int_0^s b_1(v,s)\left(d_{\gamma^h}(v) - \sigma^S_v\right)d_{g_1}(v)\,dv \\
d_{b_2}(s) &\equiv \lim_{\Gamma\to 0}\Gamma b_2(0,s) = 2\sigma^D\int_0^s b_1(v,s)\left(d_{\gamma^h}(v) - \sigma^S_v\right)b_1(0,v)\,dv \\
d_{a_2}(s) &\equiv \lim_{\Gamma\to 0}\Gamma a_2(0,s) = 2\sigma^D\int_0^s b_1(v,s)^2\left(\left(d_{\gamma^h}(v) - \sigma^S_v\right)a_1(0,v) + \widetilde{\alpha}(v)d_{g_1}(v)\right)dv
\end{aligned}$$

it follows that,

$$\lim_{\Gamma\to 0}\Gamma^2\widetilde{E}^z\left[\theta^m_v(z)^2\right] = \left(\sigma^S_v - d_{\gamma^h}(v)\right)^2 - 2\left(\sigma^S_v - d_{\gamma^h}(v)\right)\widehat{\beta}(v)d_{g_1}(v) + \widehat{\beta}(v)^2 d_{g_2}(v)$$

$$\begin{aligned}
\lim_{\Gamma\to 0}\Gamma^2 H^m_{0,T} &= -\frac{1}{2}\int_0^T \lim_{\Gamma\to 0}\Gamma^2\widetilde{E}^z\left[\theta^m_v(z)^2\right]dv \\
&= -\frac{1}{2}\int_0^T\left(\left(\sigma^S_v\right)^2 - 2\sigma^S_v d_{\gamma^h}(v) + d_{\gamma^h}(v)^2 - 2\left(\sigma^S_v - d_{\gamma^h}(v)\right)d_{g_1}(v)\widehat{\beta}(v) + \widehat{\beta}(v)^2 d_{g_2}(v)\right)dv
\end{aligned}$$

$$\lim_{\Gamma\to 0}\Gamma^2\Delta\widehat{T}^u(Z) = -\lim_{\Gamma\to 0}\Gamma^2\left(H^m_{0,T} - H^{m,ni}_{0,T}\right) = \frac{1}{2}\left(\Delta\widehat{V} + \Delta\widehat{H}^T\right)$$

where $\Delta\widehat{H}^T$ measures the effect of hedging on the trading impact,

$$\Delta\widehat{H}^T \equiv \int_0^T\left(-2\sigma^S_v d_{\gamma^h}(v) + d_{\gamma^h}(v)^2 - 2\left(\sigma^S_v - d_{\gamma^h}(v)\right)d_{g_1}(v)\widehat{\beta}(v) + \widehat{\beta}(v)^2 d_{g_2}(v)\right)dv. \tag{90}$$

Thus,

$$\lim_{\Gamma\to 0}\Gamma^2\left(\frac{\Delta\widehat{P}(Z)}{\Gamma}+\Delta\widehat{T}^u(Z)\right)=-\left(\Delta\widehat{V}+\Delta\widehat{H}^S\right)+\frac{1}{2}\left(\Delta\widehat{V}+\Delta\widehat{H}^T\right)=-\frac{1}{2}\Delta\widehat{V}-\Delta\widehat{H}^S+\frac{1}{2}\Delta\widehat{H}^T\equiv-\frac{1}{2}\left(\Delta\widehat{V}+\Delta\widehat{H}\right)$$

$$\begin{aligned}-\frac{1}{2}\Delta\widehat{H} &\equiv -\Delta\widehat{H}^S+\frac{1}{2}\Delta\widehat{H}^T\equiv\sigma^D\int_0^T\widehat{B}(v)\,d_{\gamma^h}(v)\,dv\\ &+\frac{1}{2}\int_0^T\left(-2\sigma_v^S d_{\gamma^h}(v)+d_{\gamma^h}(v)^2-2\left(\sigma_v^S-d_{\gamma^h}(v)\right)d_{g_1}(v)\,\widehat{\beta}(v)+\widehat{\beta}(v)^2 d_{g_2}(v)\right)dv.\end{aligned}$$

Note that $\Delta\mathcal{U}=\left(\exp\left(-\left(\frac{\Delta\widehat{P}(Z)}{\Gamma}+\Delta\widehat{T}^u(Z)\right)\right)-1\right)\mathcal{U}^{u,ni}=\left(\exp\left(\frac{1}{2\Gamma^2}\left(\Delta\widehat{V}+\Delta\widehat{H}\right)\right)-1\right)\mathcal{U}^{u,ni}$ and

$$\mathcal{U}^{u,ni}\equiv-\Gamma\exp\left(-\frac{X_0^{u,ni}}{\Gamma}-\frac{\left(\sigma^D\right)^2T}{2\Gamma^2}\right)=-\Gamma\exp\left(-\frac{E\left[D_T\right]}{\Gamma}+\frac{1}{2}\frac{\left(\sigma^D\right)^2T}{\Gamma^2}\right)$$

because $\theta_v^{m,ni}=\sigma^D/\Gamma$, $N_0^{u,ni}=1$ and $X_0^{u,ni}=S_0^{ni}=E\left[D_T\right]-\left(\sigma^D\right)^2T/\Gamma$. It follows that $\lim_{\Gamma\to 0}\mathcal{U}^{u,ni}=-\infty$ and $\lim_{\Gamma\to 0}\Delta\mathcal{U}=-sgn\left(\Delta\widehat{V}+\Delta\widehat{H}\right)\times\infty$. ■

Proof of Corollary 4. As,

$$\begin{aligned}\kappa_5(t) &\equiv \left[\beta(t)\frac{1-\omega}{\omega}\gamma(t)+\int_t^T\left(\beta^h(v)^2+\frac{\kappa_0(v)}{\sigma^D}\right)b_1(t,v)^2\,dv\left(\mu^D+\sigma^D\left(\gamma(t)+\gamma^D(t)\right)\right)\right]\sigma^D\\ &+\frac{1}{\Gamma}\left[\beta(t)-\sigma^D\int_t^T\left(\beta^h(v)^2+\frac{\kappa_0(v)}{\sigma^D}\right)b_1(t,v)^2\,dv\right]\sigma_t^S\sigma^D\equiv\kappa_{5,0}(t)+\frac{1}{\Gamma}\kappa_{5,1}(t)\end{aligned}$$

we can write, using (39),

$$\gamma^h(t)=\gamma_0^h(t)+\gamma_1^h(t)\frac{1}{\Gamma},\qquad \gamma_i^h(t)=-\int_t^T\exp\left(-\int_t^s\kappa_4(v)\,dv\right)\kappa_{5,i}(s)\,ds, i=0,1.$$

Likewise, using $\overline{\gamma}(t)\equiv\widetilde{\gamma}(t)-\sigma_t^S/\Gamma=\left(\gamma(t)+\gamma^h(t)+\gamma^D(t)\right)-\sigma_t^S/\Gamma$ and

$$\begin{aligned}k(t) &\equiv \mu^D+\sigma^D\overline{\gamma}(t)=\mu^D+\sigma^D\left(\gamma(t)+\gamma^h(t)+\gamma^D(t)-\frac{\sigma_t^S}{\Gamma}\right)\\ &= \mu^D+\sigma^D\left(\gamma(t)+\gamma_0^h(t)+\gamma^D(t)+\frac{\gamma_1^h(t)-\sigma_t^S}{\Gamma}\right)\equiv k_0(t)+\frac{k_1(t)}{\Gamma}\end{aligned}$$

we have,

$$g_1(t,s)\equiv\int_t^s b_1(v,s)\,k(v)\,dv=\int_t^s b_1(v,s)\left(k_0(v)+\frac{k_1(v)}{\Gamma}\right)dv\equiv g_{1,0}(t,s)+\frac{g_{1,1}(t,s)}{\Gamma}$$

$$\begin{aligned}a_2(t,s) &\equiv 2\left(\int_t^s b_1(v,s)^2k(v)\,a_1(t,v)\,dv+\sigma^D\int_t^s b_1(v,s)^2\,\widetilde{\alpha}(v)\,g_1(t,v)\,dv\right)\\ &= 2\left(\int_t^s b_1(v,s)^2k_0(v)\,a_1(t,v)\,dv+\sigma^D\int_t^s b_1(v,s)^2\,\widetilde{\alpha}(v)\,g_{1,0}(t,v)\,dv\right)\\ &\quad+2\left(\int_t^s b_1(v,s)^2k_1(v)\,a_1(t,v)\,dv+\sigma^D\int_t^s b_1(v,s)^2\,\widetilde{\alpha}(v)\,g_{1,1}(t,v)\,dv\right)\frac{1}{\Gamma}\\ &\equiv a_{2,0}(t,s)+a_{2,1}(t,s)\left(\frac{1}{\Gamma}\right)\end{aligned}$$

$$\begin{aligned}b_2(t,s) &\equiv 2\int_t^s b_1(v,s)^2k(v)\,b_1(t,v)\,dv=2\int_t^s b_1(v,s)^2k_0(t)\,b_1(t,v)\,dv+2\left(\int_t^s b_1(v,s)^2k_1(t)\,b_1(t,v)\,dv\right)\frac{1}{\Gamma}\\ &\equiv b_{2,0}(t,s)+b_{2,1}(t,s)\left(\frac{1}{\Gamma}\right)\end{aligned}$$

$$c_2(t,s) \equiv 2\sigma^D \int_t^s b_1(v,s)^2 \tilde{\alpha}(v) b_1(t,v)\,dv, \qquad d_2(t,s) \equiv 2\sigma^D \int_t^s b_1(v,s)^2 \tilde{\alpha}(v) a_1(t,v)\,dv$$

$$\begin{aligned} g_2(t,s) &\equiv 2\int_t^s b_1(v,s)^2 \left(\frac{\left(\sigma^D\right)^2}{2} + k(v) g_1(t,v) \right) dv = 2\int_t^s b_1(v,s)^2 \left(\frac{\left(\sigma^D\right)^2}{2} + k_0(v) g_{1,0}(t,v) \right) dv \\ &\quad +2\left(\int_t^s b_1(v,s)^2 \left(k_1(v) g_{1,0}(t,v) + k_0(v) g_{1,1}(t,v)\right) dv \right) \frac{1}{\Gamma} + 2\left(\int_t^s b_1(v,s)^2 k_1(v) g_{1,1}(t,v)\,dv \right) \left(\frac{1}{\Gamma}\right)^2 \\ &\equiv g_{2,0}(t,s) + g_{2,1}(t,s)\left(\frac{1}{\Gamma}\right) + g_{2,2}(t,s)\left(\frac{1}{\Gamma}\right)^2. \end{aligned}$$

and

$$\begin{aligned} \widetilde{E}^z\left[\theta_v^m(z)^2\right] &= \left(\frac{\sigma_v^S}{\Gamma} - \widehat{\gamma}(v)\right)^2 - 2\left(\frac{\sigma_v^S}{\Gamma} - \widehat{\gamma}(v)\right)\widehat{\beta}(v) g_1(0,v) + \widehat{\beta}(v)^2 g_2(0,v) \\ &\quad - \left[2\left(\frac{\sigma_v^S}{\Gamma} - \widehat{\gamma}(v)\right)\left(\widehat{\alpha}(v) + \widehat{\beta}(v) a_1(0,v)\right) - 2\widehat{\alpha}(v)\widehat{\beta}(v) g_1(0,v) - \widehat{\beta}(v)^2 a_2(0,v)\right] z \\ &\quad + \left(\widehat{\alpha}(v)^2 + 2\widehat{\alpha}(v)\widehat{\beta}(v) a_1(0,v) + \widehat{\beta}(v)^2 d_2(0,v)\right) z^2 \\ &\quad - \left[2\left(\frac{\sigma_v^S}{\Gamma} - \widehat{\gamma}(v)\right)\widehat{\beta}(v) b_1(0,v) - \widehat{\beta}(v)^2 b_2(0,v)\right] D_0 \\ &\quad + \left[2\widehat{\alpha}(v)\widehat{\beta}(v) b_1(0,v) + \widehat{\beta}(v)^2 c_2(0,v)\right] D_0 z + \widehat{\beta}(v)^2 b_1(0,v)^2 D_0^2. \end{aligned}$$

Substituting the expressions for $\widehat{\gamma}(v) = \gamma(v) + \gamma^h(v)$, $g_1(0,v)$, $g_2(0,v)$, $a_2(0,v)$, $b_2(0,v)$,

$$\begin{aligned} \widetilde{E}^z\left[\theta_v^m(z)^2\right] &= \left(\frac{\sigma_v^S}{\Gamma} - \left(\gamma(v) + \gamma_0^h(v) + \frac{\gamma_1^h(v)}{\Gamma}\right)\right)^2 \\ &\quad -2\left(\frac{\sigma_v^S}{\Gamma} - \left(\gamma(v) + \gamma_0^h(v) + \frac{\gamma_1^h(v)}{\Gamma}\right)\right)\widehat{\beta}(v)\left(g_{1,0}(0,v) + \frac{g_{1,1}(0,v)}{\Gamma}\right) \\ &\quad +\widehat{\beta}(v)^2\left(g_{2,0}(0,v) + g_{2,1}(0,v)\left(\frac{1}{\Gamma}\right) + g_{2,2}(0,v)\left(\frac{1}{\Gamma}\right)^2\right) \\ &\quad -2\left[\left(\frac{\sigma_v^S}{\Gamma} - \left(\gamma(v) + \gamma_0^h(v) + \frac{\gamma_1^h(v)}{\Gamma}\right)\right)\left(\widehat{\alpha}(v) + \widehat{\beta}(v) a_1(0,v)\right) - \widehat{\alpha}(v)\widehat{\beta}(v)\left(g_{1,0}(0,v) + \frac{g_{1,1}(0,v)}{\Gamma}\right)\right] z \\ &\quad + \left[\widehat{\beta}(v)^2\left(a_{2,0}(t,s) + \frac{a_{2,1}(t,s)}{\Gamma}\right)\right] z + \left(\widehat{\alpha}(v)^2 + 2\widehat{\alpha}(v)\widehat{\beta}(v) a_1(0,v) + \widehat{\beta}(v)^2 d_2(0,v)\right) z^2 \\ &\quad - \left[2\left(\frac{\sigma_v^S}{\Gamma} - \left(\gamma(v) + \gamma_0^h(v) + \frac{\gamma_1^h(v)}{\Gamma}\right)\right)\widehat{\beta}(v) b_1(0,v) - \widehat{\beta}(v)^2\left(b_{2,0}(t,s) + \frac{b_{2,1}(t,s)}{\Gamma}\right)\right] D_0 \\ &\quad + \left[2\widehat{\alpha}(v)\widehat{\beta}(v) b_1(0,v) + \widehat{\beta}(v)^2 c_2(0,v)\right] D_0 z + \widehat{\beta}(v)^2 b_1(0,v)^2 D_0^2 = k_{0,v}^m + k_{1,v}^m \frac{1}{\Gamma} + k_{2,v}^m \frac{1}{\Gamma^2} \end{aligned}$$

where,

$$\begin{aligned} k_{0,v}^m &= \left(\gamma(v) + \gamma_0^h(v)\right)^2 + 2\left(\gamma(v) + \gamma_0^h(v)\right)\widehat{\beta}(v) g_{1,0}(0,v) + \widehat{\beta}(v)^2 g_{2,0}(0,v) \\ &\quad + \left[2\left(\left(\gamma(v) + \gamma_0^h(v)\right)\left(\widehat{\alpha}(v) + \widehat{\beta}(v) a_1(0,v)\right) + \widehat{\alpha}(v)\widehat{\beta}(v) g_{1,0}(0,v)\right) + \widehat{\beta}(v)^2 a_{2,0}(t,s)\right] z \\ &\quad + \left(\widehat{\alpha}(v)^2 + 2\widehat{\alpha}(v)\widehat{\beta}(v) a_1(0,v) + \widehat{\beta}(v)^2 d_2(0,v)\right) z^2 + \left[2\left(\gamma(v) + \gamma_0^h(v)\right)\widehat{\beta}(v) b_1(0,v) + \widehat{\beta}(v)^2 b_{2,0}(t,s)\right] D_0 \\ &\quad + \left[2\widehat{\alpha}(v)\widehat{\beta}(v) b_1(0,v) + \widehat{\beta}(v)^2 c_2(0,v)\right] D_0 z + \widehat{\beta}(v)^2 b_1(0,v)^2 D_0^2 \end{aligned} \tag{91}$$

$$\begin{aligned} k_{1,v}^m &= -2\left(\sigma_v^S - \gamma_1^h(v)\right)\left(\gamma(v) + \gamma_0^h(v)\right) - 2\left[\left(\sigma_v^S - \gamma_1^h(v)\right) g_{1,0}(0,v) - \left(\gamma(v) + \gamma_0^h(v)\right) g_{1,1}(0,v)\right]\widehat{\beta}(v) \\ &\quad +\widehat{\beta}(v)^2 g_{2,1}(0,v) - 2\left(\sigma_v^S - \gamma_1^h(v)\right)\left[\left(\widehat{\alpha}(v) + \widehat{\beta}(v) a_1(0,v)\right) z + \widehat{\beta}(v) b_1(0,v) D_0\right] \\ &\quad +2\widehat{\alpha}(v)\widehat{\beta}(v) g_{1,1}(0,v) z + \widehat{\beta}(v)^2\left(a_{2,1}(t,s) z + b_{2,1}(t,s) D_0\right) \end{aligned}$$

$$k_{2,v}^{m}=\widehat{\beta}\left(v\right)^{2}g_{2,2}\left(0,v\right)+\left(\sigma_{v}^{S}-\gamma_{1}^{h}\left(v\right)\right)^{2}-2\left(\sigma_{v}^{S}-\gamma_{1}^{h}\left(v\right)\right)g_{1,1}\left(0,v\right)\widehat{\beta}\left(v\right). \tag{92}$$

Thus,

$$-H_{0,T}^{m}=K_{0,T}^{m}+K_{1,T}^{m}\frac{1}{\Gamma}+K_{2,T}^{m}\frac{1}{\Gamma^{2}},\qquad K_{i,T}^{m}=\frac{1}{2}\int_{0}^{T}k_{i,v}^{m}dv,\qquad i=0,1,2 \tag{93}$$

$$\Delta\widehat{T}^{u}\left(Z\right)=-\left(H_{0,T}^{m}-H_{0,T}^{m,ni}\right)=K_{0,T}^{m}+\frac{K_{1,T}^{m}}{\Gamma}+\left(K_{2,T}^{m}-\frac{\left(\sigma^{D}\right)^{2}T}{2}\right)\frac{1}{\Gamma^{2}}\equiv K_{0,T}^{\Delta\widehat{T}}+\frac{K_{1,T}^{\Delta\widehat{T}}}{\Gamma}+\frac{K_{2,T}^{\Delta\widehat{T}}}{\Gamma^{2}}.$$

Using,

$$\frac{\Delta\widehat{P}^{u}\left(Z\right)}{\Gamma}=\frac{K_{1,T}^{\Delta\widehat{P}}}{\Gamma}+\frac{K_{2,T}^{\Delta\widehat{P}}}{\Gamma^{2}},\qquad K_{2,T}^{\Delta\widehat{P}}=-\Delta\widehat{V}+\sigma^{D}\int_{0}^{T}\widehat{B}\left(s\right)\gamma_{1}^{h}\left(s\right)ds$$

$$K_{1,T}^{\Delta\widehat{P}}=\left(\widehat{B}\left(0\right)-1\right)\left(D_{0}+\mu^{D}T\right)+\widehat{A}\left(0\right)Z+\sigma^{D}\int_{0}^{T}\widehat{B}\left(s\right)\left(\gamma\left(s\right)+\gamma_{1}^{h}\left(s\right)\right)ds-\omega^{n}\widehat{I}\left(0\right)\mu^{\phi}$$

leads to,

$$\frac{\Delta\widehat{P}^{u}\left(Z\right)}{\Gamma}+\Delta\widehat{T}^{u}\left(Z\right)=K_{0,T}^{\Delta\widehat{T}}+\frac{K_{1,T}^{\Delta\widehat{T}}+K_{1,T}^{\Delta\widehat{P}}}{\Gamma}+\frac{K_{2,T}^{\Delta\widehat{T}}+K_{2,T}^{\Delta\widehat{P}}}{\Gamma^{2}}\equiv K_{0,T}^{\Delta\mathcal{U}}+\frac{K_{1,T}^{\Delta\mathcal{U}}}{\Gamma}+\frac{K_{2,T}^{\Delta\mathcal{U}}}{\Gamma^{2}}.$$

Equivalently,

$$\frac{\Delta\widehat{P}^{u}\left(Z\right)}{\Gamma}+\Delta\widehat{T}^{u}\left(Z\right)=\frac{1}{\Gamma^{2}}\left(K_{0,T}^{\Delta\mathcal{U}}\Gamma^{2}+K_{1,T}^{\Delta\mathcal{U}}\Gamma+K_{2,T}^{\Delta\mathcal{U}}\right)\equiv\frac{l\left(\Gamma\right)}{\Gamma^{2}}.$$

As $\Delta\mathcal{U}=\mathcal{U}^{u,ni}\left(\exp\left(-l\left(\Gamma\right)/\Gamma^{2}\right)-1\right)$ and $\Gamma\geqslant0$, it follows that $\Delta\mathcal{U}\geqslant0$ iff $l\left(\Gamma\right)\geqslant0$.

