

A Reexamination of Contingent Convertibles with Stock Price Triggers*

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Comments Welcome

Abstract

Initial proposals for bank contingent convertibles (CoCos) envisioned that these bonds would convert to equity when the bank's stock price declined to a pre-specified trigger. Subsequent research claimed that doing so causes the stock price to have multiple equilibria or no equilibrium. We show that when CoCos are perpetuities, which characterizes most actual CoCos, a unique stock price equilibrium exists except under unrealistic conditions. A unique equilibrium occurs when conversion terms favor or disfavor CoCo investors and when CoCos convert to equity or are written down. We illustrate these results by developing models of CoCos with perpetual or finite maturities.

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1 Introduction

The financial crisis that began in 2007 stimulated interest in contingent convertibles (CoCos). CoCos are convertible debt issued by banks that either convert to new equity shares or experience a principal write down when a particular triggering event occurs. Since Lloyds Bank issued the first CoCo in 2009, there have been over 300 different CoCo issues that, in total, have raised over \$360 billion.¹ As initially proposed by Flannery (2005), CoCos were envisioned to convert from debt to equity when the market value of the bank’s shareholders’ equity declined to a pre-specified trigger level. Subsequent academic research has considered several variations of CoCos with market value triggers.²

CoCos are potentially valuable instruments for stabilizing individual banks and the financial system as a whole. CoCos have the advantages of debt during normal times, such as lower agency costs of free cash flow (Jensen (1986)) or corporate tax-deductibility of interest expense. During financial downturns, they can possess the benefits of equity by lowering the likelihood of financial distress and bankruptcy.³ Relative to non-convertible debt, properly-designed CoCos also can reduce a bank’s risk-shifting incentives and mitigate debt overhang (the disincentive to replenish capital following losses).⁴

A key requirement for CoCos to promote financial stability is that they convert from debt to equity in a timely manner while the bank is still a going concern. Many academics and policymakers believe this can only happen if conversion is triggered at the onset of financial stress by a market measure, such as the bank’s stock price or its total market value of equity. However, in practice, all CoCos issued thus far do not have market value triggers but regulatory (book value) capital triggers, typically a core Tier 1 capital to risk-weighted assets ratio of 7%. Unfortunately, regulatory capital ratios are unlikely to trigger CoCo conversion at early stages of a bank’s distress. Haldane (2011) documents that the average regulatory capital ratios for large banks that failed or required public assistance following the 2008 Lehman Brothers bankruptcy, which he refers to as “crisis banks,”

¹Moodys Global Credit Research 10 February 2015 and 24 September 2015.

²This research includes Bulow and Klemperer (2015), Calomiris and Herring (2013), Flannery (2009), Hart and Zingales (2011), McDonald (2013), and Pennacchi et al. (2014). An exception is Glasserman and Nouri (2012a) who model CoCos with regulatory capital triggers.

³Albul et al. (2013) show that CoCos can increase firm value and reduce the chance of costly bankruptcy or bailout in a setting with corporate taxes and endogenous default.

⁴An incomplete list of research showing that suitably-designed CoCos improve risk incentives include Berg and Kaserer (2015), Calomiris and Herring (2013), Hilscher and Raviv (2014), Martynova and Perotti (2015), and Pennacchi et al. (2014). An anecdotal indication that debt overhang continues to be a problem is “Lenders’ Capital Raising Plans Hit by Sell-off,” *Financial Times*, 27 August 2015.

were virtually indistinguishable from the regulatory capital ratios of large banks that did not require assistance, referred to as “no-crisis banks.” Indeed, the average regulatory capital ratio of the crisis banks actually rose above that of the no-crisis banks shortly before Lehman’s failure. This odd behavior is consistent with evidence by Mariathasan and Merrouche (2014), Begley et al. (2015), and Plosser and Santos (2015) that distressed banks manipulate upward their regulatory capital ratios. In contrast, Haldane shows that crisis banks’ average market value of equity ratio declined substantially well before the Lehman failure, falling far below the more stable average market equity ratio of the no-crisis banks. Consequently, market equity values were more accurate and timely indicators of actual distress and potentially more suitable for triggering a CoCo conversion while the bank remains a going concern.

Yet, many policymakers and some academics have become skeptical of CoCos that have triggers linked to market values. In part, their distrust derives from the analysis of Sundaresan and Wang (2015) who conclude that if a bank issues CoCos triggered by its stock price, then the bank’s stock price will have multiple equilibria or no equilibrium.⁵ Their basic claim is that when CoCo conversion terms are advantageous to CoCo investors relative to the bank’s initial shareholders, there will be multiple equilibria for the bank’s stock price. In the opposite case where conversion terms are advantageous to the bank’s initial shareholders relative to the CoCo investors, there is no stock price equilibrium.⁶ Concerns over whether CoCos should have market price triggers have been expressed by the Basel Committee on Banking Supervision (2010), and they along with national bank supervisory authorities have called for further study of the appropriate design of CoCos, particularly going-concern CoCos with market value triggers.⁷

One recent study challenges some of the conclusions of Sundaresan and Wang (2015). Glasserman and Nouri (2012b) find that under the same continuous-time setting assumed by Sundaresan and Wang (2015), a unique equilibrium for the bank’s stock price occurs in situations where Sundaresan and Wang (2015) claim that there are multiple equilibria. However, Glasserman and Nouri (2012b) confirm the result of Sundaresan and Wang (2015) with regard to the conditions under which the bank’s stock price has no equilibrium.

⁵Examples of economists citing Sundaresan and Wang (2015) and expressing concerns over multiple stock price equilibria are Avdjiev et al. (2013) (Bank for International Settlements) and Leitner (2012) (U.S. Federal Reserve).

⁶Sundaresan and Wang (2015) find only a knife-edge case where a unique stock price exists, namely where conversion terms are neutral to shareholders and CoCo investors.

⁷See the Basel Committee on Banking Supervision (2013) and the U.S. Financial Stability Oversight Council (2012) which both recommend further review of CoCos.

Our paper re-examines several issues related to CoCos with stock price triggers. One of our main results is to identify an error in a critical proof by Sundaresan and Wang (2015) which explains why their claim of multiple equilibria differs from the unique equilibrium result in Glasserman and Nouri (2012b). We confirm Glasserman and Nouri (2012b) and provide additional intuition by deriving a CoCo valuation model that leads to a closed-form solution for the issuing bank’s unique equilibrium stock price. This model represents a clear counter-example to the proof in Sundaresan and Wang (2015).

We go further to also reexamine the conditions under which a CoCo-issuing bank’s stock price has no equilibrium, an issue where Sundaresan and Wang (2015) and Glasserman and Nouri (2012b) largely agree. These prior papers analyzed this issue mainly in the context of a CoCo that has a finite maturity. We show that if, instead, CoCos have a perpetual maturity, the conditions under which there is no stock price equilibrium shrink substantially. For realistic parameter values, our model implies that the equilibrium is unique. Counter to the finite-maturity implications of Sundaresan and Wang (2015) and Glasserman and Nouri (2012b), with a perpetual maturity uniqueness can exist when conversion terms are unfavorable to CoCo investors or when the stock price triggers a CoCo principal write down. This result has practical significance since the vast majority of CoCos issued thus far have conversion or write-down terms that harm CoCo investors.⁸ We next derive a finite-maturity counterpart to our perpetual-maturity model and show that it is the lump sum principal payment at maturity that destroys the existence of a unique stock price equilibrium when conversion terms are unfavorable to CoCo investors.