From the proof of Proposition 7, $K_{0,T}^{\Delta\mathcal{U}}\geqslant0$. If $K_{0,T}^{\Delta\mathcal{U}}>0$, it follows that $l\left(x\right)$ is a convex parabola with unique minimum at $\Gamma^{*}=-K_{1,T}^{\Delta\mathcal{U}}/\left(2K_{0,T}^{\Delta\mathcal{U}}\right)$ at which

$$l\left(\Gamma^{*}\right)=K_{2,T}^{\Delta\mathcal{U}}-\frac{\left(K_{1,T}^{\Delta\mathcal{U}}\right)^{2}}{4K_{0,T}^{\Delta\mathcal{U}}}=-\frac{J}{4K_{2,T}^{\Delta\mathcal{U}}}\qquad\text{where}\qquad J\equiv\left(K_{1,T}^{\Delta\mathcal{U}}\right)^{2}-4K_{0,T}^{\Delta\mathcal{U}}K_{2,T}^{\Delta\mathcal{U}}.$$

There are then three cases. If $J>0$, the function $l\left(\Gamma\right)$ has roots $\Gamma_{\pm}=\left(-K_{1,T}^{\Delta\mathcal{U}}\pm\sqrt{J}\right)/\left(2K_{0,T}^{\Delta\mathcal{U}}\right)$ and $l\left(\Gamma\right)\geqslant0$ for $0\leqslant\Gamma\leqslant\Gamma_{-}\vee0$ and $\Gamma_{+}\leqslant\Gamma$. If $J=0$, there is a unique root $\Gamma_{\pm}=-K_{1,T}^{\Delta\mathcal{U}}/\left(2K_{0,T}^{\Delta\mathcal{U}}\right)$ and $l\left(\Gamma\right)\geqslant0$ for $\Gamma\geqslant0$. If $J<0$, there is no root and $l\left(\Gamma\right)>0$ for $\Gamma\geqslant0$.

If $K_{0,T}^{\Delta\mathcal{U}}=0$, then $l\left(\Gamma\right)$ is linear with root $\Gamma_{0}=-K_{2,T}^{\Delta\mathcal{U}}/K_{1,T}^{\Delta\mathcal{U}}$ and increasing (decreasing) if $K_{1,T}^{\Delta\mathcal{U}}>0$ ($K_{1,T}^{\Delta\mathcal{U}}<0$). Thus $l\left(\Gamma\right)\geqslant0$ if $\Gamma\geqslant\Gamma_{0}\vee0$ and $K_{1,T}^{\Delta\mathcal{U}}>0$ or $\Gamma\leqslant\Gamma_{0}\vee0$ and $K_{1,T}^{\Delta\mathcal{U}}<0$. If $K_{0,T}^{\Delta\mathcal{U}}=K_{1,T}^{\Delta\mathcal{U}}=0$, $l\left(\Gamma\right)\geqslant0$ if $K_{2,T}^{\Delta\mathcal{U}}\geqslant0$ and $l\left(\Gamma\right)<0$ if $l\left(\Gamma\right)<0$.

Proposition 7 shows that the behavior of welfare gains for small risk tolerance depends on the sign of $\Delta\widehat{V}+\Delta\widehat{H}$. In fact, $\left(\Delta\widehat{V}+\Delta\widehat{H}\right)/2=-\lim_{\Gamma\to0}l\left(\Gamma\right)$. The sign of the right hand side is governed by $-sgn\left(K_{2,T}^{\Delta\mathcal{U}}\right)$. ■

Proof of Corollary 5. The noise trader maximizes CARA utility under the conditional beliefs $dP^{x}/dP=\left(\eta_{T}^{x}\right)^{-1}$ evaluated at $x=\phi$. The ex-ante value is $\mathcal{U}^{n}=u\left(X_{0}^{n}\right)\exp\left(-E_{0}\left[\xi_{T}^{m}\eta_{T}^{x}\log\left(\xi_{T}^{m}\eta_{T}^{x}\right)\right]\right)=\mathcal{U}^{u}\exp\left(-\mathcal{I}^{i}\left(\phi|Z\right)\right)$. Hence, $E_{0}\left[\mathcal{U}^{n}\right]=\mathcal{U}^{u}\int_{\mathbb{R}}\exp\left(-\mathcal{I}^{i}\left(x|Z\right)\right)f_{\phi|Z}\left(x\right)dx=\mathcal{U}^{u}\int_{\mathbb{R}}\exp\left(-\mathcal{I}^{i}\left(x|Z\right)\right)\mathcal{L}_{\phi,G}\left(x|Z\right)f_{G|Z}\left(x\right)dx$. The formula in the Corollary follows. As, $\mathcal{U}^{u}\leqslant0$, the interim utility is larger (less negative) if and only if $E_{0}\left[\mathcal{L}_{\phi,G}\left(G|Z\right)\exp\left(-\mathcal{I}^{i}\left(G|Z\right)\right)\right]\leqslant1$. The conditions $\omega^{i}=\omega^{n}$, $VAR\left[G\right]=VAR\left[\phi\right]$, $E\left[\phi\right]=E\left[G\right]$ imply $\mathcal{L}_{\phi,G}\left(G|Z\right)=1$ so that $E_{0}\left[\mathcal{U}^{n}\right]=E_{0}\left[\mathcal{U}^{i}\right]$. To prove the last statement, note that $\mathcal{L}_{\phi,G}\left(x|\,z\right)=h_{1}\exp\left(-h_{2}\right)$ where $h_{1}=\omega^{n}/\omega^{i}$ and

$$h_{2}=\begin{cases}\frac{1}{2}\frac{\left(\omega^{i}\right)^{2}\left(\omega^{n}\right)^{2}H\left(0\right)\left(\sigma^{\phi}\right)^{2}}{M\left(0\right)\left(\left(\omega^{n}\right)^{2}-\left(\omega^{i}\right)^{2}\right)}\left(x-\frac{\left(\omega^{n}\right)^{2}\mu^{\phi}-\left(\omega^{i}\right)^{2}E[G]+\frac{\left(\omega^{n}\right)^{3}\left(\sigma^{\phi}\right)^{2}-\left(\omega^{i}\right)^{3}H(0)}{M(0)}\left(z-E[Z]\right)}{\left(\omega^{n}\right)^{2}-\left(\omega^{i}\right)^{2}}\right)^{2} & \text{if }\omega^{i}\neq\omega^{n}\\ \frac{M(0)\left(\mu^{\phi}-E[G]+\frac{\bar{\omega}\left(\left(\sigma^{\phi}\right)^{2}-H(0)\right)}{M(0)}\left(z-E[Z]\right)\right)\left(\mu^{\phi}+E[G]+\frac{\bar{\omega}\left(\left(\sigma^{\phi}\right)^{2}+H(0)\right)}{M(0)}\left(z-E[Z]\right)-2x\right)}{2\left(\bar{\omega}\sigma^{\phi}\right)^{2}H(0)} & \text{if }\omega^{i}=\omega^{n}\equiv\bar{\omega}.\end{cases}$$

If $\omega^{i}=\omega^{n}$, $\mu^{\phi}=E\left[G\right]$ and $\left(\sigma^{\phi}\right)^{2}=VAR\left[G\right]$, then $\mathcal{L}_{\phi,G}\left(x|\,z\right)=1$. This establishes the result. ■

Proof of Corollary 6. Note that,

$$E_{0}\left[\mathcal{U}^{n}\right]-\mathcal{U}^{u,ni}=\mathcal{U}^{u,ni}\left(\exp\left(-\left(\frac{\Delta\widehat{P}\left(Z\right)}{\Gamma}+\Delta\widehat{T}^{u}\left(Z\right)\right)\right)E_{0}\left[\exp\left(-\mathcal{I}^{i}\left(G|Z\right)\mathcal{L}_{\phi,G}\left(G|Z\right)\right)\right]-1\right)$$

As $\mathcal{U}^{u,ni} < 0$, welfare improves only if,

$$\exp\left(-\left(\frac{\Delta\widehat{P}(Z)}{\Gamma} + \Delta\widehat{T}^u(Z)\right)\right) E_0\left[\exp\left(-\mathcal{I}^i(G|Z)\,\mathcal{L}_{\phi,G}(G|Z)\right)\right] < 1.$$

This condition can be expressed in terms of the certainty equivalent gain for the noise trader. The fact that $\mathcal{L}_{\phi,G}(G|\phi) = 1$ under unbiasedness, equal weights and equal variances of the noise and private signals, and the fact that the noise trader attains the interim utility of the informed are proved in Corollary 5. ■

Proof of Corollary 7. The condition $\Delta\widehat{P}(Z) + \Gamma\Delta\widehat{T}^u(Z) > \Gamma\log E_0\left[\exp\left(-\mathcal{I}^i(G|Z)\,\mathcal{L}_{\phi,G}(G|Z)\right)\right]$ is equivalent to $E_0[\mathcal{U}^n] - U^{u,ni} > 0$. It follows that the noise trader is better off. ■

Proof of Corollary 8. Let $\omega^n = 0$ and $\omega^u = 1 - \omega^i$. Consider first the case where the informed acquires information and suppose that a fully revealing equilibrium prevails. The common information filtration is then $\mathcal{F}^G_{(\cdot)} \equiv \mathcal{F}^D_{(\cdot)} \bigvee \sigma(G)$. Market clearing implies,

$$\theta^{m,wa}_t = \frac{\sigma^{S,wa}_t}{\Gamma} - h^u_t, \qquad h^u_t = -\frac{1}{2}\partial_{D_t} E_t\left[\xi^m_{t,T}\int_t^T \left(\theta^{m,wa}_v\right)^2 dv\right]$$

$$S^{wa}_t = E\left[D_T | D_t, G\right] - \int_t^T E\left[\sigma^{S,wa}_v \theta^{m,wa}_v \middle| D_t, G\right]$$

$$\sigma^{S,wa}_t = \left(\partial_{D_t} E\left[D_T | D_t, G\right] - \int_t^T \partial_{D_t} E\left[\sigma^{S,wa}_v \theta^{m,wa}_v \middle| D_t, G\right]\right)\sigma^D$$

(the superscript stands for "with acquisition"). A solution to this system of equations is,

$$\theta^{m,wa}_t = \frac{\sigma^{S,wa}_t}{\Gamma} = \frac{\partial_{D_t} E\left[D_T | D_t, G\right]}{\Gamma}\sigma^{S,wa}_t = \frac{1 - \frac{VAR[D_T|D_t]}{VAR[G|D_t]}}{\Gamma}\sigma^D = \frac{1}{\Gamma}\frac{\left(\sigma^\zeta\right)^2}{H(t)}\sigma^D$$

$$\sigma^{S,wa}_t = \left(1 - \frac{VAR\left[D_T | D_t\right]}{VAR\left[G | D_t\right]}\right)\sigma^D = \frac{\left(\sigma^\zeta\right)^2}{H(t)}\sigma^D$$

$$S^{wa}_t = E\left[D_T | D_t, G\right] - \frac{1}{\Gamma}\int_t^T \left(\sigma^{S,wa}_v\right)^2 dv = \frac{\left(\sigma^D\right)^2 (T-t)}{H(t)}G + \frac{\left(\sigma^\zeta\right)^2}{H(t)}\left(D_t + \mu^D(T-t)\right) - \frac{1}{\Gamma}\int_t^T \left(\sigma^{S,wa}_t\right)^2 dv$$

where we recall that $H(t) = \left(\sigma^D\right)^2(T-t) + \left(\sigma^\zeta\right)^2$. The price is indeed fully revealing, thus validating the initial conjecture. The wealth CE of the informed (including the information acquisition costs $C > 0$) is,

$$CE^{i,wa}_0 = S^{wa}_0 + \frac{1}{2\Gamma}\left(\int_0^T \left(\frac{\left(\sigma^\zeta\right)^2}{H(v)}\right)^2 dv\right)\left(\sigma^D\right)^2 - C.$$

Full revelation implies that $CE^{u,wa}_0 = CE^{i,wa}_0 + C > CE^{i,wa}_0$.

The ex-ante utility function $V^{i,wa} \equiv E\left[\mathcal{U}^{i,wa}\right]$ is,

$$\begin{aligned}
V^{i,wa} &= -\Gamma E\left[\exp\left(-\frac{S^{wa}_0 - C}{\Gamma}\right)\right]\exp\left(-\frac{1}{2\Gamma^2}\int_0^T \left(\sigma^{S,wa}_v\right)^2 dv\right) \\
&= -\Gamma E\left[\exp\left(-\frac{\left(\sigma^D\right)^2 T}{H(0)\,\Gamma}G\right)\right]\exp\left(-\frac{1}{\Gamma}\left(\frac{\left(\sigma^\zeta\right)^2}{H(0)}\left(D_0 + \mu^D T\right) - \frac{1}{2\Gamma}\int_0^T \left(\sigma^{S,wa}_v\right)^2 dv\right) + \frac{C}{\Gamma}\right) \\
&= -\Gamma\exp\left(-\frac{\left(\sigma^D\right)^2 T}{H(0)\,\Gamma}\left(D_0 + \mu^D T\right) + \frac{1}{2}\left(\frac{\left(\sigma^D\right)^2 T}{H(0)\,\Gamma}\right)^2 H(0) - \frac{1}{\Gamma}\left(\frac{\left(\sigma^\zeta\right)^2}{H(0)}\left(D_0 + \mu^D T\right) - \frac{1}{2\Gamma}\int_0^T \left(\sigma^{S,wa}_v\right)^2 dv - C\right)\right) \\
&= -\Gamma\exp\left(-\frac{1}{\Gamma}\left(D_0 + \mu^D T - \frac{1}{2\Gamma}\frac{\left(\left(\sigma^D\right)^2 T\right)^2}{H(0)} - \frac{1}{2\Gamma}\int_0^T \left(\sigma^{S,wa}_v\right)^2 dv\right)\right)
\end{aligned}$$

The ex-ante certainty equivalent is,

$$CE^{i,wa} = D_0 + \mu^D T - \frac{1}{2\Gamma}\frac{\left(\left(\sigma^D\right)^2 T\right)^2}{H(0)} - \frac{1}{2\Gamma}\int_0^T \left(\sigma_v^{S,wa}\right)^2 dv - C.$$

Next, consider the case where the informed does not acquire information. The common information filtration is then $\mathcal{F}^D_{(\cdot)}$. The fully revealing equilibrium is,

$$\theta_t^{m,na} = \frac{\sigma_t^{S,na}}{\Gamma} = \frac{\partial_{D_T} E\left[D_T \,|\, D_t\right]\sigma^D}{\Gamma} = \frac{\sigma^D}{\Gamma}, \qquad \sigma_t^{S,na} = \sigma^D, \qquad S_t^{na} = D_t + \mu^D (T-t) - \frac{1}{\Gamma}\left(\sigma^D\right)^2 (T-t)$$

(the superscript stands for "no acquisition"). Utility functions are the same as in a no information equilibrium,

$$\mathcal{U}^{j,na} = \mathcal{U}^{j,ni} = u\left(X_0^j\right)\exp\left(-\frac{1}{2\Gamma^2}\int_0^T \left(\sigma_t^{S,ni}\right)^2 dt\right), \qquad j \in \{u,i\}$$

$$X_0^j = N_0^j\left(D_0 + \left(\mu^D - \frac{\left(\sigma^D\right)^2}{\Gamma}\right)T\right), \qquad \sigma_t^{S,ni} = \sigma^D, \qquad N_0^j = 1.$$

The CE wealth becomes $CE^{i,na} = S_0^{na} + \frac{1}{2\Gamma}\left(\int_0^T 1dv\right)\left(\sigma^D\right)^2 = D_0 + \mu^D T - \frac{1}{2\Gamma}\left(\sigma^D\right)^2 T$. Information production is suboptimal if and only if $CE_0^{i,wa} < CE_0^{i,na}$, i.e.,

$$-\frac{1}{2\Gamma}\frac{\left(\left(\sigma^D\right)^2 T\right)^2}{H(0)} - \frac{1}{2\Gamma}\int_0^T \left(\sigma_v^{S,wa}\right)^2 dv - C < -\frac{1}{2\Gamma}\left(\sigma^D\right)^2 T.$$

This condition is equivalent to $C > C^*$ where,

$$C^* = \frac{\left(\sigma^D\right)^2}{2\Gamma}\left[\int_0^T \left(1 - \frac{\left(\sigma^\varsigma\right)^4}{H(v)^2}\right)dv - \frac{\left(\sigma^D\right)^2 T^2}{H(0)}\right] = \frac{\left(\sigma^D\right)^2}{2\Gamma}\left[T - \frac{\left(\sigma^\varsigma\right)^2 T}{H(0)} - \frac{\left(\sigma^D\right)^2 T^2}{H(0)}\right] = 0.$$

The first equality above uses $\sigma_v^{S,wa} = \left(\left(\sigma^\varsigma\right)^2 / H(v)\right)\sigma^D$. The second relies on,

$$\int_0^T \frac{\left(\sigma^D\right)^2}{H(v)^2}dv = \frac{1}{H(T)} - \frac{1}{H(0)} = \frac{H(0) - H(T)}{H(T)\,H(0)} = \frac{\left(\sigma^D\right)^2 T}{\left(\sigma^\varsigma\right)^2 H(0)}.$$

Hence, information acquisition is always suboptimal if acquisition is costly. In the absence of speculation, the no-information equilibrium prevails.