Our result that perpetual-maturity CoCos can have a unique equilibrium stock price under a variety of realistic conditions is important because, in practice, most CoCos have perpetual maturities. Under Basel III, CoCos must have a perpetual maturity to qualify as “Additional Tier 1” capital, rather than Tier 2 capital (Basel Committee on Banking Supervision (2011), page 15). Berg and Kaserer (2015) document that of the 22 different CoCos issued during the 2009 to 2013 period that they analyze, 12 had perpetual maturities while 10 had finite maturities. Similarly, Avdjiev et al. (2015) report that 57.5% of their sample of 187 CoCos issued during the September 2009 to March 2015 period have a perpetual maturity.⁹ Hence, it appears that for the foreseeable future banks may have a preference for issuing CoCos as perpetuities.

⁸See Berg and Kaserer (2015) and Avdjiev et al. (2015).

⁹See also “Investors in Asia Return to Perpetual Bonds,” *Wall Street Journal* 8 July 2014 which states that Australian bank CoCos have led the issuance of perpetual bonds in Asia. Moreover, the article “Corporate Issuance of Perpetual Debt Soars,” *Financial Times* 16 June 2015 notes that banks have increased their issuance of perpetual CoCos.

The plan of our paper is as follows. Section 2 develops a structural model of a bank that funds its assets with senior debt, CoCos with a stock (equity) price trigger, and shareholders' equity. The model fits within the same general continuous-time framework assumed by Sundaresan and Wang (2015) and Glasserman and Nouri (2012b) except that the CoCo is assumed to have a perpetual maturity. We prove that when conversion terms are favorable to CoCo investors, a unique stock price equilibrium exists, rather than multiple equilibria, confirming with a closed-form stock price solution the finding of Glasserman and Nouri (2012b). In addition, we also prove that a unique stock price equilibrium can exist when, instead, conversion terms are favorable to the bank's initial shareholders, including cases where conversion leads to a write down of the CoCo's principal. Using realistic parameter values, uniqueness appears to be likely. Moreover, extending the model to include direct costs of bankruptcy or permitting the issuing bank to call its CoCo enlarges the set of parameter values that lead to stock price uniqueness.

Section 3 critiques Sundaresan and Wang (2015) by directly identifying the error in the proof of their main theorems. We also use our model to illustrate a counter-example to their proof's logic. Section 4 provides additional intuition for the result in Glasserman and Nouri (2012b) that a stock price equilibrium does not exist when conversion terms favor the bank's initial shareholders and CoCos have a finite maturity. We modify our previous model to consider a finite-maturity CoCo and show that the candidate stock price can never be an equilibrium when conversion terms are favorable to the bank's initial shareholders. Combining this result with that of our perpetual-maturity model shows that the requirement of a lump sum principal payment eliminates many conditions where a unique equilibrium would otherwise exist. Section 5 concludes.

2 A Model of Perpetual-Maturity CoCos

This section develops a structural model of a bank that funds its assets with senior debt, a perpetual-maturity CoCo that has a stock (equity) price trigger, and shareholders' equity. It shows that a unique stock price equilibrium can exist for a broad set of conditions. Other than assuming the CoCo has a perpetual maturity, the structure of the model fits within the frameworks of Sundaresan and Wang (2015) and Glasserman and Nouri (2012b).

2.1 Model Assumptions

Assume that a bank's assets generate a continuous cashflow whose rate per unit time is a_t at date t . This cashflow rate follows the physical process

$$da_t = \mu^P a_t dt + \sigma a_t dz_t^P \quad (1)$$

where μ and σ are constants and dz_t^P is a Brownian motion process. If θ is the constant market price of risk associated with the Brownian motion dz_t^P , then the risk-neutral process for the cashflows is

$$da_t = \mu a_t dt + \sigma a_t dz_t$$

where $dz_t = dz_t^P + \theta dt$ and $\mu = \mu^P - \theta\sigma$. Let $r > 0$ be the constant risk-free rate of interest. Then assuming $\mu < r$, the market value of the bank's assets, A_t , equals

$$A_t = E^Q \left[\int_t^\infty e^{-r(s-t)} a_s ds \right] = \frac{a_t}{r - \mu} . \quad (2)$$

Thus, the value of the bank's assets, without reinvested cashflows, follows the physical process

$$\frac{dA_t}{A_t} = \mu^P dt + \sigma_t dz_t^P \quad (3)$$

and its physical return including cashflows is

$$\begin{aligned} dA_t &= (\mu^P A_t + a_t) dt + A_t \sigma dz_t^P \\ &= (\mu^P A_t + (r - \mu) A_t) dt + \sigma A_t dz_t^P \\ &= (r + \theta\sigma) A_t dt + \sigma A_t dz_t^P . \end{aligned} \quad (4)$$

It is easy to see from the process (4) that the risk-neutral rate of return for the bank's assets, including cashflows, is the risk-free rate r .

The bank's liabilities consist of shareholders' equity, senior debt, and CoCos having a perpetual maturity. Senior debt has a principal value of B and pays fixed coupon interest continuously at the annual rate of b . Prior to conversion, CoCos also pay fixed coupon interest continuously at the annual rate of c , and they have a principal value of C . It is assumed that bank regulators close the bank whenever the value of its assets reaches the default-free value of the bank's non-convertible debt. Specifically, the bank is closed when assets fall to bB/r , at which time senior debtholders own all of the assets

and equityholders are wiped out. Thus senior debt is default-free.¹⁰

The initial equityholders of the bank own n shares of stock. CoCos are assumed to convert to m additional shares when the bank's per share stock price falls to the level $L > 0$. It is assumed that equityholders receive a continuous dividend per share of $[(a_t - bB - cC) / n] dt$ prior to the CoCos' conversion and $[(a_t - bB) / (n + m)] dt$ following the CoCos' conversion.¹¹ Thus, as in Sundaresan and Wang (2015) and Glasserman and Nouri (2012b), it is assumed that the assets' cashflows, a_t , are paid out of the bank and divided between the bank's debtholders and equityholders. Let $\tau_\delta = \inf \{t \in [0, \infty) : A_t \leq bB/r\}$, which is the bank's closure (bankruptcy) date. It will be shown to occur after the date of the CoCos' conversion. Note that using equation (2) one sees that the dividend flow to equityholders just prior to the bank's closure date equals

$$[((r - \mu) bB/r - bB) / (n + m)] dt \tag{5}$$

which is positive (*negative*) if $\mu = \mu^P - \theta\sigma$ is negative (*positive*). Thus, one can rule out negative dividends by assuming $\mu^P \leq \theta\sigma$.

2.2 The Equilibrium Stock Price

Consider two hypothetical banks whose assets and senior debt are identical to the previously-discussed bank that issued CoCos. The first is referred to as a "post-conversion" bank. It has no CoCos but $n + m$ shares of equity. Its stock price per share is denoted $U_t(A_t)$. The second reference bank is called a "no-conversion" bank. It has n shares of equity and additional non-convertible debt with the same coupon interest and principal as the CoCos of the CoCo-issuing bank. This no-conversion bank's stock price is denoted $V_t(A_t)$.¹² As before, it is assumed that regulators close these banks whenever assets fall to the default-free value of the bank's non-convertible debt, which for the no-conversion bank is $(bB + cC) / r$. Since these two banks' nonconvertible debts are default-free, we can immediately derive their total equity as the residual asset value. The post-conversion bank's

¹⁰Initially, we assume there are no direct costs of bankruptcy and senior debt is default-free. Later, we extend the model to incorporate bankruptcy costs and permit default-risky senior debt.

¹¹We begin by assuming that CoCos cannot be called and, later, extend the model to allow the bank to call its CoCos at their par value.

¹²Defining this no-conversion bank is unnecessary for our derivation of the equilibrium stock price when CoCos have a perpetual maturity. However, it serves as a useful benchmark and was used in the analysis of Glasserman and Nouri (2012b).

stock price per share prior to its closure is

$$\begin{aligned}
U_t &= \frac{1}{n+m} E_t^Q \left[\int_t^{\tau_\delta} e^{-r(s-t)} (a_s - bB) ds \right] \\
&= \frac{1}{n+m} \left\{ E_t^Q \left[\int_t^\infty e^{-r(s-t)} (a_s - bB) ds \right] - E_t^Q \left[\int_{\tau_\delta}^\infty e^{-r(s-t)} (a_s - bB) ds \right] \right\} \\
&= \frac{1}{n+m} \left\{ \left(A_t - \frac{bB}{r} \right) - E_t^Q \left[e^{-r(\tau_\delta-t)} E_{\tau_\delta}^Q \left[\int_{\tau_\delta}^\infty e^{-r(s-\tau_\delta)} (a_s - bB) ds \right] \right] \right\} \\
&= \frac{1}{n+m} \left\{ \left(A_t - \frac{bB}{r} \right) - E_t^Q \left[e^{-r(\tau_\delta-t)} \left(A_{\tau_\delta} - \frac{bB}{r} \right) \right] \right\} \\
&= \frac{1}{n+m} \left(A_t - \frac{bB}{r} \right)
\end{aligned} \tag{6}$$

where the last line uses the fact that regulators close the bank when assets equal bB/r . Note that the post-conversion stock price U_t always exists, is unique, and is strictly increasing in the asset level A_t .

Similar logic implies that the no-conversion bank's stock price is

$$V_t = \frac{1}{n} \left(A_t - \frac{bB + cC}{r} \right). \tag{7}$$

Now define A_{uc} as the asset value such that the post-conversion bank's per share stock price equals L ; that is, $U(A_{uc}) = L$. Hence, from (6) we have

$$A_{uc} = L(n+m) + \frac{bB}{r}. \tag{8}$$

Similarly, define A_{vc} as the asset value such that the no-conversion bank's per share stock price equals L ; that is $V(A_{vc}) = L$. Using (7) we have

$$A_{vc} = Ln + \frac{bB + cC}{r}. \tag{9}$$

Next define $\tau_{uc} = \inf \{t \in [0, \infty) : A_t \leq A_{uc}\}$ as the date when the post-conversion bank's stock price equals L . Note that $\tau_{uc} < \tau_\delta$ when $L > 0$, which holds by assumption.

Let us now return to the CoCo-issuing bank. What we show next is that *if* this bank's equilibrium stock price exists, then the price must be continuous and, furthermore, conversion must occur at date τ_{uc} , which is when the post-conversion bank's stock price equals L . We begin by defining the conditions for an equilibrium stock price, namely, that conversion must occur when the stock price first equals L and that the stock price

must reflect the value of dividends received before and after conversion.

Definition: A pair of a conversion time, $\hat{\tau}$, and a pre-conversion per-share equity value, \hat{S}_t , is an equilibrium if $\hat{\tau}$ is a stopping time adapted to the filtration generated by the Brownian motion z_t such that

$$\hat{\tau} = \inf \left\{ t \in [0, \infty) : \hat{S}_t \leq L \right\}, \quad (10)$$

and

$$\hat{S}_t = E_t^Q \left[\int_t^{\tau_\delta} e^{-r(s-t)} \left(1_{\{s \leq \hat{\tau}\}} \frac{1}{n} (a_s - bB - cC) + 1_{\{s > \hat{\tau}\}} \frac{1}{n+m} (a_s - bB) \right) ds \right]. \quad (11)$$

Speaking intuitively, for $\hat{\tau}$ to be adapted to the filtration generated by the Brownian motion determining the bank's asset cashflow process, it should be possible to decide whether or not $\{\hat{\tau} \leq t\}$ has occurred on the basis of the knowledge of the history of the underlying Brownian motion on $[0, t]$.¹³ Equation (11) says that the value of equity per share is equal to the present value of dividends received per share before and after the conversion date $\hat{\tau}$.

We next show that since the equity value process is adapted to the Brownian motion z_t , and dividends are paid continuously, \hat{S}_t must be continuous over time.

Lemma 1: For any stopping time $\hat{\tau}$ adapted to the filtration generated by Brownian motion z_t , \hat{S}_t is continuous in t .

Proof: Consider a process equal to the per-share value of past and future dividends discounted as of an initial date 0:

$$X_t = E_t^Q \left[\int_0^{\tau_\delta} e^{-rs} \left(1_{\{s \leq \hat{\tau}\}} \frac{1}{n} (a_s - bB - cC) + 1_{\{s > \hat{\tau}\}} \frac{1}{n+m} (a_s - bB) \right) ds \right] \quad (12)$$

where $0 \leq t \leq \tau_\delta$. By the law of iterated expectations, X_t is a martingale adapted to the filtration generated by Brownian motion z_t . Hence, X_t is a continuous process since all martingales adapted to a Brownian filtration are continuous. Equation (12) can be

¹³Since the history of the Brownian motion z_t can be inferred from the history of the asset process A , $\hat{\tau}$ is a function of the history of the asset level.

rewritten as

$$X_t = \int_0^t e^{-rs} \left(1_{\{s \leq \hat{\tau}\}} \frac{1}{n} (a_s - bB - cC) + 1_{\{s > \hat{\tau}\}} \frac{1}{n+m} (a_s - bB) \right) ds \quad (13)$$

$$+ e^{-rt} E_t^Q \left[\int_t^{\tau_\delta} e^{-r(s-t)} \left(1_{\{s \leq \hat{\tau}\}} \frac{1}{n} (a_s - bB - cC) + 1_{\{s > \hat{\tau}\}} \frac{1}{n+m} (a_s - bB) \right) ds \right].$$

Applying (11) yields

$$X_t = \int_0^t e^{-rs} \left(1_{\{s \leq \hat{\tau}\}} \frac{1}{n} (a_s - bB - cC) + 1_{\{s > \hat{\tau}\}} \frac{1}{n+m} (a_s - bB) \right) ds + e^{-rt} \hat{S}_t \quad (14)$$

Since X_t and the time integral are continuous in t , \hat{S}_t must be continuous in t .