To evaluate the welfare implications of speculation, it remains to compare interim utilities with and without speculation. But, the equilibrium without speculation corresponds to the equilibrium without information. Thus, welfare improves under the conditions of Corollary 7. ■

Proof of Proposition 10. Consider the information infrastructure $\Im(n)$ with n signals equally spaced in time $\{t_0^n, \ldots, t_n^n\}$. The interim utility is as in the one period model,

$$\mathcal{U}^i_{t^n_{n-1}} = -\Gamma\exp\left(-\frac{X^i_{t^n_{n-1}}}{\Gamma} - E\left[\xi^{G_n}_{t^n_{n-1},t^n_n}\log\xi^{G_n}_{t^n_{n-1},t^n_n}\,\middle|\,\mathcal{G}_{t^n_{n-1}}\right]\right).$$

Choosing $X^n_{t_{n-1}}$ optimally at time t^n_{n-2} leads to the first order condition and optimal wealth,

$$\exp\left(-\frac{X_{t^n_{n-1}}}{\Gamma}\right) = y_{t^n_{n-2}}\xi^{G_{n-1}}_{t^n_{n-2},t^n_{n-1}}\exp\left(E\left[\xi^{G_n}_{t^n_{n-1},t^n_n}\log\xi^{G_n}_{t^n_{n-1},t^n_n}\,\middle|\,\mathcal{G}_{t^n_{n-1}}\right]\right)$$

$$\begin{aligned}
-\frac{X_{t^n_{n-2}}}{\Gamma} &= \log y_{t^n_{n-2}} + E\left[\xi^{G_{n-1}}_{t^n_{n-2},t^n_{n-1}}\log\xi^{G_{n-1}}_{t^n_{n-2},t^n_{n-1}} + \xi^{G_{n-1}}_{t^n_{n-2},t^n_{n-1}}E\left[\xi^{G_n}_{t^n_{n-1},t^n_n}\log\xi^{G_n}_{t^n_{n-1},t^n_n}\,\middle|\,\mathcal{G}_{t^n_{n-1}}\right]\,\middle|\,\mathcal{G}_{t^n_{n-1}}\right] \\
&= \log y_{t^n_{n-2}} + E\left[\sum_{k=n-2}^{n-1}\prod_{i=n-2}^{k}\xi^{G_{i+1}}_{t^n_i,t^n_{i+1}}\log\xi^{G_{k+1}}_{t^n_k,t^n_{k+1}}\,\middle|\,\mathcal{G}_{t^n_{n-2}}\right].
\end{aligned}$$

Interim utility at t_{n-2}^n is,

$$\mathcal{U}_{t_{n-2}^n}^i = -\Gamma \exp\left(-\frac{X_{t_{n-2}^n}^i}{\Gamma} - E\left[\sum_{k=n-2}^{n-1}\prod_{i=n-2}^{k} \xi_{t_i^n,t_{i+1}^n}^{G_{i+1}} \log \xi_{t_k^n,t_{k+1}^n}^{G_{k+1}} \middle| \mathcal{G}_{t_{n-2}^n}\right]\right).$$

Iterating this argument gives,

$$\mathcal{U}_{t_0}^i = -\Gamma \exp\left(-\frac{X_{t_0^n}^i}{\Gamma} - E\left[\sum_{k=0}^{n-1}\prod_{i=0}^{k} \xi_{t_i^n,t_{i+1}^n}^{G_{i+1}} \log \xi_{t_k^n,t_{k+1}^n}^{G_{k+1}} \middle| \mathcal{G}_{t_0^n}\right]\right).$$

As $X_{t_0^n}^i = S_0^{m,n}$ where $S_0^{m,n} = E\left[\prod_{i=0}^n \xi_{t_i^n,t_{i+1}^n}^m D_T \middle| \mathcal{F}_0^{m,n}\right]$, the ex-ante utility associated with $\Im(n)$ is,

$$V^i(n) = -\Gamma E\left[\exp\left(-\frac{S_0^{m,n}}{\Gamma} - E\left[\sum_{k=0}^{n-1}\prod_{i=0}^{k} \xi_{t_i^n,t_{i+1}^n}^{G_{i+1}} \log \xi_{t_k^n,t_{k+1}^n}^{G_{k+1}} \middle| \mathcal{G}_0\right]\right)\right] - C(n)$$

Using,

$$E\left[\xi_{t_k,t_{k+1}}^{G_{k+1}} \log \xi_{t_k^n,t_{k+1}^n}^{G_{k+1}} \middle| \mathcal{G}_{t_k^n}\right] = E\left[\xi_{t_k^n,t_{k+1}^n}^m \log \xi_{t_k^n,t_{k+1}^n}^m \middle| \mathcal{F}_{t_k^n}^{m,n}\right] + E\left[\xi_{t_k^n,t_{k+1}^n}^m \log \eta_{t_k^n,t_{k+1}^n}^x \middle| \mathcal{F}_{t_k^n}^{m,n}\right]_{x=G_{k+1}}$$

and defining $K_0^n \equiv E\left[\sum_{k=0}^{n-1}\prod_{i=0}^k \xi_{t_i^n,t_{i+1}^n}^m \log \xi_{t_k^n,t_{k+1}^n}^m \middle| \mathcal{F}_0^{m,n}\right]$ gives,

$$V^i(n) = -\Gamma E\left[\exp\left(-\frac{S_0^{m,n}}{\Gamma} - K_0^n - E\left[\sum_{k=0}^{n-1}\prod_{i=0}^{k} \xi_{t_i^n,t_{i+1}^n}^m \int_{-\infty}^{+\infty} \log \eta_{t_k^n,t_{k+1}^n}^x P\left(G_{k+1} \in dx \middle| \mathcal{F}_0^{m,n}\right) \middle| \mathcal{F}_0^{m,n}\right]\right)\right] - C(n).$$

As $\eta_{t_k^n,t_{k+1}^n}^x = \frac{P\left(G_{k+1}\in dx \middle| \mathcal{F}_{t_{k+1}^n}^{m,n}\right)}{P\left(G_{k+1}\in dx \middle| \mathcal{F}_{t_k^n}^{m,n}\right)}$ it follows that $\Phi \equiv \int_{-\infty}^{+\infty} \log \eta_{t_k^n,t_{k+1}^n}^x P\left(G_{k+1} \in dx \middle| \mathcal{F}_0^{m,n}\right)$ becomes,

$$\Phi = \int_{-\infty}^{+\infty}\left(\log \frac{P\left(G_{k+1} \in dx \middle| \mathcal{F}_0^{m,n}\right)}{P\left(G_{k+1} \in dx \middle| \mathcal{F}_{t_{k+1}^n}^{m,n}\right)} - \log \frac{P\left(G_{k+1} \in dx \middle| \mathcal{F}_0^{m,n}\right)}{P\left(G_{k+1} \in dx \middle| \mathcal{F}_{t_k^n}^{m,n}\right)}\right) P\left(G_{k+1} \in dx \middle| \mathcal{F}_0^{m,n}\right) \equiv D_{t_{k+1}^n}^{KL} - D_{t_k^n}^{KL} \equiv \Delta D_{t_k^n}^{KL}$$

where D_t^{KL} is the Kullback-Leibler divergence between the signal density conditional on initial information $\mathcal{F}_0^{m,n}$ and information $\mathcal{F}_t^{m,n}$. Ex-ante utility is,

$$V^i(n) = -\Gamma E\left[\exp\left(-\frac{S_0^{m,n}}{\Gamma} - K_0^n - E\left[\sum_{k=0}^{n-1}\prod_{i=0}^{k} \xi_{t_i^n,t_{i+1}^n}^m \Delta D_{t_k^n}^{KL} \middle| \mathcal{F}_0^{m,n}\right]\right)\right].$$

As densities are Gaussian,

$$D_t^{KL} = \frac{1}{2}\left(\log \frac{VAR\left[G_{k+1} \middle| \mathcal{F}_t^{m,n}\right]}{VAR\left[G_{k+1} \middle| \mathcal{F}_0^{m,n}\right]} - 1 + \frac{VAR\left[G_{k+1} \middle| \mathcal{F}_0^{m,n}\right] + \left(E\left[G_{k+1} \middle| \mathcal{F}_0^{m,n}\right] - E\left[G_{k+1} \middle| \mathcal{F}_t^{m,n}\right]\right)^2}{VAR\left[G_{k+1} \middle| \mathcal{F}_t^{m,n}\right]}\right)$$

$$\begin{aligned}\Delta D_{t_k^n}^{KL} &= \frac{1}{2}\log \frac{VAR\left[G_{k+1} \middle| \mathcal{F}_{t_{k+1}^n}^{m,n}\right]}{VAR\left[G_{k+1} \middle| \mathcal{F}_{t_k^n}^{m,n}\right]} + \frac{1}{2}\left(\frac{1}{VAR\left[G_{k+1} \middle| \mathcal{F}_{t_{k+1}^n}^{m,n}\right]} - \frac{1}{VAR\left[G_{k+1} \middle| \mathcal{F}_{t_k^n}^{m,n}\right]}\right) VAR\left[G_{k+1} \middle| \mathcal{F}_0^{m,n}\right] \\ &+\frac{1}{2}\left(\left(\frac{E\left[G_{k+1} \middle| \mathcal{F}_{t_{k+1}^n}^{m,n}\right] - E\left[G_{k+1} \middle| \mathcal{F}_0^{m,n}\right]}{\sqrt{VAR\left[G_{k+1} \middle| \mathcal{F}_{t_{k+1}^n}^{m,n}\right]}}\right)^2 - \left(\frac{E\left[G_{k+1} \middle| \mathcal{F}_{t_k^n}^{m,n}\right] - E\left[G_{k+1} \middle| \mathcal{F}_0^{m,n}\right]}{\sqrt{VAR\left[G_{k+1} \middle| \mathcal{F}_{t_k^n}^{m,n}\right]}}\right)^2\right).\end{aligned}$$

The corresponding ex-ante certainty equivalent is,

$$\begin{aligned} CE^{i}(n) &= -\Gamma \log E\left[\exp\left(-\frac{S_0^{m,n}}{\Gamma} - E\left[\sum_{k=0}^{n-1}\prod_{i=0}^{k}\xi^{G_{i+1}}_{t_i^n,t_{i+1}^n}\log \xi^{G_{k+1}}_{t_k^n,t_{k+1}^n}\middle| \mathcal{G}_0\right]\right)\right] - C(n) \\ &= -\Gamma \log E\left[\exp\left(-\frac{S_0^{m,n}}{\Gamma} - K_0^n - E\left[\sum_{k=0}^{n-1}\prod_{i=0}^{k}\xi^{m}_{t_i^n,t_{i+1}^n}\Delta D^{KL}_{t_k^n}\middle| \mathcal{F}_0^{m,n}\right]\right)\right] - C(n). \end{aligned}$$

The informed is a price-taker and takes $S_0^{m,n}$ and $\xi^m_{t_i^n,t_{i+1}^n}$ as given when making the information acquisition decision. Only the impacts of $\log \eta^x_{t_k^n,t_{k+1}^n}$ and therefore $\Delta D^{KL}_{t_k^n}$ on ex-ante utility will be taken into account when choosing the optimal infrastructure. Thus, the informed only considers the value of all Kullback-Leibler divergence gains of the infrastructure for a given state price density. The optimal choice is the maximal solution such that,

$$-\Gamma \log\left(\frac{E\left[\exp\left(-\frac{S_0^{m,n^*}}{\Gamma} - K_0^{n^*} - E\left[\sum_{k=0}^{n^*}\prod_{i=0}^{k}\xi^m_{t_i^{n^*},t_{i+1}^{n^*}}\Delta D^{KL}_{t_k^{n^*+1}}\middle|\mathcal{F}_0^{m,n^*}\right]\right)\right]}{E\left[\exp\left(-\frac{S_0^{m,n^*}}{\Gamma} - K_0^{n^*} - E\left[\sum_{k=0}^{n^*-1}\prod_{i=0}^{k}\xi^m_{t_i^{n^*},t_{i+1}^{n^*}}\Delta D^{KL}_{t_k^{n^*}}\middle|\mathcal{F}_0^{m,n^*}\right]\right)\right]}\right) \leqslant C(n^*+1) - C(n^*).$$

It remains to show that the optimal number of signals n^* (i.e., the frequency) is finite. As $\lim_{n\to\infty} C(n) = +\infty$ by assumption, it is shown that given the market price of risk and the state price density, the certainty equivalent gain of the informed is bounded. For this, note that $-\log$ is a convex function so that, by Jensen's inequality,

$$CE^i(n) \leqslant E\left[S_0^{m,n}\right] + \Gamma E\left[K_0^n\right] + \Gamma E\left[\sum_{k=0}^{n-1}\prod_{i=0}^{k}\xi^m_{t_i^n,t_{i+1}^n}\Delta D^{KL}_{t_k^n}\right] - C(n).$$

By Proposition 11, the information structure is isomorphic to one with signal process $G_t = D_T + \widetilde{\zeta}_t$ where $\widetilde{\zeta}_t = \frac{\zeta'_{N_t} VAR\left[\zeta_{N_t}\right]^{-1} 1_{N_t}}{1'_{N_t} VAR\left[\zeta_{N_t}\right]^{-1} 1_{N_t}}$ and $VAR\left[\widetilde{\zeta}_t\right] = 1/\left(1'_{N_t} VAR\left[\zeta_{N_t}\right]^{-1} 1_{N_t}\right)$. The assumption $\lim_{N_t\to\infty} VAR\left[\widetilde{\zeta}_t\right] = \left(\sigma^{\zeta}\right)^2$ from above, implies,

$$\begin{aligned} E\left[K_0^n\right] + E\left[\sum_{k=0}^{n-1}\prod_{i=0}^{k}\xi^m_{t_i^n,t_{i+1}^n}\Delta D^{KL}_{t_k^n}\right] &= \frac{1}{2}E\left[\int_0^T \widetilde{\xi}^m_{0,v}\int_{-\infty}^{+\infty}\left(\theta_v^{G|m}(x) + \widetilde{\theta}_v^m\right)^2 P\left(G_v \in dx \middle| \mathcal{F}_0^m\right) dv\right] \\ &\leqslant \frac{1}{2}E\left[\int_0^T \widetilde{\xi}^m_{0,v}\int_{-\infty}^{+\infty}\left(\overline{\theta}_v^{G|m}(x) + \widetilde{\theta}_v^m\right)^2 P\left(G_v \in dx \middle| \mathcal{F}_0^m\right) dv\right] \end{aligned}$$

where $\overline{\theta}_v^{G|m}(x) = \sigma^D \frac{x - E\left[D_T | \mathcal{F}_v^m\right]}{\left(\sigma^D\right)^2 (T-v) + \left(\sigma^{\zeta}\right)^2}$. ■

Proof of Proposition 11. The private information process is $G_t = D_T 1_{N_t} + \zeta_{N_t}$ where 1_{N_t} is an $N_t \times 1$ vector and ζ_{N_t} is an $N_t \times 1$ vector of independent noises. The PIPR is,

$$\begin{aligned} \theta_t^{G|m}(G_t) &= \left(G_t - E_t\left[D_T\right] 1_{N_t}\right)' \left(VAR_t\left[D_T\right] 1_{N_t} 1'_{N_t} + VAR\left[\zeta_{N_t}\right]\right)^{-1} 1_{N_t} \partial_{D_t} E_t\left[D_T\right] \\ &= \left(\left(D_T - E_t\left[D_T\right]\right) 1_{N_t} + \zeta_{N_t}\right)' \left(VAR_t\left[D_T\right] 1_{N_t} 1'_{N_t} + VAR\left[\zeta_{N_t}\right]\right)^{-1} 1_{N_t} \partial_{D_t} E_t\left[D_T\right] \\ &= \left(\frac{\left(D_T - E_t\left[D_T\right]\right) 1_{N_t} + \zeta_{N_t}}{VAR_t\left[D_T\right]}\right)' \left(1_{N_t} 1'_{N_t} + \Psi_t\right)^{-1} 1_{N_t} \partial_{D_t} E_t\left[D_T\right] \end{aligned}$$

where $E_t[\cdot] \equiv E_t\left[\cdot | \mathcal{F}_t^m\right], VAR_t\left[D_T\right] \equiv VAR\left[\cdot | \mathcal{F}_t^m\right]$ and $\Psi_t \equiv \frac{VAR\left[\zeta_{N_t}\right]}{VAR_t\left[D_T\right]}$. The Sherman-Morrison formula (Golub and van Loan (1996)) gives $\left(1_{N_t} 1'_{N_t} + \Psi_t\right)^{-1} 1_{N_t} = \left(1 + 1'_{N_t}\Psi_t^{-1} 1_{N_t}\right)^{-1} \Psi_t^{-1} 1_{N_t}$, so that,

$$\begin{aligned} \theta_t^{G|m}(G_t) &= \left(\left(\frac{D_T - E_t\left[D_T\right]}{VAR_t\left[D_T\right]}\right) 1'_{N_t}\Psi_t^{-1} 1_{N_t}\partial_{D_t} E_t\left[D_T\right] + \frac{\zeta'_{N_t}\Psi_t^{-1} 1_{N_t}\partial_{D_t} E_t\left[D_T\right]}{VAR_t\left[D_T\right]}\right)\left(\frac{1}{1 + 1'_{N_t}\Psi_t^{-1} 1_{N_t}}\right) \\ &= \frac{D_T - E_t\left[D_T\right] + \widetilde{\zeta}_t}{VAR_t\left[D_T\right] + VAR\left[\widetilde{\zeta}_t\right]}\partial_{D_t} E_t\left[D_T\right] \equiv \frac{\widetilde{G}_t - E_t\left[\widetilde{G}_t\right]}{VAR_t\left[\widetilde{G}_t\right]}\partial_{D_t} E_t\left[D_T\right] \end{aligned}$$

$$\widetilde{\zeta}_t = \frac{1'_{N_T} VAR\left[\zeta_{N_t}\right]^{-1} \zeta_{N_t}}{1'_{N_T} VAR\left[\zeta_{N_t}\right]^{-1} 1_{N_t}} \quad \text{and} \quad \widetilde{G}_t = D_T + \widetilde{\zeta}_t.$$

The noise trader's demand depends on $\theta_t^{G|m}\left(\widetilde{\phi}_t\right) = \frac{\widetilde{\phi}_t - E_t\left[\widetilde{G}_t\right]}{VAR_t\left[\widetilde{G}_t\right]} \partial_{D_t} E_t\left[D_T\right]$. The WAPR is therefore based on the public signal $\widetilde{Z}_t = \omega^i \widetilde{G}_t + \omega^n \widetilde{\phi}_t$. ■

Proof of Corollary 9. To simplify notation, let $v_t^\zeta \equiv \left(\sigma_t^\zeta\right)^2 = VAR\left[\widetilde{\zeta}_t\right]$ and note that v_t^ζ is constant for $t \neq t_j$ and jumps down for $t = t_j$,

$$\Delta v_t^\zeta = \begin{cases} 0 & \text{for } t \neq t_j \\ -v_{t_{j-1}}^\zeta \left(\frac{\left(k_{1,t_{j-1}}^\zeta\right)^2}{\left(k_{1,t_{j-1}}^\zeta\right)^2 + v_{t_{j-1}}^\zeta k_{2,t_{j-1}}^\zeta} \right) & \text{for } t = t_j. \end{cases}$$

where

$$k_{1,t_{j-1}}^\zeta \equiv 1 - 1'_{N_{t_{j-1}}} VAR\left[\zeta_{N_{t_{j-1}}}\right]^{-1} COV\left[\zeta_{N_{t_{j-1}}}, \zeta_j\right]$$

$$k_{2,t_{j-1}}^\zeta \equiv VAR\left[\zeta_j\right]^2 - COV\left[\zeta_{N_{t_{j-1}}}, \zeta_j\right]' VAR\left[\zeta_{N_{t_{j-1}}}\right]^{-1} COV\left[\zeta_{N_{t_{j-1}}}, \zeta_j\right]$$

$\zeta_{N_{t_{j-1}}}$ is an $N_{t_{j-1}} \times 1$ vector of error noises of the private signals received before time t_j and ζ_j is the error noise of the private signal received at time t_j. As $e'_{N_{t_j}} VAR\left[\zeta_{N_{t_j}}\right]^{-1} e_{N_{t_j}} = 1/k_{2,t_{j-1}}^\zeta$ where $e'_{N_{t_j}}$ is the $N_{t_j}^{th}$ unit vector and as $VAR\left[\zeta_{N_{t_j}}\right]$ and therefore $VAR\left[\zeta_{N_{t_j}}\right]^{-1}$ are positive definite, it follows that $k_{2,t_{j-1}}^\zeta > 0$ and $\Delta v_t^\zeta < 0$.

In between information arrival times, as v_t^ζ is constant, the time-behavior of the stock price volatility is determined by the same factors as in the benchmark model with a single private signal. Price volatility therefore increases. At information arrival times, the jump in volatility is determined by the jump in v_t^ζ and the impact of v_t^ζ on volatility.

To find the latter, note that, in the absence of hedging,

$$\partial_{v_t^\zeta} B(t) = B(t) \left(\frac{\omega}{H(T)} - \frac{\omega}{H(t)} + \frac{(1-\omega)\left(\omega^i\right)^2}{M(T)} - \frac{(1-\omega)\left(\omega^i\right)^2}{M(t)} \right) > 0$$

implying that price volatility jumps down. The effect due to hedging is $\partial_{v_t^\zeta} B^h(t) = \sigma^D B^h(t) \int_t^T \partial_{v_t^\zeta} \beta^h(v)\, dv$ where,

$$\partial_{v_t^\zeta} \beta^h(v) = \int_t^T \exp\left(\int_t^v \left(\kappa_1(s) + 2\left(1 + 2\sigma^D\right) \beta^h(s) \right) ds \right) \left(\partial_{v_t^\zeta} \kappa_0(v) + \partial_{v_t^\zeta} \kappa_1(v) \beta^h(v) \right)$$

$$\partial_{v_t^\zeta} \kappa_0(t) = 2\left(\frac{1-\omega}{\omega}\right) \beta(t)\, \partial_{v_t^\zeta} \beta(t), \qquad \partial_{v_t^\zeta} \kappa_1(t) = 2\sigma^D \left(\partial_{v_t^\zeta} \beta(t) + \partial_{v_t^\zeta} \beta^D(t) \right)$$

$$\partial_{v_t^\zeta} \beta(t) = \omega \left(\frac{\left(\omega^n\right)^2 \left(\sigma^\phi\right)^2}{\left(M(t) H(t)\right)^2} \right) M(t) \left(\left(\omega^i\right)^2 H(t) + 1 \right) > 0, \qquad \partial_{v_t^\zeta} \beta^D(t) = \frac{\left(\omega^i\right)^4 \sigma^D}{M(t)} > 0.$$

Thus $\partial_{v_t^\zeta} B^h(t) > 0$ implying that the hedging factor jumps down as well. Hence, $\partial_{v^\zeta} \sigma_t^S > 0$ and price volatility decreases at information arrival times $t_j, j = 1, ..., J$. ■

Proof of Proposition 12. Assume heterogeneous risk tolerances. The market clearing condition gives,

$$\sum_{j \in \{u,i,n\}} \omega^j \Gamma^j \left(\theta_t^m + h_t^u \sigma^D \right) + \omega^i \Gamma^i \left(\theta_t^{G|m}(G) + h_t^{G|m}(G|Z)\, \sigma^D \right) + \omega^n \Gamma^n \left(\theta_t^{G|m}(\phi) + h_t^{G|m}(\phi|Z)\, \sigma^D \right) = \sigma_t^S.$$

Rearranging terms and using the definitions of $\widetilde{\omega}^i, \widetilde{\omega}^n$ leads to,

$$\theta_t^m + h_t^u \sigma^D + \widetilde{\omega}^i \left(\theta_t^{G|m}(G) + h_t^{G|m}(G|Z)\, \sigma^D \right) + \widetilde{\omega}^n \left(\theta_t^{G|m}(\phi) + h_t^{G|m}(\phi|Z)\, \sigma^D \right) = \frac{\sigma_t^S}{\Gamma^a}.$$

The WAPR and weighted average of the informational hedging components follow. The equilibrium market price of risk is identical to the one under homogeneous information modulo the parameter substitution announced. ■

Proof of Corollary 10. Under the conditions of the corollary $\partial\Gamma^a \equiv \omega^i \partial\Gamma^i + \omega^n \partial\Gamma^n = 0 \Longleftrightarrow \partial\Gamma^n / \partial\Gamma^i = -\left(\omega^i / \omega^n\right) <$

0 and $\partial\widetilde{\omega} = \partial\widetilde{\omega}^i + \partial\widetilde{\omega}^n = \left(\omega^i\partial\Gamma^i + \omega^n\partial\Gamma^n\right)/\Gamma^a = 0$. Suppose $\partial\Gamma^n > 0$. Simple calculations show,

$$\partial\frac{\widetilde{\omega}^i}{\widetilde{\omega}^n} = \partial\frac{\omega^i\Gamma^i}{\omega^n\Gamma^n} = \frac{\omega^i}{\omega^n}\frac{\partial\Gamma^i\Gamma^n - \Gamma^i\partial\Gamma^n}{(\Gamma^n)^2} = -\frac{1}{\omega^n}\frac{\omega^n\Gamma^n + \omega^i\Gamma^i}{(\Gamma^n)^2}\partial\Gamma^n < 0$$

$$\frac{M(T)}{M(t)} = \frac{(\widetilde{\omega}^n)^2(\sigma^\phi)^2}{(\widetilde{\omega}^i)^2 H(t) + (\widetilde{\omega}^n)^2(\sigma^\phi)^2} = \frac{(\sigma^\phi)^2}{\left(\frac{\widetilde{\omega}^i}{\widetilde{\omega}^n}\right)^2 H(t) + (\sigma^\phi)^2}, \qquad \partial\frac{M(T)}{M(t)} = -\left(\frac{M(T)}{M(t)}\right)^2 2\left(\frac{\widetilde{\omega}^i}{\widetilde{\omega}^n}\right)\left(\frac{H(t)}{(\sigma^\phi)^2}\right)\partial\frac{\widetilde{\omega}^i}{\widetilde{\omega}^n} > 0$$

$$\partial B(t) = \partial\left(\frac{H(T)}{H(t)}\right)^{\widetilde{\omega}}\left(\frac{M(T)}{M(t)}\right)^{1-\widetilde{\omega}} = (1-\widetilde{\omega})\left(\frac{H(T)}{H(t)}\right)^{\widetilde{\omega}}\left(\frac{M(T)}{M(t)}\right)^{-\widetilde{\omega}}\partial\frac{M(T)}{M(t)} > 0$$

$$B^h(t) = e^{\sigma^D\int_t^T\beta^h(v)dv}, \qquad \partial B^h(t) = B^h(t)\,\sigma^D\int_t^T\partial\beta^h(v)\,dv$$

$$\beta(t) = -\widetilde{\omega}\frac{1-\kappa_t\widetilde{\omega}^i}{H(t)}\sigma^D = -\widetilde{\omega}\frac{(\widetilde{\omega}^n)^2(\sigma^\phi)^2}{M(t)H(t)}\sigma^D = -\widetilde{\omega}\frac{M(T)}{M(t)H(t)}\sigma^D$$

$$\alpha^D(t) \equiv \frac{\omega^i\sigma^D}{M(t)}, \qquad \beta^D(t) \equiv -\widetilde{\omega}^i\alpha^D(t) = -\frac{(\widetilde{\omega}^i)^2\sigma^D}{M(t)}$$

$$\beta(t) + \beta^D(t) = -\frac{\widetilde{\omega}(\widetilde{\omega}^n)^2(\sigma^\phi)^2 + (\widetilde{\omega}^i)^2 H(t)}{M(t)}\frac{\sigma^D}{H(t)} = -\left[\frac{H(t) + \widetilde{\omega}\left(\frac{\widetilde{\omega}^n}{\widetilde{\omega}^i}\right)^2(\sigma^\phi)^2}{H(t) + \left(\frac{\widetilde{\omega}^n}{\widetilde{\omega}^i}\right)^2(\sigma^\phi)^2}\right]\frac{\sigma^D}{H(t)}$$

$$\partial\left(\beta(t) + \beta^D(t)\right) = (1-\widetilde{\omega})\frac{2H(t)\left(\frac{\widetilde{\omega}^n}{\widetilde{\omega}^i}\right)(\sigma^\phi)^2\partial\left(\frac{\widetilde{\omega}^n}{\widetilde{\omega}^i}\right)}{\left[H(t) + \left(\frac{\widetilde{\omega}^n}{\widetilde{\omega}^i}\right)^2(\sigma^\phi)^2\right]^2}\frac{\sigma^D}{H(t)} > 0$$

$$\kappa_0(t) \equiv \left(\frac{1-\widetilde{\omega}}{\widetilde{\omega}}\right)\beta(t)^2\sigma^D, \qquad \partial\kappa_0(t) \equiv 2\left(\frac{1-\widetilde{\omega}}{\widetilde{\omega}}\right)\beta(t)\,\partial\beta(t)\,\sigma^D > 0$$

$$\kappa_1(t) \equiv -2\sigma^D\left(\beta(t) + \beta^D(t)\right), \qquad \partial\kappa_1(t) \equiv -2\sigma^D\partial\left(\beta(t) + \beta^D(t)\right) < 0.$$

As β^h solves $\dot{\beta}^h(t) = \kappa_0(t) + \kappa_1(t)\beta^h(t) - \sigma^D\beta^h(t)^2$ with terminal condition $\beta^h(T) = 0$, the derivative $\partial\beta^h$ relative to the risk tolerance perturbation described above satisfies, $\partial\dot{\beta}^h(t) = \partial\kappa_0(t) + \partial\kappa_1(t)\beta^h(t) + \left[\kappa_1(t) - 2\sigma^D\beta^h(t)\right]\partial\beta^h(t)$ with boundary condition $\partial\beta^h(T) = 0$. The solution is,

$$\partial\beta^h(t) = -\int_t^T e^{-\int_t^v\left(\kappa_1(s) - 2\sigma^D\beta^h(s)\right)ds}\left(\partial\kappa_0(v) + \partial\kappa_1(v)\beta^h(v)\right)dv$$

which is strictly negative if $\partial\kappa_0(v) + \partial\kappa_1(v)\beta^h(v) > 0$ for all $v \in [0,T]$. As $\partial\kappa_0(v) > 0$, $\partial\kappa_1(v) < 0$, $\beta^h(v) < 0$ for all $v \in [0,T]$, the result stated follows. ■

Proof of Proposition 13. It is first shown that in equilibrium the informed can infer $Z = \omega^i G + \omega^n\phi$ from the residual demand, where $G = D_T + \zeta$. For this note that for arbitrary $\iota \in [0,1]$, $\mu(d\iota)$, a.e.,

$$\theta_t^{G^\iota|m}(G^\iota) + h_t^{G^\iota|m}(G^\iota) = \frac{\check{\sigma}^S}{\Gamma} - \check{\theta}_t^m - \omega^i\int_{[0,1]/\{\iota\}}\left(\theta_t^{G^\iota|m}(G^\iota) + h_t^{G^\iota|m}(G^\iota)\right)\mu(d\iota) - \omega^n\int_{[0,1]}\left(\theta_t^{G^\iota|m}(\phi^\iota) + h_t^{G^\iota|m}(\phi^\iota)\right)\mu(d\iota).$$

The Gaussian structure of Z, D_t implies that $\theta_t^{G^\iota|m}(G^\iota) + h_t^{G^\iota|m}(G^\iota)$ is an affine function of G^ι, Z, D_t and $\theta_t^{G^\iota|m}(\phi^\iota) + h_t^{G^\iota|m}(\phi^\iota)$ an affine function of $\phi^{\ \iota}, Z, D_t$. With a continuum of informed, each individual is negligible $\mu(d\iota)$ a.e. on $[0,1]$. Thus, $\int_{[0,1]/\{\iota\}}\left(\theta_t^{G^\iota|m}(G^\iota) + h_t^{G^\iota|m}(G^\iota)\right)\mu(d\iota) = \int_{[0,1]}\left(\theta_t^{G^\iota|m}(G^\iota) + h_t^{G^\iota|m}(G^\iota)\right)\mu(d\iota)$ and there exist coefficients $\breve{\alpha}(t)$, $\breve{\beta}(t)$, and $\breve{\gamma}(t)$ such that $\mu(d\iota)$ a.e., $\theta_t^{G^\iota|m}(G^\iota) + h_t^{G^\iota|m}(G^\iota) = \check{\sigma}^S/\Gamma - \check{\theta}_t^m - \breve{\alpha}(t)Z - \breve{\beta}(t)D_t - \breve{\gamma}(t)$. It follows that the residual demand of each privately informed individual reveals $Z = \omega^i G + \omega^n\phi$. Defining, $\varrho_t^\iota \equiv \frac{VAR_t[G]}{VAR_t[G] + VAR[\zeta^\iota]}$, individual PIPRs are,

$$\theta_t^{G^\iota|m}(x) = \frac{x - E_t[D_T]}{VAR_t[G] + VAR[\zeta^\iota]}\partial_{D_t}E_t[D_T]\sigma^D = \varrho_t^\iota\left(\frac{x - E_t[D_T]}{VAR_t[G]}\right)\partial_{D_t}E_t[D_T]\sigma^D \equiv \varrho_t^\iota\theta_t^{G|m}(x)$$

$$\theta_v^{G|m}(x) = \left(\frac{x - E_t[D_T]}{VAR_t[G]}\right)\partial_{D_t}E_t[D_T]\sigma^D = \alpha_1(v)x + \alpha_2(v)z + \beta_0(v)D_v + \gamma_0(v).$$

Define the population moments $\varrho_t^{[n]} \equiv \int_0^1 (\varrho_t^\iota)^n \mu(d\iota)$ for $n = 1, 2$, and the transformed variables,

$$\check{x}(t) \equiv \varrho_t^{[1]} x(t) \text{ and } \overline{x}(t) \equiv \check{x}(t) + \check{x}^h(t) \quad \text{for } x \in \{\alpha_1, \alpha_2, \alpha, \beta_0, \beta, \gamma_0, \gamma\}.$$

Under the conjecture that volatility $\check{\sigma}_t^S$ is deterministic and that the market price of risk is an affine function of Z, D_t given by, $\check{\theta}_t^m = \check{\sigma}_t^S/\Gamma - \overline{\alpha}(t) Z - \overline{\beta}(t) D_t - \overline{\gamma}(t)$, it holds that,

$$\begin{aligned}
\partial_{D_t} E_t\left[\check{\xi}_{t,v}^m \theta_v^{G|m}(x)^2\right] &= \partial_{D_t} E_t\left[\check{\xi}_{t,v}^m \left((\alpha_1(v)x + \alpha_2(v)Z + \beta_0(v)D_v + \gamma_0(v))\right)^2\right] \\
&= 2(\alpha_1(v)x + \alpha_2(v)Z + \gamma_0(v))\beta_0(v)\partial_{D_t} E_t\left[\check{\xi}_{t,v}^m D_v\right] + \beta_0(v)^2 \partial_{D_t} E_t\left[\check{\xi}_{t,v}^m D_v^2\right]
\end{aligned}$$

$$\begin{aligned}
\partial_{D_t} E_t\left[\check{\xi}_{t,v}^m \check{\theta}_v^m \theta_v^{G|m}(x)\right] &= \partial_{D_t} E_t\left[\check{\xi}_{t,v}^m \left(\frac{\check{\sigma}_v^S}{\Gamma} - \overline{\alpha}(v)Z - \overline{\beta}(v)D_v - \overline{\gamma}(v)\right)(\alpha_1(v)x + \alpha_2(v)Z + \beta_0(v)D_v + \gamma_0(v))\right] \\
&= \left(\left(\frac{\check{\sigma}_v^S}{\Gamma} - \overline{\alpha}(v)Z - \overline{\gamma}(v)\right)\beta_0(v) - (\alpha_1(v)x + \alpha_2(v)Z + \gamma_0(v))\overline{\beta}(v)\right)\partial_{D_t} E_t\left[\check{\xi}_{t,v}^m D_v\right] \\
&\quad -\overline{\beta}(v)\beta_0(v)\partial_{D_t} E_t\left[\check{\xi}_{t,v}^m D_v^2\right]
\end{aligned}$$

$$\partial_{D_t}\widetilde{E}^z\left[D_s \middle| \mathcal{F}_t^D\right] = \check{b}_1(t,s), \qquad \partial_{D_t}\widetilde{E}^z\left[D_v^2 \middle| \mathcal{F}_t^D\right] = \check{b}_2(t,v) + \check{c}_2(t,v)Z + 2\check{b}_1(t,v)^2 D_t.$$