Because \hat{S}_t is a continuous process, the equity value just before and after conversion must be the same. Now note that at the time of conversion, the CoCo-issuing bank becomes identical to the post-conversion bank. Therefore, the CoCo-issuing bank's stock price must equal that of the post-conversion bank at date $\hat{\tau}$:

$$\hat{S}_{\hat{\tau}} = \frac{1}{n+m} E_{\hat{\tau}}^Q \left[\int_{\hat{\tau}}^{\tau_\delta} e^{-r(s-\hat{\tau})} (a_s - bB) ds \right] = U_{\hat{\tau}} = \frac{1}{n+m} \left(A_{\hat{\tau}} - \frac{bB}{r} \right). \quad (15)$$

An implication of Lemma 1 and (15) is that when an equilibrium stock price, \hat{S}_t , exists, it must lead to conversion only when $A_t = A_{uc}$. That is because from (15) and (8), $\hat{S}_{\hat{\tau}} = L$ if and only if $A_{\hat{\tau}} = A_{uc}$. If conversion happens when $A_t > A_{uc}$, then according to (15) $\hat{S}_{\hat{\tau}} > L$, meaning that conversion cannot happen at A_t . If conversion happens when $A_t < A_{uc}$, then $\hat{S}_{\hat{\tau}} < L$, and because of the continuity of \hat{S}_t the stock value before conversion must be below L , implying that conversion should have happened earlier. Thus, we have proven the following proposition.

Proposition 1: *If there is an equilibrium, then conversion happens when $A_t = A_{uc}$, that is,*

$$\hat{\tau} = \tau_{uc} = \inf \{ t \in [0, \infty) : A_t \leq A_{uc} \} .$$

We next describe a ‘‘candidate’’ value for the per-share stock price of the bank that issues CoCos. It equals the value of dividends received per share where conversion is assumed to happen at date τ_{uc} . If this candidate price is, indeed, consistent with conversion first happening at date τ_{uc} , then it must be an equilibrium price. For $t < \tau_{uc}$, this

candidate price equals

$$\begin{aligned}
S_t(A_t) &= \frac{1}{n} E_t^Q \left[\int_t^{\tau_{uc}} e^{-r(s-t)} (a_s - bB - cC) ds \right] + \frac{1}{n+m} E_t^Q \left[\int_{\tau_{uc}}^{\tau_\delta} e^{-r(s-t)} (a_s - bB) ds \right] \\
&= \frac{1}{n} \left\{ E_t^Q \left[\int_t^{\tau_\delta} e^{-r(s-t)} (a_s - bB) ds \right] - E_t^Q \left[\int_t^{\tau_{uc}} e^{-r(s-t)} cC ds \right] \right. \\
&\quad \left. - \frac{m}{n+m} E_t^Q \left[\int_{\tau_{uc}}^{\tau_\delta} e^{-r(s-t)} (a_s - bB) ds \right] \right\} \\
&= \frac{1}{n} \left\{ E_t^Q \left[\int_t^{\tau_\delta} e^{-r(s-t)} (a_s - bB) ds \right] - E_t^Q \left[\int_t^{\tau_{uc}} e^{-r(s-t)} cC ds \right] - E_t^Q \left[e^{-r(\tau_{uc}-t)} mL \right] \right\}
\end{aligned} \tag{16}$$

The first line of (16) equates the value of equity to the value of dividends received per share before and after the conversion date τ_{uc} . The second line uses equivalent cashflow accounting, but where the first term inside the curly brackets is the value of total cashflows per n shares after subtracting out the coupon payments paid to senior debt holders, which occurs until the bank's closure date when assets equal the value of senior debt. The second term subtracts out the value of coupons paid to CoCo investors until the conversion date τ_{uc} , while the third term subtracts out the proportion of total dividends paid to CoCos investors after the time of conversion. The last equality in (16) follows from the fact that

$$\begin{aligned}
\frac{m}{n+m} E_t^Q \left[\int_{\tau_{uc}}^{\tau_\delta} e^{-r(s-t)} (a_s - bB) ds \right] &= \frac{m}{n+m} E_t^Q \left[E_{\tau_{uc}}^Q \left[\int_{\tau_{uc}}^{\tau_\delta} e^{-r(s-t)} (a_s - bB) ds \right] \right] \\
&= \frac{m}{n+m} E_t^Q \left[e^{-r(\tau_{uc}-t)} E_{\tau_{uc}}^Q \left[\int_{\tau_{uc}}^{\tau_\delta} e^{-r(s-\tau_{uc})} (a_s - bB) ds \right] \right] \\
&= m E_t^Q \left[e^{-r(\tau_{uc}-t)} L \right]
\end{aligned} \tag{17}$$

since $\frac{1}{n+m} E_{\tau_{uc}}^Q \left[\int_{\tau_{uc}}^{\tau_\delta} e^{-r(s-\tau_{uc})} (a_s - bB) ds \right]$ is the stock price immediately after conversion and is equal to L . Evaluating the three terms in the last line of (16), one obtains

$$S_t(A_t) = \frac{1}{n} \left\{ A_t - \frac{bB}{r} - \frac{cC}{r} \left[1 - \left(\frac{A_t}{A_{uc}} \right)^{-\gamma} \right] - mL \left(\frac{A_t}{A_{uc}} \right)^{-\gamma} \right\} \tag{18}$$

where we use the fact that

$$E_t^Q \left[e^{-r(\tau_{uc}-t)} \right] = \left(\frac{A_t}{A_{uc}} \right)^{-\gamma} \tag{19}$$

where

$$\gamma \equiv \frac{1}{\sigma^2} \left[\mu - \frac{1}{2}\sigma^2 + \sqrt{\left(\mu - \frac{1}{2}\sigma^2\right)^2 + 2r\sigma^2} \right] > 0. \quad (20)$$

We now consider whether this candidate stock price is an increasing function of the bank's assets. While this is generally true for most firms, it might not always be the case in our setting of a bank that issues CoCos. When conversion terms favor the bank's shareholders, i.e., $mL < \frac{cC}{r}$, our proposed equilibrium stock price may increase when the asset level is decreasing towards A_{uc} . As we argue later, an equilibrium stock price must be increasing in value of the bank's assets. In the meantime, Lemma 2 characterizes the set of parameters for which $S_t(A_t)$ is strictly increasing in A_t for all $A_t \geq A_{uc}$.

Lemma 2: *If one of the following is true:*

(i) $mL \geq \frac{cC}{r}$ or

(ii) $mL < \frac{cC}{r}$ and

$$\sigma^2 \geq \frac{2(r + \mu\gamma^*)}{\gamma^*(1 + \gamma^*)}, \quad (21)$$

where

$$\gamma^* \equiv \frac{\frac{bB}{r} + L(n + m)}{\frac{cC}{r} - Lm}, \quad (22)$$

then $S_t(A_{uc}) = L$ and $S_t(A_t)$ is strictly increasing in A_t for all $A_t \geq A_{uc}$. Otherwise, $S_t(A_t) < L$ for some $A_t > A_{uc}$.