As $1 > \varrho_t^\iota > 0$, it follows from Kolmogorov's strong law of large number that $\int_0^1 (\varrho_v^\iota)^n \zeta^\iota \mu(d\iota) = 0$ for $n = 1, 2$. Then, also using $G^\iota = G + \zeta^\iota$ and the expressions above leads to,

$$\begin{aligned}
k_{1,t,v} &\equiv \int_0^1 \partial_{D_t} E_t\left[\check{\xi}_{t,v}^m \theta_v^{G^\iota|m}(x)^2\right]_{|x=G^\iota} \mu(d\iota) = \int_0^1 (\varrho_v^\iota)^2 \partial_{D_t} E_t\left[\check{\xi}_{t,v}^m \theta_v^{G|m}(x)^2\right]_{|x=G^\iota} \mu(d\iota) \\
&= 2(\alpha_1(v)G + \alpha_2(v)Z + \gamma_0(v))\beta_0(v)\check{b}_1(t,v)\int_0^1 (\varrho_v^\iota)^2 \mu(d\iota) \\
&\quad +2\alpha_1(v)\beta_0(v)\check{b}_1(t,v)\int_0^1 (\varrho_v^\iota)^2 \zeta^\iota \mu(d\iota) + \beta_0(v)^2\left(\check{b}_2(t,v) + \check{c}_2(t,v)Z + 2\check{b}_1(t,v)^2 D_t\right)\int_0^1 (\varrho_v^\iota)^2 \mu(d\iota) \\
&= \left[2(\alpha_1(v)G + \alpha_2(v)Z + \gamma_0(v))\beta_0(v)\check{b}_1(t,s) + \beta_0(v)^2\left(\check{b}_2(t,v) + \check{c}_2(t,v)Z + 2\check{b}_1(t,v)^2 D_t\right)\right]\varrho_v^{[2]}
\end{aligned}$$

$$\begin{aligned}
k_{2,t,v} &\equiv \int_0^1 \partial_{D_t} E_t\left[\check{\xi}_{t,v}^m \check{\theta}_v^m \theta_v^{G^\iota|m}(x)\right]_{|x=G^\iota} \mu(d\iota) = \int_0^1 \varrho_v^\iota \partial_{D_t} E_t\left[\check{\xi}_{t,v}^m \check{\theta}_v^m \theta_v^{G|m}(x)\right]_{|x=G^\iota} \mu(d\iota) \\
&= \left(\left(\frac{\check{\sigma}_v^S}{\Gamma} - \overline{\alpha}(v)Z - \overline{\gamma}(v)\right)\beta_0(v) - (\alpha_1(v)G + \alpha_2(v)Z + \gamma_0(v))\overline{\beta}(v)\right)\check{b}_1(t,v)\int_0^1 \varrho_v^\iota \mu(d\iota) \\
&\quad -\alpha_1(v)\overline{\beta}(v)\check{b}_1(t,v)\int_0^1 \varrho_v^\iota \zeta^\iota \mu(d\iota) - \overline{\beta}(v)\beta_0(v)\left(\check{b}_2(t,v) + \check{c}_2(t,v)Z + 2\check{b}_1(t,v)^2 D_t\right)\int_0^1 \varrho_v^\iota \mu(d\iota) \\
&= \left(\left(\frac{\check{\sigma}_v^S}{\Gamma} - \overline{\alpha}(v)Z - \overline{\gamma}(v)\right)\beta_0(v) - (\alpha_1(v)G + \alpha_2(v)Z + \gamma_0(v))\overline{\beta}(v)\right)\check{b}_1(t,v)\varrho_v^{[1]} \\
&\quad -\overline{\beta}(v)\beta_0(v)\left(\check{b}_2(t,v) + \check{c}_2(t,v)Z + 2\check{b}_1(t,v)^2 D_t\right)\varrho_v^{[1]}
\end{aligned}$$

and the informed hedging components related to the PIPR become,

$$\int_0^1 h_{1t}^{G^\iota|m}(G^\iota|Z)\mu(d\iota) = -\frac{1}{2}\int_t^T k_{1,t,v}dv = \check{\psi}_{11}^{i,G}(t)G + \check{\psi}_{11}^{i,Z}(t)Z + \check{\psi}_{21}^i(t)D_t + \check{\psi}_{31}^i(t)$$

$$\int_0^1 h_{2t}^{G^\iota|m}(G^\iota|Z)\mu(d\iota) = \check{\psi}_{12}^{i,G}(t)G + \psi_{12}^{i,Z}(t)Z + \check{\psi}_{22}^i(t)D_t + \check{\psi}_{32}^i(t)$$

where,

$$\check{\psi}_{11}^{i,G}(t) = -\int_t^T \alpha_1(v)\beta_0(v)\check{b}_1(t,v)\varrho_v^{[2]}dv, \quad \check{\psi}_{11}^{i,Z}(t) = -\int_t^T \left(\alpha_2(v)\beta_0(v)\check{b}_1(t,v) + \frac{1}{2}\beta_0(v)^2\check{c}_2(t,v)\right)\varrho_v^{[2]}dv$$

$$\check{\psi}_{21}^i(t) = -\int_t^T \beta_0(v)^2\check{b}_1(t,v)^2\varrho_v^{[2]}dv, \quad \check{\psi}_{31}^i(t) = -\int_t^T \left(\gamma_0(v)\beta_0(v)\check{b}_1(t,v) + \frac{1}{2}\beta_0(v)^2\check{b}_2(t,v)\right)\varrho_v^{[2]}dv$$

$$\check{\psi}_{12}^{i,G}(t) = \int_t^T \overline{\beta}(v)\,\alpha_1(v)\,\check{b}_1(t,v)\,\varrho_v^{[1]}dv,\;\; \psi_{12}^{i,Z}(t) = \int_t^T \left(\left(\overline{\beta}(v)\,\alpha_2(v) + \beta_0(v)\,\overline{\alpha}(v)\right)\check{b}_1(t,v) + \overline{\beta}(v)\,\beta_0(v)\,\check{c}_2(t,v)\right)\varrho_v^{[1]}dv$$

$$\check{\psi}_{22}^{i}(t) = 2\int_t^T \beta_0(v)\,\overline{\beta}(v)\,\check{b}_1(t,v)^2\,\varrho_v^{[1]}dv$$

$$\check{\psi}_{32}^{i}(t) = \int_t^T \left(\left(\gamma_0(v)\,\overline{\beta}(v) + \left(\overline{\gamma}(v) - \frac{\check{\sigma}_v^S}{\Gamma}\right)\beta_0(v)\right)\check{b}_1(t,v) + \beta_0(v)\,\overline{\beta}(v)\,\check{b}_2(t,v)\right)\varrho_v^{[1]}dv.$$

The hedging components associated with the MPR depend on $h_t^u(Z) = \check{\psi}_{1t}^u Z + \check{\psi}_{2t}^u D_t + \check{\psi}_{3t}^u$ where,

$$\check{\psi}_{1t}^u = -\int_t^T \left(\overline{\alpha}(v)\,\check{b}_1(t,v) + \frac{1}{2}\overline{\beta}(v)^2\,\check{c}_2(t,v)\right)dv, \qquad \check{\psi}_{2t}^u = -\int_t^T \overline{\beta}(v)^2\,\check{b}_1(t,v)^2\,dv$$

$$\check{\psi}_{3t}^u = -\int_t^T \left(\overline{\beta}(v)\,\overline{\gamma}(v)\,\check{b}_1(t,v) + \frac{1}{2}\overline{\beta}(v)^2\,\check{b}_2(t,v)\right)dv.$$

The aggregate hedging demand is the same as in Lemma 4 with coefficients,

$$\check{\psi}_{1t}^m = \check{\psi}_{1t}^u + \check{\psi}_{11}^{i,G}(t) + \check{\psi}_{12}^{i,G}(t) + \omega\left(\check{\psi}_{11}^{i,Z}(t) + \check{\psi}_{12}^{i,Z}(t)\right)$$

$$\check{\psi}_{2t}^m = \check{\psi}_{2t}^u + \omega\left(\check{\psi}_{21}^{i}(t) + \check{\psi}_{22}^{i}(t)\right), \quad \check{\psi}_{3t}^m = \check{\psi}_{3t}^u + \omega\left(\check{\psi}_{31}^{i}(t) + \check{\psi}_{32}^{i}(t)\right)$$

still to be identified. Given the conjecture about the aggregate structure of hedging demands, it must be that $\check{\alpha}^h(t) = \check{\psi}_{1t}^m\sigma^D$, $\check{\beta}^h(t) = \check{\psi}_{2t}^m\sigma^D$ and $\check{\gamma}^h(t) = \check{\psi}_{3t}^m\sigma^D$. Therefore,

$$\begin{aligned}
\frac{\check{\alpha}^h(t)}{\sigma^D} \;=\;& -\int_t^T \left(\left(1 - \beta(v)\,\varrho_v^{[1]}\right)\overline{\alpha}(v) + \alpha(v)\left(\beta_0(v)\,\varrho_v^{[2]} - \overline{\beta}(v)\,\varrho_v^{[1]}\right)\right)\check{b}_1(t,v)\,dv \\
& -\frac{1}{2}\int_t^T \left(\overline{\beta}(v)^2 - 2\omega\beta_0(v)\left(\overline{\beta}(v)\,\varrho_v^{[1]} - \frac{1}{2}\beta_0(v)\,\varrho_v^{[2]}\right)\right)\check{c}_2(t,v)\,dv
\end{aligned}$$

$$\frac{\check{\beta}^h(t)}{\sigma^D} = -\int_t^T \left(\overline{\beta}(v)^2 - 2\omega\beta_0(v)\left(\overline{\beta}(v)\,\varrho_v^{[1]} - \frac{1}{2}\beta_0(v)\,\varrho_v^{[2]}\right)\right)\check{b}_1(t,v)^2\,dv$$

$$\begin{aligned}
\frac{\check{\gamma}^h(t)}{\sigma^D} \;=\;& -\int_t^T \left(\overline{\beta}(v)\,\overline{\gamma}(v) - \omega\left(\gamma_0(v)\,\overline{\beta}(v)\,\varrho_v^{[1]} + \left(\left(\overline{\gamma}(v) - \frac{\check{\sigma}_v^S}{\Gamma}\right)\varrho_v^{[1]} - \gamma_0(v)\,\varrho_v^{[2]}\right)\beta_0(v)\right)\right)\check{b}_1(t,v)\,dv \\
& -\int_t^T \left(\frac{1}{2}\overline{\beta}(v)^2 - \omega\beta_0(v)\left(\overline{\beta}(v)\,\varrho_v^{[1]} - \frac{1}{2}\beta_0(v)\,\varrho_v^{[2]}\right)\right)\check{b}_2(t,v)\,dv.
\end{aligned}$$

Using $\partial_t\check{b}_1(t,v)^2 = -2\sigma^D\check{b}_1(t,v)^2\left(\beta(t) + \check{\beta}^h(t) + \beta^D(t)\right)$ and $\beta(t) = \omega\beta_0(t)$ leads to,

$$\begin{aligned}
\frac{\partial_t\check{\beta}^h(t)}{\sigma^D} \;=\;& \overline{\beta}(t)^2 - 2\omega\beta_0(t)\left(\overline{\beta}(t)\,\varrho_t^{[1]} - \frac{1}{2}\beta_0(t)\,\varrho_t^{[2]}\right) \\
& -2\left(-\int_t^T \left(\overline{\beta}(v)^2 - 2\omega\beta_0(v)\left(\overline{\beta}(v)\,\varrho_t^{[1]} - \frac{1}{2}\beta_0(v)\,\varrho_t^{[2]}\right)\right)\check{b}_1(t,v)^2\,dv\right)\sigma^D\left(\beta(t) + \check{\beta}^h(t) + \beta^D(t)\right) \\
=\;& \overline{\beta}(t)^2 - 2\omega\beta_0(t)\left(\overline{\beta}(t)\,\varrho_t^{[1]} - \frac{1}{2}\beta_0(t)\,\varrho_t^{[2]}\right) - 2\check{\beta}^h(t)\left(\beta(t) + \check{\beta}^h(t) + \beta^D(t)\right) \\
=\;& \overline{\beta}(t)^2 - 2\beta(t)\left(\overline{\beta}(t)\,\varrho_t^{[1]} - \frac{1}{2}\beta_0(t)\,\varrho_t^{[2]}\right) - 2\check{\beta}^h(t)\left(\beta(t) + \beta^D(t)\right) - 2\check{\beta}^h(t)^2.
\end{aligned}$$

Straightforward simplifications, with the definitions $\overline{\beta}(t) = \beta(t) + \check{\beta}^h(t)$ and $\beta(t) = \omega\beta_0(t)$, give,

$$\begin{aligned}
\frac{\partial_t\check{\beta}^h(t)}{\sigma^D} \;=\;& \left(\beta(t) + \check{\beta}^h(t)\right)^2 - 2\beta(t)\left(\left(\beta(t) + \check{\beta}^h(t)\right)\varrho_t^{[1]} - \frac{1}{2}\beta_0(t)\,\varrho_t^{[2]}\right) - 2\check{\beta}^h(t)\left(\beta(t) + \beta^D(t)\right) - 2\check{\beta}^h(t)^2 \\
=\;& \left(1 - 2\varrho_t^{[1]}\right)\beta(t)^2 + \beta(t)\,\beta_0(t)\,\varrho_t^{[2]} - 2\check{\beta}^h(t)\left(\beta^D(t) + \beta(t)\,\varrho_t^{[1]}\right) - \check{\beta}^h(t)^2 \\
=\;& \left(1 - 2\varrho_t^{[1]} + \frac{\varrho_t^{[2]}}{\omega}\right)\beta(t)^2 - 2\check{\beta}^h(t)\left(\beta^D(t) + \beta(t)\,\varrho_t^{[1]}\right) - \check{\beta}^h(t)^2 \equiv \frac{\check{\kappa}_0(t)}{\sigma^D} + \frac{\check{\kappa}_1(t)}{\sigma^D}\beta^h(t) - \beta^h(t)^2 \quad (94)
\end{aligned}$$

where $\breve{\kappa}_0(t) \equiv \left(1 - 2\varrho_t^{[1]} + \frac{1}{\omega}\varrho_t^{[2]}\right)\beta(t)^2\sigma^D$ and $\breve{\kappa}_1(t) = -2\sigma^D\left(\beta(t)\varrho_t^{[1]} + \beta^D(t)\right)$.

Using $\breve{b}_1(t,t) = 1$, $\breve{c}_2(t,t) = 0$, $\partial_t b_1(t,v) = -\sigma^D b_1(t,v)\left(\beta(t) + \breve{\beta}^h(t) + \beta^D(t)\right)$ and

$$\partial_t \breve{c}_2(t,v) = -2\sigma^D \breve{b}_1(t,v)^2\left(\alpha(t) + \breve{\alpha}^h(t) + \alpha^D(t)\right) - \sigma^D \breve{c}_2(t,v)\left(\beta(t) + \breve{\beta}^h(t) + \beta^D(t)\right)$$

gives,

$$\begin{aligned}
\frac{\partial_t \breve{\alpha}^h(t)}{\sigma^D} &= \left(1 - \beta(t)\varrho_t^{[1]}\right)\left(\alpha(t) + \breve{\alpha}^h(t)\right) + \alpha(t)\left(\beta_0(t)\varrho_t^{[2]} - \overline{\beta}(t)\varrho_t^{[1]}\right) \\
&\quad - \int_t^T \left(\left(1 - \beta(v)\varrho_v^{[1]}\right)\overline{\alpha}(v) + \alpha(v)\left(\beta_0(v)\varrho_v^{[2]} - \overline{\beta}(v)\varrho_v^{[1]}\right)\right)\left(-\sigma^D b_1(t,v)\right)dv\left(\beta(t) + \breve{\beta}^h(t) + \beta^D(t)\right) \\
&\quad - \frac{1}{2}\int_t^T \left(\overline{\beta}(v)^2 - 2\omega\beta_0(v)\left(\overline{\beta}(v)\varrho_v^{[1]} - \frac{1}{2}\beta_0(v)\varrho_v^{[2]}\right)\right)\left(-\sigma^D \breve{c}_2(t,v)\right)dv\left(\beta(t) + \breve{\beta}^h(t) + \beta^D(t)\right) \\
&\quad - \frac{1}{2}\int_t^T \left(\overline{\beta}(v)^2 - 2\omega\beta_0(v)\left(\overline{\beta}(v)\varrho_v^{[1]} - \frac{1}{2}\beta_0(v)\varrho_v^{[2]}\right)\right)\left(-2\sigma^D \breve{b}_1(t,v)^2\right)dv\left(\alpha(t) + \breve{\alpha}^h(t) + \alpha^D(t)\right) \\
&= \left(1 - \beta(t)\varrho_t^{[1]}\right)\left(\alpha(t) + \breve{\alpha}^h(t)\right) + \alpha(t)\left(\beta_0(t)\varrho_t^{[2]} - \overline{\beta}(t)\varrho_t^{[1]}\right) - \sigma^D\left(\frac{\breve{\alpha}^h(t)}{\sigma^D}\right)\left(\beta(t) + \breve{\beta}^h(t) + \beta^D(t)\right) \\
&\quad + 2\sigma^D \int_t^T \left(\frac{1}{2}\overline{\beta}(v)^2 + \beta(v)\left(\frac{\beta(v)}{2\omega}\varrho_v^{[2]} - \overline{\beta}(v)\varrho_v^{[1]}\right)\right)\breve{b}_1(t,v)^2\,dv\left(\alpha(t) + \breve{\alpha}^h(t) + \alpha^D(t)\right) \\
&= \frac{\breve{\kappa}_2(t)}{\sigma^D}\breve{\alpha}^h(t) + \frac{\breve{\kappa}_3(t)}{\sigma^D}
\end{aligned}$$

where,

$$\frac{\breve{\kappa}_2(t)}{\sigma^D} \equiv 1 - \beta(t)\varrho_t^{[1]} - \left(\beta(t) + \breve{\beta}^h(t) + \beta^D(t)\right) + 2\sigma^D \int_t^T \left(\frac{1}{2}\overline{\beta}(v)^2 + \beta(v)\left(\frac{\beta(v)}{2\omega}\varrho_v^{[2]} - \overline{\beta}(v)\varrho_v^{[1]}\right)\right)\breve{b}_1(t,v)^2\,dv \quad (95)$$

$$\begin{aligned}
\frac{\breve{\kappa}_3(t)}{\sigma^D} &\equiv \alpha(t)\left(1 - \beta(t)\varrho_t^{[1]} + \frac{\beta(t)}{\omega}\varrho_t^{[2]} - \overline{\beta}(t)\varrho_t^{[1]}\right) \\
&\quad + 2\sigma^D \int_t^T \left(\frac{1}{2}\overline{\beta}(v)^2 + \beta(v)\left(\frac{\beta(v)}{2\omega}\varrho_v^{[2]} - \overline{\beta}(v)\varrho_v^{[1]}\right)\right)\breve{b}_1(t,v)^2\,dv\left(\alpha(t) + \alpha^D(t)\right). \quad (96)
\end{aligned}$$