Note that an alternative way of writing the inequality (21) is in terms of the CoCo trigger level, L :

$$L \geq \frac{\gamma cC - bB}{r(n + (1 + \gamma)m)}. \quad (23)$$

Proof: *One can easily verify from (18) and (8) that $S_t(A_{uc}) = L$. Also from (18) for $t < \tau_{uc}$*

$$\frac{\partial S_t}{\partial A_t} = \frac{1}{n} \left\{ 1 - \frac{\gamma}{A_{uc}} \left(\frac{cC}{r} - mL \right) \left(\frac{A_{uc}}{A_t} \right)^{1+\gamma} \right\}. \quad (24)$$

From (24) it is clear that when $\frac{cC}{r} \leq mL$, $\frac{\partial S_t}{\partial A_t}$ is always positive so that S_t is increasing in A_t . When $\frac{cC}{r} > mL$, note from (24) that $\frac{\partial S_t}{\partial A_t}$ is increasing in A_t . As a result, when $\frac{cC}{r} > mL$, S_t is increasing in A_t for all $A_t \geq A_{uc}$ if and only if $\left. \frac{\partial S_t}{\partial A_t} \right|_{A_t=A_{uc}} \geq 0$. Since

$$\begin{aligned}
\left. \frac{\partial S_t}{\partial A_t} \right|_{A_t=A_{uc}} &= \frac{1}{n} \left\{ 1 - \frac{\gamma}{A_{uc}} \left(\frac{cC}{r} - mL \right) \right\} \\
&= \frac{1}{n} \left\{ 1 - \gamma \frac{\frac{cC}{r} - mL}{L(n+m) + \frac{bB}{r}} \right\}, \tag{25}
\end{aligned}$$

one can see that $\left. \frac{\partial S_t}{\partial A_t} \right|_{A_t=A_{uc}} \geq 0$ is equivalent to (23). On the other hand, when (23) does not hold, $\left. \frac{\partial S_t}{\partial A_t} \right|_{A_t=A_{uc}} < 0$, and $S_t(A_t) < L$ in the right neighborhood of A_{uc} .

According to equation (20), γ is the positive root of the following quadratic equation:

$$\frac{1}{2}\sigma^2\gamma^2 - \left(\mu - \frac{1}{2}\sigma^2\right)\gamma - r = 0.$$

On the other hand, inequality (23) can be rewritten as follows:

$$\gamma \leq \frac{\frac{bB}{r} + L(n+m)}{\frac{cC}{r} - mL} = \gamma^*.$$

Since $\gamma^* > 0$, inequality $\gamma \leq \gamma^*$ is equivalent to

$$\frac{1}{2}\sigma^2(\gamma^*)^2 - \left(\mu - \frac{1}{2}\sigma^2\right)\gamma^* - r \geq 0,$$

which gives (21).

Lemma 2 shows that for asset values exceeding A_{uc} , the candidate stock price is always an increasing function of the bank's assets when conversion terms are favorable to CoCo investors ($mL \geq cC/r$). When conversion terms are, instead, favorable to the initial stockholders ($mL < cC/r$), the parametric region under which the stock price continues to increase can be characterized in various ways. Condition (23) is in terms of the conversion trigger, L , while condition (21) is in terms of the bank's asset return variance.

Note that since $\mu < r$ for the value of the bank's assets to be finite, setting $\mu = r$ in condition (21) implies a lower bound on σ^2 that is sufficient to guarantee the monotonicity of $S_t(A_t)$:

$$\sigma^2 \geq \frac{2r}{\gamma^*}. \tag{26}$$

While condition (26) is stronger than condition (21), its advantage is that it holds for

any cash flow growth rate. Substituting (22) into (26) and rearranging terms allows us to rewrite the sufficient condition as a bound on the conversion trigger, which can be useful for implementation purposes.

Corollary 1: If $mL < \frac{cC}{r}$ and

$$L \geq \frac{\frac{cC}{r} - \frac{\sigma^2 bB}{2r}}{m + \frac{\sigma^2}{2r}(n+m)} \quad (27)$$

then $S_t(A_{uc}) = L$ and $S_t(A_t)$ is strictly increasing in A_t for all $A_t \geq A_{uc}$.

Now we can state the theorem for the existence and uniqueness of a stock price equilibrium:

Theorem 1: If either condition (i) or (ii) in Lemma 2 is satisfied, then there exists a unique equilibrium in which the CoCo converts when the bank's asset level drops to A_{uc} for the first time and where the equilibrium stock prices per share before and after conversion are given by (18) and (6), respectively. If neither condition (i) nor (ii) in Lemma 2 is satisfied, then there is no equilibrium.

Proof: Equation (18) says that $S_t(A_t)$ equals the present value of the dividends per share before conversion assuming that CoCos convert when the asset level drops to A_{uc} for the first time. Moreover, $S_t(A_{uc}) = L$. When either condition (i) or (ii) in Lemma 2 is satisfied, then $S_t(A_t) > L$ for all $A_t > A_{uc}$, and $S_t(A_t)$ is an equilibrium price. On the other hand, when neither condition (i) nor (ii) in Lemma 2 is satisfied, $S_t(A_t) < L$ for some $A_t > A_{uc}$, which is inconsistent with conversion occurring the first time that $A_t = A_{uc}$, so that $S_t(A_t)$ cannot be an equilibrium stock price. Proposition 1 ensures that there is no other equilibrium.

It is interesting to note that a unique equilibrium stock price exists whenever condition (23) or, equivalently, condition (21) holds. We will discuss the reasonableness of this parametric condition later when presenting numerical examples. Note also that (18) can be rewritten as

$$\begin{aligned} S_t(A_t) &= \frac{1}{n} \left\{ A_t - \frac{bB + cC}{r} + \left(\frac{cC}{r} - mL \right) \left(\frac{A_{uc}}{A_t} \right)^\gamma \right\} \\ &= V_t + \frac{1}{n} \left(\frac{cC}{r} - mL \right) \left(\frac{A_{uc}}{A_t} \right)^\gamma, \quad A_t > A_{uc}. \end{aligned} \quad (28)$$

As one would expect, $S_t(A_t)$ converges to the no-conversion stock price V_t as $A_t \rightarrow \infty$. Equation (28) also shows that the equilibrium stock price is greater (*less*) than V_t when the conversion terms $(\frac{cC}{r} - mL)$ favor (*disfavor*) the initial shareholders.

Once a unique stock price is determined, then the value of the CoCo bonds before conversion, C_t , equals the asset value A_t minus the values of stock, nS_t , and straight debt, $\frac{bB}{r}$. Thus,

$$\begin{aligned} C_t &= \frac{cC}{r} \left[1 - \left(\frac{A_t}{A_{uc}} \right)^{-\gamma} \right] + mL \left(\frac{A_t}{A_{uc}} \right)^{-\gamma} \\ &= \frac{cC}{r} + \left(mL - \frac{cC}{r} \right) \left(\frac{A_{uc}}{A_t} \right)^\gamma. \end{aligned} \tag{29}$$

The first term on the right-hand side of the first line of (29) represents the value of the CoCo coupon flow until conversion, and the last one is the present value of the CoCo payoff at conversion. The second line shows that the CoCo's value is greater (*less*) than an equivalent non-convertible bond when the conversion terms $(mL - \frac{cC}{r})$ favor (*disfavor*) the CoCo investors.

2.3 Principal Write Down

CoCos that have their principal automatically written down, rather than convert to new equity, are increasingly common.¹⁴ This section considers a modification of the previous model in which the bank's stock price triggers an automatic principal write-down instead of a conversion to new equity. As before, senior debt pays fixed coupon interest of b and has a principal value of B . However, contingent debt pays fixed coupon interest of c and has a principal value of C as long as the stock price remains above L , and its principal is reduced to αC when the stock price falls to L for the first time. We assume senior debt and post-write-down contingent debt are default-free, since bank regulators close the bank whenever assets fall to $bB/r + \alpha cC/r$, the default-free value of the bank's debt. As we demonstrate below, this setting is virtually identical to a setting with CoCos having a trigger stock price L and $m = \alpha \frac{cC}{rL}$.

¹⁴Boermans et al. (2014) state that CoCos with principal write downs have become more popular with European banks. Avdjiev et al. (2015) report that 55% of their sample of CoCos issued from 2009 to 2014 have principal write downs.

The stock price per share following a write down but prior to the bank's closure equals

$$U_t^w = \frac{1}{n} \left(A_t - \frac{bB}{r} - \alpha \frac{cC}{r} \right) . \quad (30)$$

Now define A_{uc}^w as the asset value such that the post-write-down bank's per share stock price equals L ; that is, $U^w(A_{uc}^w) = L$. From (30):

$$A_{uc}^w = nL + \frac{bB}{r} + \alpha \frac{cC}{r} . \quad (31)$$

Assuming the write-down occurs when the asset level drops to A_{uc}^w for the first time, one can easily derive the share price S_t^w before the debt write-down following the same approach employed in derivation of (18)

$$S_t^w(A_t) = \frac{1}{n} \left\{ A_t - \frac{bB}{r} - \frac{cC}{r} \left[1 - \left(\frac{A_t}{A_{uc}^w} \right)^{-\gamma} \right] - \alpha \frac{cC}{r} \left(\frac{A_t}{A_{uc}^w} \right)^{-\gamma} \right\} . \quad (32)$$

Comparing equations (31) with (8) and (32) with (18), one sees that $A_{uc}^w = A_{uc}$ and $S_t^w(A_t) = S_t(A_t)$ when $\alpha \frac{cC}{r} = mL$. As a result, after a minor modification the results of Lemma 2 and Theorem 1 continue to hold in this setting with debt write-downs:

Lemma 3: *If one of the following is true:*

(i) $\alpha \geq 1$ or

(ii) $\alpha < 1$ and

$$L \geq \frac{\gamma cC - bB}{r(n + (1 + \gamma)\alpha \frac{cC}{rL})} \quad (33)$$

then $S_t^w(A_{uc}^w) = L$ and $S_t^w(A_t)$ is strictly increasing in A_t for all $A_t \geq A_{uc}^w$. Otherwise, $S_t^w(A_t) < L$ for some $A_t > A_{uc}^w$.

Note that by solving (33) for α , condition (ii) of Lemma 3 can be re-written as

$$(ii) \quad 1 > \alpha > \frac{\gamma cC - bB - nrL}{cC(1 + \gamma)} . \quad (34)$$

Theorem 2: *If either condition (i) or (ii) of Lemma 3 is satisfied, then there exists a unique equilibrium in which the contingent debt is written-down when the bank's asset level drops to A_{uc}^w for the first time and where the equilibrium stock prices per share before and after the write-down are given by (32) and (30), respectively. If neither condition (i)*

nor (ii) of Lemma 3 is satisfied, there is no equilibrium.

Note from condition (34) that a unique equilibrium stock price exists for any non-negative value of α when $\gamma cC < (bB + nrL)$. Furthermore, the logic of Lemma 2 with $m = \alpha \frac{cC}{rL}$ can be applied to show that a unique equilibrium exists whenever

$$\sigma^2 \geq \frac{2(r + \mu\gamma^{**})}{\gamma^{**}(1 + \gamma^{**})}, \quad (35)$$

where

$$\gamma^{**} \equiv \frac{\frac{bB}{r} + \alpha \frac{cC}{r} + Ln}{\frac{cC}{r}(1 - \alpha)}. \quad (36)$$

2.4 Numerical Examples

This section illustrates the model's basic results using numerical examples. It shows that a unique stock price equilibrium exists for all but unrealistic parameter values. The following are benchmark parameter values:

<u>Parameter</u>	<u>Value</u>
Senior Debt Principal, B	90
Senior Debt Coupon, b	3.2%
CoCo Principal, C	5
CoCo Coupon, c	3.6%
Initial Equityholder Shares, n	1
CoCo Conversion Shares, m	1
Risk-neutral Cashflow Growth, μ	0.0%
Volatility of Asset Returns, σ	4.0%
Risk-free Interest Rate, r	3.0%

The above parameter values imply that the value of senior debt equals $bB/r = 96$, which is also the value of assets at which regulators would close the bank. Non-convertible debt with the same principal and coupon as CoCos would be worth $cC/r = 6$. If CoCos do convert, it is assumed that they receive 50% of the post conversion total shares ($m = n = 1$). The volatility of asset returns, $\sigma = 4\%$, equals what Pennacchi et al. (2014) estimate to be the average asset return volatility for Bank of America, Citigroup, and

JPMorgan Chase over the period 2003 to 2012.¹⁵ The risk-neutral cashflow growth rate of $\mu = 0$ implies from (5) that dividends paid to equity holders decline to equal zero at the time that the bank is closed by regulators. Given these values of μ , σ , and $r = 3\%$, the implied value of γ from (20) is 5.62.

2.4.1 Conversion to Equity

Consider, first, the equilibrium stock price when the conversion trigger is a stock price threshold of $L = 8$. In this case condition (i) of Lemma 2 is satisfied since $mL = 8 > 6 = cC/r$. This case represents conversion terms that favor the CoCo investors and, hence, are disadvantageous to the bank's initial shareholders. The result is illustrated in Figure 1 which graphs stock prices as a function of the bank's asset values.

The dotted blue line in Figure 1 is the stock price given in equation (6) of the hypothetical post-conversion bank that has only senior debt and a total of $n + m = 2$ shares of equity. Based on our parameter values, the asset value at which this stock price equals $L = 8$ is $A_{uc} = L(n + m) + \frac{bB}{r} = 112$. The dashed red line in Figure 1 is the stock price given in equation (7) of the hypothetical no-conversion bank that has non-convertible debt with the same coupon and principal as the CoCos. The asset value at which this bank's stock price equals $L = 8$ is $A_{vc} = Ln + \frac{bB+cC}{r} = 110$. Note that post- and no-conversion stock prices are equal at a value of 6 or an asset value of 108. Finally, the solid green line is the stock price given in equation (18) for $A_t \geq A_{uc}$. This is the equilibrium stock price prior to the CoCos' conversion. Hence, the complete equilibrium stock price is the solid green line for stock prices greater than 8 (or asset values greater than 112) and then the dotted blue line (post-conversion stock price) for stock prices less than 8 (or asset values less than 112).