Likewise, using $\breve{b}_2(t,t) = 0$, $\overline{b}(t) = \beta(t) + \breve{\beta}^h(t)$, $\beta(t) = \omega\beta_0(t)$, $\gamma(t) = \omega\gamma_0(t)$ and

$$\partial_t b_1(t,v) = -\sigma^D b_1(t,v)\left(\beta(t) + \breve{\beta}^h(t) + \beta^D(t)\right)$$

$$\partial_t \breve{b}_2(t,v) = -\sigma^D \breve{b}_2(t,v)\left(\beta(t) + \breve{\beta}^h(t) + \beta^D(t)\right) - 2\breve{b}_1(t,v)^2\left(\mu^D + \sigma^D\left(\gamma(t) + \gamma^D(t) - \frac{\breve{\sigma}_t^S}{\Gamma}\right)\right) - 2\sigma^D \breve{b}_1(t,v)^2\breve{\gamma}^h(t)$$

leads to,

$$\begin{aligned}
\frac{\partial_t \breve{\gamma}^h(t)}{\sigma^D} &= \overline{\beta}(t)\overline{\gamma}(t) - \omega\left(\gamma_0(t)\overline{\beta}(t)\varrho_t^{[1]} + \left(\left(\overline{\gamma}(t) - \frac{\breve{\sigma}_t^S}{\Gamma}\right)\varrho_t^{[1]} - \gamma_0(t)\varrho_t^{[2]}\right)\beta_0(t)\right) \\
&\quad - \int_t^T \left(\overline{\beta}(v)\overline{\gamma}(v) - \omega\left(\gamma_0(v)\overline{\beta}(v)\varrho_v^{[1]} + \left(\left(\overline{\gamma}(v) - \frac{\breve{\sigma}_v^S}{\Gamma}\right)\varrho_v^{[1]} - \gamma_0(v)\varrho_v^{[2]}\right)\beta_0(v)\right)\right)\partial_t \breve{b}_1(t,v)\,dv \\
&\quad - \int_t^T \left(\frac{1}{2}\overline{\beta}(v)^2 - \omega\beta_0(v)\left(\overline{\beta}(v)\varrho_v^{[1]} - \frac{1}{2}\beta_0(v)\varrho_v^{[2]}\right)\right)\partial_t \breve{b}_2(t,v)\,dv \\
&= \overline{\beta}(t)\overline{\gamma}(t) - \omega\left(\gamma_0(t)\overline{\beta}(t)\varrho_t^{[1]} + \left(\left(\overline{\gamma}(t) - \frac{\breve{\sigma}_t^S}{\Gamma}\right)\varrho_t^{[1]} - \gamma_0(t)\varrho_t^{[2]}\right)\beta_0(t)\right) - \sigma^D\left(\frac{\breve{\gamma}^h(t)}{\sigma^D}\right)\left(\overline{\beta}(t) + \beta^D(t)\right) \\
&\quad + 2\int_t^T \left(\frac{1}{2}\overline{\beta}(v)^2 - \beta(v)\left(\overline{\beta}(v)\varrho_v^{[1]} - \frac{\beta(v)}{2\omega}\varrho_v^{[2]}\right)\right)\breve{b}_1(t,v)^2\,dv\left(\mu^D + \sigma^D\left(\gamma(t) + \breve{\gamma}^h(t) + \gamma^D(t) - \frac{\breve{\sigma}_t^S}{\Gamma}\right)\right) \\
&= \frac{\breve{\kappa}_4(t)}{\sigma^D}\breve{\gamma}^h(t) + \frac{\breve{\kappa}_5(t)}{\sigma^D}
\end{aligned}$$

where,

$$\frac{\check{\kappa}_4(t)}{\sigma^D} \equiv -\beta^D(t) - \beta(t)\varrho_t^{[1]} + 2\sigma^D \int_t^T \left(\frac{1}{2}\overline{\beta}(v)^2 - \beta(v)\left(\overline{\beta}(v)\varrho_v^{[1]} - \frac{\beta(v)}{2\omega}\varrho_v^{[2]}\right)\right)\check{b}_1(t,v)^2\,dv \tag{97}$$

$$\begin{aligned}\frac{\check{\kappa}_5(t)}{\sigma^D} \equiv\; & \overline{\beta}(t)\gamma(t) - \left(\gamma(t)\overline{\beta}(t)\varrho_t^{[1]} + \left(\left(\gamma(t) - \frac{\check{\sigma}_t^S}{\Gamma}\right)\varrho_t^{[1]} - \frac{\gamma(t)}{\omega}\varrho_t^{[2]}\right)\beta(t)\right) \\ & + 2\int_t^T \left(\frac{1}{2}\overline{\beta}(v)^2 - \beta(v)\left(\overline{\beta}(v)\varrho_v^{[1]} - \frac{\beta(v)}{2\omega}\varrho_v^{[2]}\right)\right)\check{b}_1(t,v)^2\,dv\left(\mu^D + \sigma^D\left(\gamma(t) + \gamma^D(t) - \frac{\check{\sigma}_t^S}{\Gamma}\right)\right)\end{aligned} \tag{98}$$

and where $\check{\sigma}_t^S/\Gamma$ is as σ_t^S in (15) but with β^h replaced by $\check{\beta}^h$. ■

Proof of Proposition 14. Continuity of the fundamental process D implies $D_{T-} = D_T$, at time T_-. At T_-, the informational advantage of the informed has disappeared. The Gaussian structure and the conditional independence of dividend jumps, implies that optimal demands at T_- are $N_{T-}^{\iota} = \Gamma\left(D_{T-} + E[\epsilon] - \widetilde{S}_{T-}\right)/VAR[\epsilon]$ for $\iota \in \{u,i,n\}$. Market clearing, $\sum_{\iota\in\{u,i,n\}} N_{T-}^{\iota} = 1$, gives $VAR[\epsilon]/\Gamma = D_{T-} + E[\epsilon] - \widetilde{S}_{T-}$, or equivalently, $\widetilde{S}_{T-} = D_{T-} + E[\epsilon] - VAR[\epsilon]/\Gamma$. The value function of an informed is,

$$\mathcal{U}^{\iota} = \sup_{X_{T-}^{\iota},N_{T-}^{\iota}} \left(-\Gamma E\left[\exp\left(-\frac{X_{T-}^{\iota}}{\Gamma}\right) E\left[\exp\left(-\frac{\Delta X_T^{\iota}}{\Gamma}\right)\middle|\mathcal{F}_{T-}^m\right]\middle|\mathcal{F}_0^{\iota}\right]\right)$$

where $\Delta X_T^{\iota} = N_{T-}^{\iota}\left(D_{T-} + \epsilon - \widetilde{S}_{T-}\right) = N_{T-}^{\iota}\left(\epsilon - E[\epsilon] + VAR[\epsilon]/\Gamma\right)$. At the equilibrium price, optimal holdings are $N_{T-}^{\iota} = 1$, $\Delta X_T^{\iota} = \Delta\tilde{S}_T = \epsilon - E[\epsilon] + VAR[\epsilon]/\Gamma$. As $E_{T-}\left[\exp\left(-\frac{\Delta X_T^{\iota}}{\Gamma}\right)\right] = \exp\left(\frac{1}{2\Gamma^2}VAR[\epsilon]\right)$, it follows that,

$$\mathcal{U}^{\iota} = \sup_{X_{T-}^{\iota}} \left(-E\left[\Gamma\exp\left(-\frac{X_{T-}^{\iota}}{\Gamma}\right)\middle|\mathcal{F}_0^{\iota}\right]\right)\exp\left(\frac{1}{2\Gamma^2}VAR[\epsilon]\right).$$

For $t \in [0,T)$, the signal structure of the informed is the same as in Section 2. The optimal X_{T-}^{ι} is the same as the optimal X_T^{ι} in the model without jumps. The same holds for the optimal portfolio demands, the WAPR and the market price of risk. As $\widetilde{S}_{T-} = D_{T-} + E[\epsilon] - VAR[\epsilon]/\Gamma$, the stock price now solves the BSDE, $d\widetilde{S}_t = \sigma_t^{\tilde{S}}(\theta_t^m dt + dW_t^m)$ subject to $\widetilde{S}_{T-} = D_{T-} + E[\epsilon] - VAR[\epsilon]/\Gamma$. Therefore, for $t < T$,

$$\widetilde{S}_t = E_t\left[\widetilde{S}_{T-}\right] - \int_t^T \sigma_v^{\tilde{S}} E_t[\theta_v^m]\,dv = E_t[D_{T-}] + E[\epsilon] - VAR[\epsilon]/\Gamma - \int_t^T \sigma_v^{\tilde{S}} E_t[\theta_v^m]\,dv.$$

Volatility solves the BSDE $\sigma_t^{\tilde{S}} = \left(\partial_{D_t} E_t[D_{T-}] - \int_t^T \sigma_v^{\tilde{S}}\partial_{D_t} E_t[\theta_v^m]\,dv\right)\sigma^D$. As $E_t[D_{T-}] = E_t[D_T]$, by continuity of the fundamental process, this BSDE is the same as without dividend surprises. It follows that $\sigma_t^{\tilde{S}} = \sigma_t^S$ with σ_t^S as in Proposition 5. For $t < T$, the only effect of the surprise is that the price shifts by the constant $E[\epsilon] - VAR[\epsilon]/\Gamma$. Given the price jump $\Delta\widetilde{S}_T = \widetilde{S}_T - \widetilde{S}_{T-} = \epsilon - E[\epsilon] + VAR[\epsilon]/\Gamma$ at T, the formulas stated follow. ■

Proof of Proposition 15. The residual demand and market clearing imply $Z_t \equiv \omega^i G + \omega^n\phi_0 + \omega^n V_t = E\left[\omega^i G + \omega^n\phi_0 + \omega^n V_t\middle|\mathcal{F}_t^m\right]$ and therefore $Z_t \in \mathcal{F}_t^m$. As G, ϕ_0, V_t are mutually independent and $\mathcal{F}_{(\cdot)}^D$ is independent of $\mathcal{F}_{(\cdot)}^{W^\phi}$, it follows that the endogenous initial signal is $Z_0 = \omega^i G + \omega^n\phi_0$ and, as $dZ_t = \omega^n dV_t$, the public filtration is $\mathcal{F}_{(\cdot)}^m = \mathcal{F}_{(\cdot)}^{D,W^\phi}\bigvee\sigma(Z_0)$. The stock price $S_t = E\left[\xi_{t,T}^{m,0}D_T\middle|\mathcal{F}_t^m\right]$ becomes,

$$S_t = D_t + \mu^D(T-t) - \sigma^D\int_t^T E\left[\Psi_{t,v}^z\xi_{t,v}^{m,0}\left(\frac{\mu_v^S\sigma_v^{S,D}}{\|\sigma_v\|^2} - \theta_v^{Z|D}(z)\right)\middle|\mathcal{F}_t^{D,W^\phi}\right]_{|z=Z_0} dv \tag{99}$$

where, as Z_0 is independent of $\mathcal{F}_t^{W^\phi}$, $\theta_t^{Z|D}(z) = \sigma^D\frac{z - E[Z_0|D_t]}{VAR[Z_0|D_t]} = \alpha^D(t)z + \beta^D(t)D_t + \gamma^D(t)$ with coefficients $\alpha^D, \beta^D, \gamma^D$ as in the model with $\phi_t = \phi$, and

$$\Psi_{t,T}^z \equiv \mathcal{E}\left(\int_t^{\cdot}\theta_v^{Z|D}(z)\,dW_v^D\right)_T, \quad \eta_{t,T}^x \equiv \mathcal{E}\left(\int_t^{\cdot}\theta_v^{G|m}(x)\left(dW_v^D - \theta_v^{Z|D}(z)\,dv\right)\right)$$

$$\xi_{t,T}^{m,0} \equiv \mathcal{E}\left(-\int_t^{\cdot}\frac{\mu_v^S\sigma_v^{S,D}}{\|\sigma_v\|^2}\left(dW_v^D - \theta_v^{Z|D}(z)\,dv\right)\right)_T \mathcal{E}\left(-\int_t^{\cdot}\frac{\mu_v^S\sigma_v^{S,W^\phi}}{\|\sigma_v\|^2}dW_v^\phi\right)_T.$$

Thus $\Psi^{z}_{t,T}\xi^{m,0}_{t,T}=\mathcal{E}\left(-\int_{t}^{\cdot}\left(\frac{\mu^{S}_{v}\sigma^{S,D}_{v}}{\|\sigma_{v}\|^{2}}-\theta^{Z|D}_{v}\left(z\right)\right)dW^{D}_{v}\right)_{T}\mathcal{E}\left(-\int_{t}^{\cdot}\frac{\mu^{S}_{v}\sigma^{S,W^{\phi}}_{v}}{\|\sigma_{v}\|^{2}}dW^{\phi}_{v}\right)_{T}$ and therefore

$$\sigma'_{t}=\left[\sigma^{D},0\right]-\sigma^{D}\int_{t}^{T}\left(\mathcal{D}_{t}E\left[\Psi^{z}_{t,v}\xi^{m,0}_{t,v}\left(\frac{\mu^{S}_{v}\sigma^{S,D}_{v}}{\|\sigma_{v}\|^{2}}-\theta^{Z|D}_{v}\left(z\right)\right)\middle|\mathcal{F}^{D,W^{\phi}}_{t}\right]\right)_{|z=Z_{0}}dv.$$

The uninformed and informed hedging demands are,

$$N^{u,h}_{t}=\Gamma\int_{t}^{T}\left(h^{m}_{t,v}\left(z\right)\right)'_{|z=Z_{0}}dv\frac{\sigma_{t}}{\|\sigma_{t}\|^{2}},\quad N^{i,h}_{t}\left(G\right)=\Gamma\int_{t}^{T}\left(h^{G}_{t,v}\left(z,x\right)\right)'_{|z=Z_{0},x=G}dv\frac{\sigma_{t}}{\|\sigma_{t}\|^{2}}$$

with,

$$\left(h^{m}_{t,v}\left(z\right)\right)'\equiv-\frac{1}{2}\mathcal{D}_{t}E\left[\psi^{z}_{t,v}\xi^{m,0}_{t,v}\left(\frac{\mu^{S}_{v}}{\sigma^{S}_{v}}\right)^{2}\middle|\mathcal{F}^{D,\phi}_{t}\right]$$

$$\left(h^{G}_{t,v}\left(z,x\right)\right)'\equiv-\frac{1}{2}\mathcal{D}_{t}E\left[\psi^{z}_{t,v}\xi^{m,0}_{t,v}\theta^{G|m}_{v}\left(x\right)\left(\theta^{G|m}_{v}\left(x\right)+2\frac{\mu^{S}_{v}\rho^{S,D}_{v}}{\sigma^{S}_{v}}\right)\middle|\mathcal{F}^{D,\phi}_{t}\right]$$

and $\sigma^{S}_{v}=\|\sigma_{v}\|$. The mean-variance demands are $N^{u,m}_{t}=\Gamma\frac{\mu^{S}_{t}}{\|\sigma_{t}\|^{2}}$ and $N^{i,m}_{t}\left(G\right)=\Gamma\frac{\theta^{G|m}_{t}(G)\sigma^{S,D}_{t}}{\|\sigma_{t}\|^{2}}$, where,

$$\theta^{G|m}_{t}\left(G\right)=\sigma^{D}\left(\frac{G-E\left[G|\,Z_{0},D_{t}\right]}{VAR\left[G|\,Z_{0},D_{t}\right]}\right)=\alpha_{0}\left(t\right)G+\alpha_{1}\left(t\right)Z_{0}+\beta_{0}\left(t\right)D_{t}+\gamma_{0}\left(t\right)$$

by independence of G and $\mathcal{F}^{W^{\phi}}_{t}$. The coefficients $\alpha_{0},\alpha_{1},\beta_{0},\gamma_{0}$ are the same as in the model with $\phi_{t}=\phi$. Define the correlation coefficients $\rho^{S,D}_{t}=\sigma^{S,D}_{t}/\sigma^{S}_{t}$ and $\rho^{S,W^{\phi}}_{t}=\sigma^{S,W^{\phi}}_{t}/\sigma^{S}_{t}$. Market clearing implies,

$$\frac{\sigma^{S}_{t}}{\Gamma}=\frac{\mu^{S}_{t}}{\sigma^{S}_{t}}+\Theta_{t}\left(Z_{0};\omega\right)\rho^{S,D}_{t}+\omega^{n}\alpha_{0}\left(t\right)\rho^{S,D}_{t}V_{t}+\int_{t}^{T}h_{t,v}\left(z,x\right)'_{|z=Z_{0},x=G}dv\begin{bmatrix}\rho^{S,D}_{t}\\ \rho^{S,W^{\phi}}_{t}\end{bmatrix}\tag{100}$$

$$\Theta_{t}\left(Z_{0};\omega\right)=\left(\frac{Z_{0}-\omega E\left[G|\,D_{t},Z_{0}\right]}{VAR\left[G|\,D_{t},Z\right]}\right)\sigma^{D}=\alpha\left(t\right)Z_{0}+\beta\left(t\right)D_{t}+\gamma\left(t\right)\tag{101}$$

$$h_{t,v}\left(z,x\right)=\omega h^{G}_{t,v}\left(z,x\right)+h^{m}_{t,v}\left(z\right),\qquad\omega=\omega^{i}+\omega^{n}\tag{102}$$

with coefficients α,β,γ as in the model with $\phi_{t}=\phi$. Next, conjecture that $\frac{\mu^{S}_{t}}{\sigma^{S}_{t}}=\varphi^{Z}\left(t\right)Z_{0}+\varphi^{D}\left(t\right)D_{t}+\varphi^{V}\left(t\right)V_{t}+\varphi^{0}\left(t\right)$ for deterministic functions $\varphi^{Z},\varphi^{D},\varphi^{V},\varphi^{0}$. Then,

$$h^{m}_{t,v}\left(z\right)=-K_{t}\left(\begin{bmatrix}\varphi^{Z}z+\varphi^{0}\\ \varphi^{Z}z+\varphi^{0}\end{bmatrix}\left(v\right)+L_{v}\begin{bmatrix}E\left[\Psi^{z}_{t,v}\xi^{m,0}_{t,v}V_{v}\middle|\,\mathcal{F}^{D,W^{\phi}}_{t}\right]_{|z=Z_{0}}\\ E\left[\Psi^{z}_{t,v}\xi^{m,0}_{t,v}D_{v}\middle|\,\mathcal{F}^{D,W^{\phi}}_{t}\right]_{|z=Z_{0}}\end{bmatrix}\right)$$

$$\begin{aligned}h^{G}_{t,v}\left(z,x\right)&=-K_{t}\begin{bmatrix}\frac{\left(\alpha_{0}x+\alpha_{1}z+\gamma_{0}\right)\left(\beta_{0}+2\varphi^{D}\rho^{S,D}\right)+\beta_{0}\left(\alpha_{0}x+\left(\alpha_{1}+2\varphi^{Z}\rho^{S,D}\right)z+\gamma_{0}+2\varphi^{0}\rho^{S,D}\right)}{2}\\ \left(\alpha_{0}x+\alpha_{1}z+\beta_{0}+\gamma_{0}\right)\varphi^{V}\rho^{S,D}\end{bmatrix}\left(v\right)\\&\quad-K_{t}\begin{bmatrix}\beta_{0}\varphi^{V}\rho^{S,D}&\beta_{0}\left(\beta_{0}+2\varphi^{D}\rho^{S,D}\right)\\ 0&\beta_{0}\varphi^{V}\rho^{S,D}\end{bmatrix}\left(v\right)\begin{bmatrix}E\left[\Psi^{z}_{t,v}\xi^{m,0}_{t,v}V_{v}\middle|\,\mathcal{F}^{D,W^{\phi}}_{t}\right]_{|z=Z_{0}}\\ E\left[\Psi^{z}_{t,v}\xi^{m,0}_{t,v}D_{v}\middle|\,\mathcal{F}^{D,W^{\phi}}_{t}\right]_{|z=Z_{0}}\end{bmatrix}\end{aligned}$$

$$K_{t}\equiv\begin{bmatrix}\sigma^{D}&0\\ 0&\sigma^{\phi}\left(t,t\right)\end{bmatrix},\quad L_{v}=\begin{bmatrix}\varphi^{V}&\varphi^{D}+\left(\varphi^{D}\right)^{2}\\ \varphi^{V}+\left(\varphi^{V}\right)^{2}&\varphi^{D}\end{bmatrix}\left(v\right).\tag{103}$$

The volatility now solves,

$$\sigma'_{t}=\left[\sigma^{D},0\right]-\sigma^{D}\int_{t}^{T}\left(h^{\sigma}_{t,v}\right)'dv\quad\text{where}\quad h^{\sigma}_{t,v}=K_{t}\begin{bmatrix}\varphi^{D}\rho^{S,D}-\beta^{D}\\ \varphi^{V}\rho^{S,D}\end{bmatrix}\left(v\right).$$

Finally, $M_{t,v}(z)' \equiv \left[E\left[\Psi^z_{t,v}\xi^{m,0}_{t,v}V_v \middle| \mathcal{F}^{D,W^\phi}_t\right],\ E\left[\Psi^z_{t,v}\xi^{m,0}_{t,v}D_v \middle| \mathcal{F}^{D,W^\phi}_t\right]\right]$ solves the linear ODE,

$$\begin{aligned} M_{t,v}(z) &= \begin{bmatrix} V_t \\ D_t + \mu^D(v-t) \end{bmatrix} - \int_t^T \begin{bmatrix} \sigma^\phi(\cdot,v)\rho^{S,W^\phi}\left(\varphi^Z z + \varphi^0\right) \\ \sigma^D\left(\left(\varphi^Z\rho^{S,D} - \alpha^Z\right)z + \varphi^0\rho^{S,D} - \gamma_0\right) \end{bmatrix}(s)\,ds \\ &\quad - \int_t^T \begin{bmatrix} \sigma^\phi(\cdot,v)\rho^{S,W^\phi} & 0 \\ 0 & \sigma^D \end{bmatrix}(s) \begin{bmatrix} \varphi^V & \varphi^D \\ \varphi^V\rho^{S,D} & \varphi^D\rho^{S,D} - \beta^D \end{bmatrix}(s)\,M_{t,s}(z)\,ds. \end{aligned}$$

Given market clearing in (100) and the conjecture $\mu^S_t/\sigma^S_t = \varphi^Z(t)Z_0 + \varphi^D(t)D_t + \varphi^V(t)V_t + \varphi^0(t)$, the NREE is fully characterized by the system of backward ODEs,

$$\varphi^D(t) = -\beta(t)\rho^{S,D}_t + \frac{\omega K_t\sigma'_t}{\sigma^S_t}\int_t^T Y^\varphi_v N^D_{t,v}dv \tag{104}$$

$$\varphi^V(t) = -\omega^n\alpha_0(t)\rho^{S,D}_t + \frac{\omega K_t\sigma'_t}{\sigma^S_t}\int_t^T Y^\varphi_v N^V_{t,v}dv \tag{105}$$

$$\varphi^Z(t) = -\alpha(t)\rho^{S,D}_t + \frac{\omega K_t\sigma'_t}{\sigma^S_t}\int_t^T Y^\varphi_v N^Z_{t,v}dv \tag{106}$$

$$\varphi^0(t) = -\gamma(t)\rho^{S,D}_t - \frac{\sigma^S_t}{\Gamma} + \frac{\omega K_t\sigma'_t}{\sigma^S_t}\int_t^T Y^\varphi_v N^0_{t,v}dv \tag{107}$$

where,

$$Y^\varphi_v = \begin{bmatrix} \beta_0\varphi^V\rho^{S,D} & \beta_0\left(\beta_0 + 2\varphi^D\rho^{S,D}\right) \\ 0 & \beta_0\varphi^V\rho^{S,D} \end{bmatrix}(v) + \begin{bmatrix} \varphi^V & \varphi^D + \left(\varphi^D\right)^2 \\ \varphi^V + \left(\varphi^V\right)^2 & \varphi^D \end{bmatrix}(v)$$

and $N^D_{t,v} \equiv \partial_{D_t}M_{t,v}$, $N^V_{t,V} \equiv \partial_{V_t}M_{t,v}$, $N^Z_{t,v} \equiv \partial_Z M_{t,v}$ and $N^0_{t,v} \equiv M_{t,v} - N^Z_{t,v}Z_0 - N^D_{t,v}D_t - N^V_{t,v}V_t$ are given by,

$$N^D_{t,V} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \int_t^T Y^N_s N^D_{t,s}ds, \qquad N^V_{t,v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \int_t^T Y^N_s N^V_{t,s}ds \tag{108}$$

$$N^Z_{t,v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \int_t^T Y^N_s N^Z_{t,s}ds - \int_t^T \begin{bmatrix} \sigma^\phi(\cdot,v)\rho^{S,W^\phi}\varphi^Z \\ \sigma^D\left(\varphi^Z\rho^{S,D} - \alpha^Z\right) \end{bmatrix}(s)\,ds \tag{109}$$

$$N^0_{t,v} = \begin{bmatrix} 0 \\ \mu^D(v-t) \end{bmatrix} - \int_t^T \begin{bmatrix} \sigma^\phi(\cdot,v)\rho^{S,W^\phi}\varphi^0 \\ \sigma^D\left(\varphi^0\rho^{S,D} - \gamma_0\right) \end{bmatrix}(s)\,ds - \int_t^T Y^N_s N^0_{t,s}ds \tag{110}$$

with

$$Y^N_s = \begin{bmatrix} \sigma^\phi(\cdot,v)\rho^{S,W^\phi} & 0 \\ 0 & \sigma^D \end{bmatrix}(s) \begin{bmatrix} \varphi^V & \varphi^D \\ \varphi^V\rho^{S,D} & \varphi^D\rho^{S,D} - \beta^D \end{bmatrix}(s).$$

Thus,

$$\sigma_t = \sigma^D\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} - K_t\int_t^T \begin{bmatrix} \varphi^D\rho^{S,D} - \beta^D \\ \varphi^V\rho^{S,D} \end{bmatrix}(v)\,dv\right) \tag{111}$$

with $\sigma^S_t = \|\sigma_t\|$, $\rho^{S,D}_t = \sigma^{S,D}_t/\sigma^S_t$, $\rho^{S,W^\phi}_t = \sigma^{S,W^\phi}_t/\sigma^S_t$ and K_t as in (103). If the system (104)-(111) has a solution, a NREE exists. Using the coefficients $\varphi^Z, \varphi^D, \varphi^V, \varphi^0$ in (104)-(108) and the valuation formula (99), the stock price becomes $S_t = \Upsilon^Z(t)Z_0 + \Upsilon^D(t)D_t + \Upsilon^V(t)V_t + \Upsilon^0(t)$ where,

$$\Upsilon^Z(t) = -\sigma^D\int_t^T\left(\varphi^Z(v)\rho^{S,D}_v - \alpha^D(v)\right)dv, \qquad \Upsilon^D(t) = 1 - \sigma^D\int_t^T\left(\varphi^D(v)\rho^{S,D}_v - \beta^D(v)\right)N^{D,D}_{t,v}dv \tag{112}$$

$$\Upsilon^V(t) = -\sigma^D\int_t^T\varphi^V(v)\rho^{S,D}_v N^{V,V}_{t,v}dv, \qquad \Upsilon^0(t) = \mu^D(T-t) - \sigma^D\int_t^T\left(\varphi^0(v)\rho^{S,D}_v - \gamma^D(v)\right)dv \tag{113}$$

with $N^{D,D}_{t,v} = e'_2N^D_{t,v}$, $N^{V,V}_{t,v} = e'_1N^V_{t,v}, e'_1 = [1,0]$ and $e'_2 = [0,1]$. ■

Proof of Corollary 11. Using $\dot{\sigma}^S_t = \sigma'_t\dot{\sigma}_t/\sigma^S_t$ gives $\dot{\sigma}^S_t = \left(\rho^{S,D}_t\right)^2\varphi^D(t)\sigma^D - \sigma^D\beta^D(t)\rho^{S,D}_t + \rho^{S,D}\sqrt{1-\left(\rho^{S,D}_t\right)^2}\varphi^V(t)\sigma^V$ with boundary condition $\sigma^S_T = \sigma^D$. As $\lim_{t\uparrow T}\rho^{S,D}_t = 1$, $\lim_{t\uparrow T}\dot{\sigma}^S_t = \left(\varphi^D(T) - \beta^D(T)\right)\sigma^D = -\left(\beta(T) + \beta^D(T)\right)\sigma^D >$ 0. As $\lim_{t\uparrow T}\sigma^S_t = \sigma^D$, the stock volatility converges to σ^D from below. If $\sigma^S_t > \sigma^D$ for some $t < T$, necessarily $\dot{\sigma}^S_v < 0$

for some $v \geqslant t$. ■

Proof of Remark 12. Consider the economy without private information and where the noise trader speculates based solely on the extraneous noise V. In this economy, the WAPR is null. Market clearing gives,

$$\frac{\sigma_t^{S,ni}}{\Gamma} = \frac{\mu_t^{S,ni}}{\sigma_t^{S,ni}} + \omega^n \alpha_0^{ni}(t)\rho_t^{S,D,ni} V_t + \left(\frac{\sigma_t^{ni}}{\sigma_t^{S,ni}}\right)' \int_t^T \left(\omega h_{t,v}^{G,ni}(z,x) + h_{t,v}^{m,ni}\right)'_{|z=Z_0, x=G} dv \tag{114}$$

where $\theta_t^{S,ni} = \varphi^{V,ni}(t) + \varphi^{0,ni}(t)$ and

$$h_{t,v}^{m,ni}(z) = -K_t \left(\left[\begin{array}{c} \varphi^{0,ni} \\ \varphi^{0,ni} \end{array} \right](v) + \left[\begin{array}{c} \varphi^{V,ni}(v) E\left[\Psi_{t,v,ni}^{z} \xi_{t,v}^{m,0,ni} V_v \middle| \mathcal{F}_t^{D,W^\phi} \right]_{|z=Z_0} \\ \left(\varphi^{V,ni}(v) + \varphi^{V,ni}(v)^2\right) E\left[\Psi_{t,v}^{z,ni} \xi_{t,v}^{m,0,ni} D_v \middle| \mathcal{F}_t^{D,W^\phi} \right]_{|z=Z_0} \end{array} \right] \right)$$

$$h_{t,v}^{G,ni}(z,x) = -K_t \left[\begin{array}{c} 0 \\ (\alpha_0 x + \alpha_1 z + \beta_0 + \gamma_0)\varphi^{V,ni} \rho^{S,D,ni} \end{array} \right](v)$$

The vector of volatility coefficients now solves,

$$\sigma_t^{ni} = \left[\begin{array}{c} \sigma^D \\ 0 \end{array} \right] - \sigma^D \int_t^T K_t \left[\begin{array}{c} 0 \\ \varphi^{V,ni} \rho^{S,D,ni} \end{array} \right](v)\, dv = \left[\begin{array}{c} \sigma^D \\ -\left(\sigma^D\right) \sigma^\phi(t,t) \int_t^T \varphi^{V,ni}(v) \rho_v^{S,D,ni} dv \end{array} \right]$$

and the stock price volatility becomes,

$$\sigma_t^{S,ni} = \sigma^D \sqrt{1 + \left(\sigma^D\right)^2 \left(\int_t^T \varphi^{V,ni}(v) \rho_v^{S,D,ni} dv \right)^2} \geqslant \sigma^D \tag{115}$$

with strict inequality for $t < T$. This establishes the behavior of the volatility in the absence of private information. The equilibrium coefficients satisfy,

$$\varphi^{V,ni}(t) = -\omega^n \alpha_0(t) \rho_t^{S,D,ni} + \frac{\omega K_t \left(\sigma_t^{ni}\right)'}{\sigma_t^{S,ni}} \int_t^T Y_v^{\varphi,ni} N_{t,v}^{V,ni} dv \tag{116}$$

$$\varphi^{0,ni}(t) = -\gamma(t) \rho_t^{S,D,ni} - \frac{\sigma_t^{S,ni}}{\Gamma} + \frac{\omega K_t \left(\sigma_t^{ni}\right)'}{\sigma_t^{S,ni}} \int_t^T Y_v^{\varphi,ni} N_{t,v}^{0,ni} dv \tag{117}$$

where,

$$Y_v^{\varphi,ni} = \left[\begin{array}{cc} \varphi^{V,ni} & 0 \\ \varphi^{V,ni} + \left(\varphi^{V,ni}\right)^2 & 0 \end{array} \right](v), \qquad N_{t,v}^{V,ni} = \left[\begin{array}{c} 1 \\ 0 \end{array} \right] - \int_t^T Y_s^{N,ni} N_{t,s}^{V,ni} ds \tag{118}$$

$$N_{t,v}^{0,ni} = \left[\begin{array}{c} 0 \\ \mu^D (v-t) \end{array} \right] - \int_t^T \left[\begin{array}{c} \sigma^\phi(\cdot, v) \rho^{S,W^\phi,ni} \varphi^{0,ni} \\ \sigma^D \left(\varphi^{0,ni} \rho^{S,D,ni} - \gamma_0\right) \end{array} \right](s)\, ds - \int_t^T Y_s^{N,ni} N_{t,s}^{0,ni} ds \tag{119}$$

$$Y_s^{N,ni} = \left[\begin{array}{cc} \sigma^\phi(\cdot, v) \rho^{S,W^\phi,ni} & 0 \\ 0 & \sigma^D \end{array} \right](s) \left[\begin{array}{cc} \varphi^{V,ni} & 0 \\ \varphi^{V,ni} \rho^{S,D,ni} & 0 \end{array} \right](s). \tag{120}$$

The corresponding price coefficients are,

$$\Upsilon^{V,ni}(t) = -\sigma^D \int_t^T \varphi^{V,ni}(v) \rho_v^{S,D,ni} N_{t,v}^{V,V,ni} dv, \qquad \Upsilon^{0,ni}(t) = \mu^D (T-t) - \sigma^D \int_t^T \varphi^{0,ni}(v) \rho_v^{S,D,ni} dv. \tag{121}$$

■

Appendix C: Proofs of non-existence with exogenous noise trading

Proof of Proposition 3. Optimal informed and uninformed demands of CARA investors are related by $N_t^i = N_t^u + N_t^{i,G}(G)$ for $t \in [0,T]$. The residual demand is $N_t^a = \omega^i N_t^i + \omega^n \phi$. By market clearing, $\omega^u N_t^u + N_t^a = 1$, so that $N_t^a = 1 - \omega^u N_t^u$ for $t \in [0,T]$. Thus, N_t^a is $\mathcal{F}_t^m$-measurable for $t \in [0,T]$, i.e., N^a is adapted to the public filtration $\mathcal{F}_{(\cdot)}^m$. It follows that the process Z, where $Z_t \equiv \omega^i \left(N_t^{i,G}(G) - E\left[N_t^{i,G}(G) \middle| \mathcal{F}_t^m \right] \right) + \omega^n \phi$ is adapted to $\mathcal{F}_{(\cdot)}^m$ and therefore

$Z_t = E[Z_t | \mathcal{F}_t^m] = \omega^n E[\phi | \mathcal{F}_t^m]$. The joint Gaussianity of Z_t, G, D_t, ϕ implies that $Z_t = \omega^i \delta(t) G + \omega^n \phi$ for some function of time $\delta(t)$. Define $\mathcal{T} \equiv \{t \in [0,T] : \delta(t) \neq \delta(0)\}$ and note that if (5) does not hold, necessarily $leb(\mathcal{T}) > 0$. Then, for $t \in \mathcal{T}$,

$$\begin{bmatrix} G \\ \phi \end{bmatrix} = \begin{bmatrix} \omega^i \delta(0) & \omega^n \\ \omega^i \delta(t) & \omega^n \end{bmatrix}^{-1} \begin{bmatrix} Z_0 \\ Z_t \end{bmatrix} = \frac{1}{\Delta_t} \begin{bmatrix} \omega^n (Z_0 - Z_t) \\ -\omega^i (\delta(t) Z_0 - \delta(0) Z_t) \end{bmatrix}$$

where $\Delta_t \equiv \omega^n \omega^i (\delta(0) - \delta(t))$. Hence $G, \phi \in \mathcal{F}_t^Z$ and the equilibrium is fully revealing for $t > \underline{t}$ where $\underline{t} = \inf\{t \in [0,T] : t \in \mathcal{T}\}$. ■

Proof of Proposition 4. If (5) holds, then the equilibrium filtration becomes $\mathcal{F}_{(\cdot)}^m = \mathcal{F}_{(\cdot)}^D \bigvee \sigma(Z_0)$ where $Z_0 = \omega^i \delta(0) G + \omega^n \phi$. The proof of the proposition uses the next lemma identifying admissible endogenous signals Z_0.

Lemma 12 *Suppose that $\nabla_\phi N_t^n$ is constant for all $t \in [0,T]$. A necessary condition for the existence of a NREE is that $\delta(t) \equiv \nabla_G N_t^i(G) / (\sigma^D \Gamma)$ be constant for all $t \in [0,T]$. Under this condition, if a NREE exists, then the equilibrium filtration is $\mathcal{F}_{(\cdot)}^m = \mathcal{F}_{(\cdot)}^D \bigvee \sigma(Z_\pm)$ with initial endogenous signal $Z_0 = Z_\pm$ where,*

$$Z_\pm = \omega^i \left(\left(\frac{(\omega^n)^2 (\sigma^\phi)^2}{2(\omega^i)^2} \right) \left(1 \pm \sqrt{1 - 4 \frac{(\omega^i)^2}{(\omega^n)^2 (\sigma^\phi)^2 (\sigma^\varsigma)^2}} \right) \right) G + \omega^n \phi. \tag{122}$$

Moreover, the condition $\omega^n \geqslant 2\omega^i / (\sigma^\phi \sigma^\varsigma)$ is then necessary for the existence of a NREE.