Next, consider the equilibrium stock price when the conversion trigger is a lower stock price threshold of $L = 4$ such that conversion terms favor the bank's original shareholders and disfavor the CoCo investors. In this case condition (ii) of Lemma 2 is satisfied since $mL = 4 < 6 = cC/r$ and $L = 4 \geq \frac{\gamma cC - bB}{r(n + (1 + \gamma)m)} = -8.17$. The result is illustrated in Figure 2 which graphs stock prices as a function of the bank's asset values. The stock prices for the hypothetical post-conversion bank (dotted blue line) and no-conversion bank (dashed red line) are exactly the same as in Figure 1 since their values in equations (6) and (7)

¹⁵The model in Pennacchi et al. (2014) assumes assets follow a jump-diffusion process. Their estimate of a total return volatility of 4% is broken down between a diffusion component volatility of 3% and a jump component volatility of 1%. The asset return volatility estimates for individual banks are 4.2%, 4.4%, and 3.3% for Bank of America, Citigroup, and JPMorgan Chase, respectively.

do not depend on a conversion trigger. Now, however, $A_{uc} = L(n + m) + \frac{bB}{r} = 104$ and $A_{vc} = Ln + \frac{bB+cC}{r} = 106$.

The equilibrium stock price prior to the conversion of the CoCos is, again, given by equation (18) and graphed by the solid green line. Similar to previous case, the complete equilibrium stock price is the solid green line for stock prices greater than 4 (or asset values greater than 104) and then the dotted blue line (post conversion stock price) for stock prices less than 4 (or asset values less than 104). Note that this example confirms Lemma 2 in that the equilibrium stock price prior to conversion is a monotonically increasing function of the bank's asset level.

Our next example is one where neither conditions (i) or (ii) of Lemma 2 are met. As in the previous example, let $L = 4$ so that $mL = 4 < 6 = cC/r$ and conversion terms are, again, advantageous to the initial shareholders. However, now consider an example where $L = 4 < \frac{\gamma cC - bB}{r(n + (1 + \gamma)m)}$. This is done by raising the value of γ . It is straightforward to show from (20) that γ is a decreasing function of the bank's asset return volatility σ ; that is, $\partial\gamma/\partial\sigma < 0$. So consider lowering σ from its benchmark value of 4.0% to $\sigma = 0.25\%$, which implies $\gamma = 97.5$ and $L = 4 < \frac{\gamma cC - bB}{r(n + (1 + \gamma)m)} = 4.84$. Equivalently from (21), $\sigma < \sqrt{2(r + \mu\gamma^*) / [\gamma^*(1 + \gamma^*)]} = 0.47\%$.

This example is graphed in Figure 3. The stock prices of the hypothetical post- and no-conversion banks are, as before, unchanged since they do not depend on σ . The "candidate" equilibrium stock price for the CoCo-issuing bank is again given by equation (18) and graphed by the solid green line, which is only slightly above the no-conversion stock price (red dashed line) for high asset values. However, unlike in Figure 2, now the candidate stock price is not monotonically increasing prior to the asset level $A_{uc} = L(n + m) + \frac{bB}{r} = 104$. This candidate stock price dips below 4 first starting at the asset level 105.5. Hence, in this example, there is no stock price equilibrium consistent with conversion.

It is noteworthy that a unique equilibrium stock price exists for banks with high asset volatility exceeding 0.47%, the types of banks for which CoCos would be most valuable. Note that one also could generate a higher value of γ that would lead to a no-equilibrium stock price by raising the risk-neutral cashflow growth rate, μ . However, there are reasonable limits to doing so. First, in order to prevent equityholders from receiving negative dividends prior to the banks closure, one needs $\mu \leq 0$. Negative dividends, or requiring equityholders to make cash contributions to the bank, is inconsistent with the limited liability of shareholders' equity. Second, even if negative dividends are accepted,

equation (2) implies that a finite asset value requires $\mu < r$. From Corollary 1 that sets $\mu = r$, when $L = 4$ an equilibrium lower bound for σ is $\sqrt{2r/\gamma^*} = \sqrt{0.06/52} = 3.4\%$, less than the $\sigma = 4\%$ estimated for large banks. Thus, under this highly unrealistic upper bound for the cash flow growth rate, the implied asset return volatility starts to become realistic. Consequently, given our model of perpetual maturity CoCos, it is not easy to find reasonable parameter values that lead to a situation where there is no equilibrium stock price.

2.4.2 Principal Write Down

This section continues to use the parameter values of the previous section but now illustrates the model results for the case of a CoCo principal write down, rather than a conversion to equity. Figure 4 indicates the case of a stock price trigger of $L = 8$ and a write down of $\alpha = 50\%$. From (34) one sees that condition (ii) of Lemma 3 is satisfied: $\alpha = \frac{1}{2} > (\gamma cC - bB - nrL) / (cC(1 + \gamma)) = -1.77$, so that the unique equilibrium stock price exists. As with the case of a CoCo that converts to equity, the equilibrium price equals the post-conversion price at the trigger level of $L = 8$, equivalent to the asset value of $A_{uc}^w = nL + (bB + \alpha cC) / r = 107$.

Figure 5 is similar except that the trigger stock price is the lower value of $L = 4$. The no- and post-conversion stock prices are the same, but $\alpha = \frac{1}{2} > (\gamma cC - bB - nrL) / (cC(1 + \gamma)) = -1.67$, which still satisfies condition (ii) of Lemma 3 that ensures a unique stock price equilibrium. In this case, the asset value at which the write-down occurs is $A_{uc}^w = nL + (bB + \alpha cC) / r = 103$.

Figure 6 maintains the stock trigger level of $L = 4$ but assumes a complete write-down of $\alpha = 0$, so that the CoCo becomes worthless at the time of conversion. Still there exists a unique stock price equilibrium since from condition (ii) of Lemma 3 $\alpha = 0 > (\gamma cC - bB - nrL) / (cC(1 + \gamma)) = -1.67$. For this case, the write down occurs at the asset value $A_{uc}^w = nL + (bB + \alpha cC) / r = 100$.

Finally, Figure 7 maintains the same $L = 4$, $\alpha = 0$ assumptions of Figure 6 but lowers the bank's asset return volatility to $\sigma = 1.0\%$ which results in a value of $\gamma = 24.0$. Now, there is no equilibrium stock price since $\alpha = 0 < (\gamma cC - bB - nrL) / (cC(1 + \gamma)) = 0.29$ and neither of Lemma 3's conditions are not satisfied. We see in this case that the "candidate" equilibrium stock price first declines to the \$4 trigger level at an asset value of 103.15, prior to $A_{uc}^w = 100$. Based on (35) when $L = 4$ and $\alpha = 0$ there is no equilibrium stock price when $\sigma < \sqrt{2(r + \mu\gamma^{**}) / [\gamma^{**}(1 + \gamma^{**})]} = 1.43\%$.