Proof of Lemma 12. Under (5), if $\nabla_\phi N_t^n$ is constant for all $t \in [0,T]$, necessarily,

$$\delta(0) = \delta(t) = \delta(T) = \frac{\nabla_G \theta_T^{G|m}(G)}{\sigma^D} = \frac{1}{VAR[G | \mathcal{F}_T^m]} = \frac{1}{(\sigma^\varsigma)^2} \left(\frac{(\omega^i)^2 (\sigma^\varsigma)^2 \delta(0)^2 + (\omega^n)^2 (\sigma^\phi)^2}{(\omega^n)^2 (\sigma^\phi)^2} \right).$$

Therefore $\delta(0)$ must correspond to the root of $(\omega^i)^2 (\sigma^\varsigma)^2 \delta(0)^2 - (\omega^n)^2 (\sigma^\phi)^2 (\sigma^\varsigma)^2 \delta(0) + (\omega^n)^2 (\sigma^\phi)^2 = 0$. Hence,

$$\delta(0) = \left(\frac{(\omega^n)^2 (\sigma^\phi)^2}{2(\omega^i)^2} \right) \left(1 \pm \sqrt{1 - 4 \frac{(\omega^i)^2}{(\omega^n)^2 (\sigma^\phi)^2 (\sigma^\varsigma)^2}} \right) \tag{123}$$

and $\omega^n \sigma^\phi \sigma^\varsigma \geqslant 2\omega^i$ is necessary for existence. ■

Proof of Proposition 4 (continued). Let $\delta(0)$ be any real solution of (123) with associated signal $Z_\pm$ defined in (122). As $P(G \in dx | \mathcal{F}_t^m) = P(G \in dx | \mathcal{F}_t^D \bigvee \sigma(Z_\pm))$, the PIPR is,

$$\theta_t^{G|m}(G) = \sigma^D \frac{G - E[G | \mathcal{F}_t^D \bigvee \sigma(Z_\pm)]}{VAR[G | \mathcal{F}_t^D \bigvee \sigma(Z_\pm)]} = \alpha_0(t) G + \alpha_1(t) Z_\pm + \beta_0(t) D_t + \gamma_0(t)$$

with coefficients as in (19). The public filtration $\mathcal{F}_t^m = \mathcal{F}_t^D \bigvee \sigma(Z_\pm)$ is the filtration $\mathcal{F}_{(\cdot)}^D$ enlarged by $Z_\pm$. The PIPR $\theta_t^{G|m}(x)$ and the information price of risk of the endogenous signal, $\theta_t^{Z|D}(z)$, are as in the model with endogenous noise trading except that Z is replaced by $Z_\pm$. The informational hedging demand has the same structure as well.

The informational component of the informed demand is $N_t^{i,m}(G) + N_t^{i,h}(G)$ with $N_t^{i,m}(G) = \Gamma \frac{\theta_t^{G|m}(G)}{\sigma_t^S}$ and

$$N_t^{i,h}(G) = -\frac{\Gamma}{2} \int_t^T \left(\partial_{D_t} E\left[\Psi_{t,v}^z \xi_{t,v}^m \theta_v^{G|m}(x) \left(\theta_v^{G|m}(x) + 2\theta_v^m \right) \Big| D_t \right] \right)_{|z=Z_\pm, x=G} dv \frac{\sigma^D}{\sigma_t^S}$$

where $\Psi_{t,T}^z \equiv \mathcal{E}\left(\int_t \theta_v^{Z|D}(z) dW_v^D \right)_T$. Thus, $\delta(t) = \nabla_G N_t^{i,m}(G)/\Gamma + \nabla_G N_t^{i,h}(G)/\Gamma \equiv \delta^m(t) + \delta^h(t)$ where,

$$\delta^m(t) = \frac{\sigma^D}{VAR[G | \mathcal{F}_t^m] \sigma_t^S} = \left(\frac{1}{H(t)} + \left(\frac{\omega^i \delta(0)}{\omega^n \sigma^\phi} \right)^2 \right) \frac{\sigma^D}{\sigma_t^S}, \quad \text{and} \quad \delta^h(t) = -\int_t^T g(t,v) \delta^m(v) dv$$

where, $g(t,v) \equiv \partial_{D_t} E\left[\psi_{t,v}^z \xi_{t,v}^m \left(\theta_v^{G|m}(x) + \theta_v^m \right) \Big| D_t \right]_{|z=Z_\pm, x=G} \left(\frac{\sigma^D \sigma_v^S}{\sigma_t^S} \right)$.

As $\delta(t) = \delta(0)$ for all $t \in [0,T]$, it must be that $0 = \dot{\delta}^m(t) + g(t,t) \delta^m(t) - \int_t^T \partial_t g(t,v) \delta^m(v) dv$ and therefore that $\dot{\delta}^m(T) + g(T,T) \delta^m(T) = 0$. The limit expression in this equation are determined next. First, given that

$\theta_T^m = \varphi^Z(T) Z_\pm + \varphi^D(T) D_T + \varphi^0(T)$, it follows that $g(T,T) = (\beta_0(T) + \varphi^D(T))\sigma^D$, $\delta^m(T) = \frac{1}{(\sigma^\varsigma)^2} + \left(\frac{\omega^i \delta(0)}{\omega^n \sigma^\phi}\right)^2$. Second, from the definition of δ^m, $\lim_{t\uparrow T} \dot{\delta}^m(t) = \frac{(\sigma^D)^2}{(\sigma^\varsigma)^4} - \delta^m(T) \lim_{t\uparrow T} \partial_t \log(\sigma_t^S)$. Differentiating

$$S_t = E\left[\xi_{t,T}^m D_T \middle| \mathcal{F}_t^m\right] = D_t + \mu^D(T-t) - \sigma^D \int_t^T E\left[\Psi_{t,v}^z \xi_{t,v}^m \left(\theta_v^m - \theta_v^{Z|D}(z)\right) \middle| D_t\right]_{|z=Z_\pm} dv$$

leads to the volatility,

$$\sigma_t^S = \sigma^D - \left(\sigma^D\right)^2 \int_t^T \left(\partial_{D_t} E\left[\Psi_{t,v}^z \xi_{t,v}^m \left(\theta_v^m - \theta_v^{Z|D}(z)\right) \middle| D_t\right]\right)_{|z=Z_\pm} dv.$$

Thus, $\lim_{t\uparrow T} \sigma_t^S = \sigma^D$ and as

$$\partial_t \sigma_t^S = \left(\sigma^D\right)^2 \left(\varphi^D(t) - \beta^D(t)\right) - \left(\sigma^D\right)^2 \int_t^T \partial_t \left(\partial_{D_t} E\left[\Psi_{t,v}^z \xi_{t,v}^m \left(\theta_v^m - \theta_v^{Z|D}(z)\right) \middle| D_t\right]\right)_{|z=Z_\pm} dv$$

where $\beta^D(t)$ is defined in (31), it follows that $\lim_{t\uparrow T} \partial_t \sigma_t^S / \sigma_t^S = \sigma^D(\varphi^D(T) - \beta^D(T))$. Substituting these limits in $\dot{\delta}^m(T) = \frac{(\sigma^D)^2}{(\sigma^\varsigma)^4} - \delta^m(T) \lim_{t\uparrow T} \partial_t \log(\sigma_t^S)$ and using $\dot{\delta}^m(T) + g(T,T)\delta^m(T) = 0$, shows that $\delta(0)$ must solve

$$0 = \frac{(\sigma^D)^2}{(\sigma^\varsigma)^4} + \sigma^D \left(\frac{1}{(\sigma^\varsigma)^2} + \left(\frac{\omega^i \delta(0)}{\omega^n \sigma^\phi}\right)^2\right)\left(\beta^D(T) + \beta_0(T)\right) \tag{124}$$

where,

$$\beta^D(T) = -\frac{(\omega^i)^2 \sigma^D}{\left((\omega^i)^2 (\sigma^\varsigma)^2 + (\omega^n)^2 (\sigma^\phi)^2\right)}, \quad \beta_0(T) = -\frac{\sigma^D (\omega^n)^2 (\sigma^\phi)^2}{\left((\omega^n)^2 (\sigma^\phi)^2 + (\omega^i)^2 (\sigma^\varsigma)^2\right)(\sigma^\varsigma)^2}.$$

If $\omega^i > 0$, $\omega^n > 0$ and $\sigma^\phi > 0$, then $\delta(0)$ given in (123) does not solve (124). Therefore, $\delta(t)$ cannot be constant and a NREE does not exist. ■

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Figure 1: **Sensitivity of WAPR to private information:** The figure shows the dynamic behavior of $\alpha(t)$ for $t \in [0, T]$. Parameter values are $T = 1$, $\sigma^D = 0.2$, $\mu^D = 0.05$, $\mu^\phi = E[G]$, $\sigma^\phi = STD[G]$, $\sigma^\zeta = 0.1$. The weight of the informed is $\omega^i = 1/10$. The weight of the speculating noise trader varies between $\omega^n = 1/125$ (left panel), $\omega^n = 1/60$ (middle panel) and $\omega^n = 1/4$ (right panel).

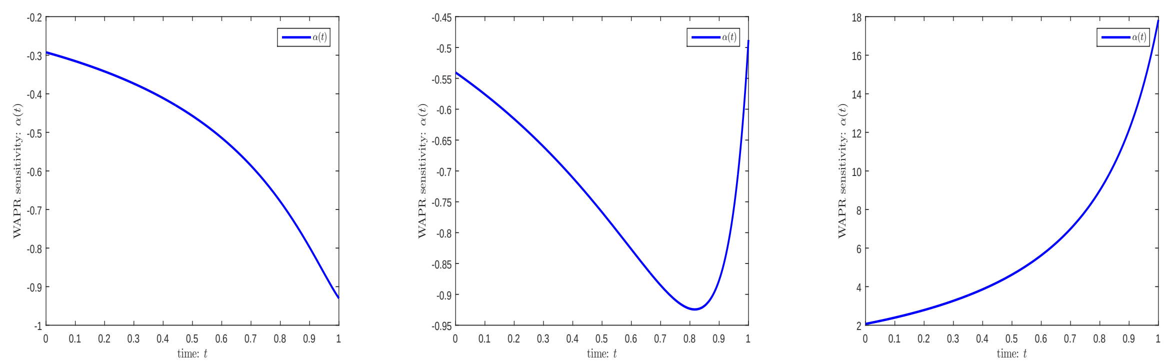

Figure 2: **Stock volatility with and without hedging, Markovian volatility and sensitivity of WAPR and stock price to fundamental information:** The figure shows the dynamic behavior of σ_t^S (with hedging), $\sigma_t^{S,nh}$ (no hedging), $\sigma_t^{S,M}$ (Markovian), $\widehat{\beta}(t)$, $\beta(t)$ and $\widehat{B}(t)$, $B(t)$ for $t \in [0,T]$. The Markovian volatility is $\sigma_t^{S,M} \equiv \sqrt{\lim_{h\to 0} VAR\left[S_{t+h} - S_t \middle| S_t\right]} = \sigma_t^S \sqrt{1 + 2\omega^i \widehat{A}(t) / \widehat{B}(t)}$. Parameter values are $\Gamma = 1/4$, $T = 1$, $\sigma^D = 0.2$, $\mu^D = 0.05$, $\mu^\phi = E[G]$, $\sigma^\phi = STD[G]$. The weights of informed and the speculating noise traders are $\omega^i = 1/10$ and $\omega^n = 1/5$. Signal noise volatility varies between $\sigma^\zeta = 0.25$ (left panel), $\sigma^\zeta = 0.1$ (middle panel), and $\sigma^\zeta = 0.05$ (right panel).

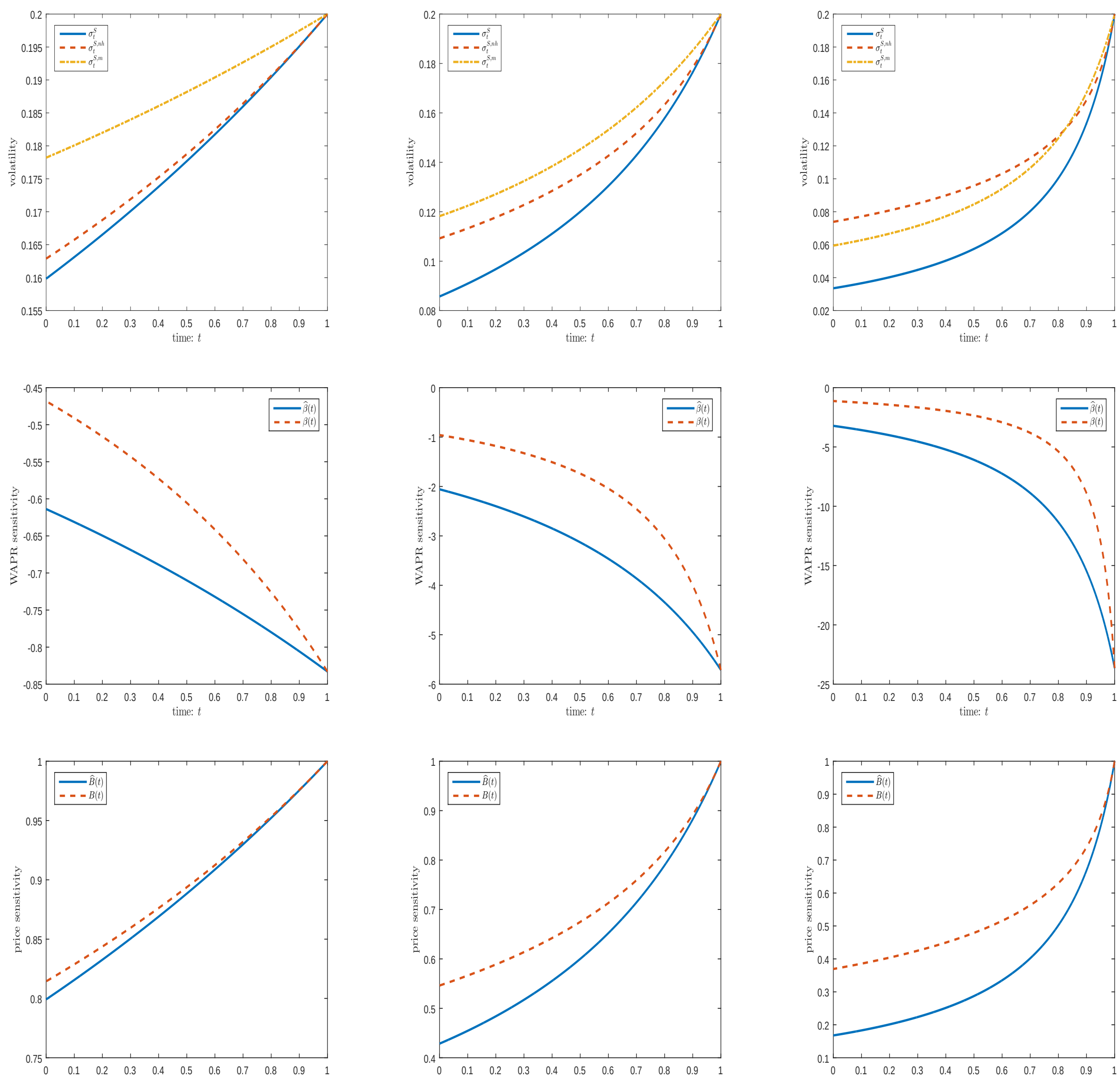

Figure 3: **Pareto ranking of equilibria:** This figure shows the different components of $\Delta\hat{P}/\Gamma + \Delta\hat{T}^u$ as a function of risk tolerance Γ. Parameter values are $T = 1/24$, $D_0 = 1$, $\omega^i = 1/10$, $\omega^n = 1/5$, $\sigma^D = 1.18$, $\mu^D = 0.05$, $\mu^\phi = E[G]$, $\sigma^\phi = STD[G]$. The precision of the private information signal varies between $\sigma^\zeta = 0.5$ (left panels), $\sigma^\zeta = 0.1$, (middle panels) and $\sigma^\zeta = 0.05$ (right panels). The top row shows welfare components based on optimal demands. The bottom row shows welfare components based on demands excluding hedging components.

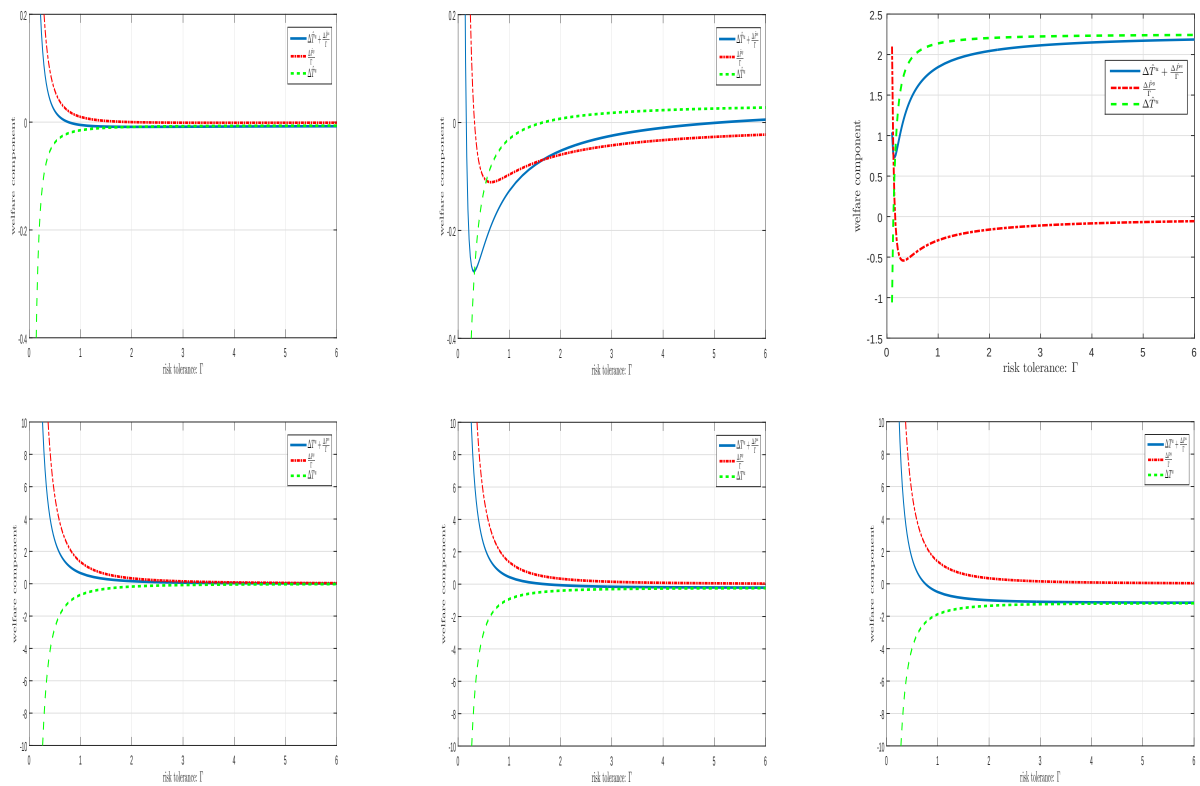

Figure 4: **Information acquisition gains without speculative noise trading:** This figure shows the gains from information acquisition in the absence of speculative noise traders as a function of risk tolerance Γ ($GA = CE^{i,wa} - CE^{i,na}$) where $CE^{i,wa}$ and $CE^{i,na}$ are the certainty equivalents of the informed with and without information production. Parameter values are $T = 1$, $D_0 = 1$, $\omega^i = 1/10$, $\sigma^D = 0.2$, $\mu^D = 0.05$, $\mu^\phi = E[G]$, $\sigma^\phi = STD[G]$. The precision of the private information signal is $\sigma^\zeta = 0.25$ (left panel), $\sigma^\zeta = 0.10$ (middle panel) and $\sigma^\zeta = 0.05$ (right panel). Certainty equivalent acquisition costs are $C = 0.5$