2.5 Model Extensions

For simplicity, our analysis assumed that there were no direct costs to resolving a bank's failure and senior debt was default-free. The Appendix extends the basic model to consider positive direct costs of bankruptcy and their potential to cause greater losses. Importantly, it shows that accounting for these costs increases the likelihood that a unique stock price equilibrium exists. The intuition is that, in equilibrium, the bank's shareholders ultimately "pay" for the higher bankruptcy costs. If direct costs of bankruptcy lower senior debtholders' recovery value, these senior debt investors will require a higher equilibrium coupon rate, b , as compensation. But from inequality (23) that follows Lemma 2, a higher coupon rate on senior debt increases the likelihood of a unique stock price equilibrium.¹⁶ Alternatively, direct costs of resolving the bank's failure might lead bank regulators to close the bank earlier, that is, at a higher asset value, relative to the case of no direct bankruptcy costs. Under this assumption, the Appendix shows that the parameter space for which a unique stock price equilibrium exists is greater compared to the case where there are no direct bankruptcy costs.

In practice, many actual issues of perpetual maturity CoCos are callable by the issuing bank.¹⁷ The Appendix also considers an extension of the basic model where the issuing bank has the option to call its CoCos at their par value, C , and exercises this option to maximize the value of shareholders' equity. It derives the equilibrium stock price and CoCo value when a stock price equilibrium exists. There are four different cases. Perhaps the least interesting one is where $C \leq mL$ and $c > r$, so that the CoCo conversion value is greater than par and the CoCo coupon exceeds the risk-free rate. For this case, it is always optimal for the bank to redeem the CoCo immediately. A second case is where $C \leq mL$ and $c \leq r$ in which case it is optimal to redeem CoCos just before the stock price declines to L in order to pay off the CoCo at its par value rather than convert it to a higher value of equity.

A third case is $C > mL$ and $c \leq r$, so that conversion is at less than par and the CoCo coupon is less than the risk-free rate. Here, it is never optimal to redeem the CoCo, so that the CoCo value and stock price are exactly the same as when the bank issues an equivalent non-callable CoCo. The fourth case is $C > mL$ and $c > r$ so that CoCos are converted

¹⁶The result is the same if senior debt is government-insured and the (deposit) insurer requires that banks pay higher insurance premiums to cover the greater losses to the government from direct bankruptcy costs.

¹⁷We thank Martijn Boermans of the De Nederlandsche Bank for making us aware of this call feature. For a sample of 42 perpetual CoCos issued during the 2011 to 2014 period, he finds that at least 28 of them were callable.

at less than par but their coupon exceeds the risk-free rate. It is optimal to allow CoCo to convert at the same asset level, A_{uc} , as in the basic model but to redeem them at a higher asset level determined by the condition that this higher asset boundary maximizes shareholders' equity. In summary, taking these four cases together, the Appendix also shows that the bank's stock price has a unique equilibrium for a somewhat greater set of parameter values compared to the set of values when CoCos are non-callable. Hence, the conclusion is that extensions to consider direct costs of bankruptcy and a CoCo call feature tend to make a unique stock price equilibrium more likely.

3 Critical Analysis of Sundaresan and Wang (2015)

This section critiques the main theorems in Sundaresan and Wang (2015), hereafter SW, by identifying an error in their proof. We also show that our model provides a counterexample to their theorems' claims. Consequently, where SW claim that there can be multiple stock price equilibria in a continuous-time setting, there is instead a unique stock price equilibrium.

3.1 Assumptions

We begin by outlining the main assumptions of the "Dynamic Continuous-Time Model" in Section II of SW that starts on page 892. Our description of their assumptions and results are by no means complete, and we encourage the reader to refer to the original SW article. While our setting and notation are similar to those in SW, they are not identical. In order to avoid possible confusion, we will use a tilde "˜" to indicate the SW variables. The table below explains the notation of some major variables in SW and compares them to our variables.

SW Notation	SW Description	Our Equivalents
\tilde{S}_t	Per-share value of common equity	$\tilde{S}_t = S_t$
\tilde{U}_t	Total equity value without CoCo	$\tilde{U}_t = (n + m)U_t$
\tilde{C}_t	CoCo value before conversion	$\tilde{C}_t = C_t$
\tilde{n}	Number of shares before conversion	$\tilde{n} = n$
\tilde{m}_t	Conversion ratio	$\tilde{m}_t = m$
\tilde{K}_t	Total equity value that triggers conversion	$\tilde{K}_t = nL$
$\tilde{\Lambda}$	Time when conversion is allowed	$\tilde{\Lambda} = [0, \infty)$

Note that our model has a constant conversion trigger, L , and a constant conversion ratio, m , that is a special case of SW's more general time-varying conversion policy that uses the corresponding parameters \tilde{K}_t and \tilde{m}_t .

3.2 Error in the Proof

In Theorem 1 and Theorem 2 of SW (page 896), necessary and sufficient conditions are claimed in order for a bank's stock price to have a unique equilibrium. We first restate their theorems and then point out where their proof is incorrect. We use our model's results as a counter-example to their claims.

Theorems 1 and 2 of SW are:

Theorem 1 (SW) *For any given trigger \tilde{K}_t and conversion ratio \tilde{m}_t , a necessary condition for the existence of a unique equilibrium $(\tilde{S}_t; \tilde{C}_t)$ is $\tilde{n}\tilde{C}_t = \tilde{m}_t\tilde{K}_t$ for every $t \in \tilde{\Lambda}$.*

This necessary condition is also sufficient in the following sense:

Theorem 2 (SW) *For any given trigger \tilde{K}_t , there exists a conversion ratio \tilde{m}_t and a unique equilibrium $(\tilde{S}_t; \tilde{C}_t)$ satisfying $\tilde{n}\tilde{C}_t = \tilde{m}_t\tilde{K}_t$ for every $t \in \tilde{\Lambda}$.*

Rewriting their necessary and sufficient condition for the existence of a unique equilibrium, $\tilde{n}\tilde{C}_t = \tilde{m}_t\tilde{K}_t$ for every $t \in \tilde{\Lambda}$, in terms of our notation and setting with a constant conversion trigger and a constant conversion ratio leads to $nC_t = mnL$ for every $t \in \tilde{\Lambda}$, or $C_t = mL$ for all $t \geq 0$ prior to conversion. As a preview, note that this is clearly not the case in our setting. According to our equation (29), $C_t = mL$ for all $t \geq 0$ prior to conversion only if $mL = \frac{cC}{r}$. However, according to our Theorem 1, there exists unique equilibria even when $mL \neq \frac{cC}{r}$ as long as either condition (i) or (ii) of Lemma 2 is satisfied, e.g., whenever $L \geq (\gamma cC - bB) / [r(n + (1 + \gamma)m)]$.

The critical error in their proof occurs in the last paragraph of the proof of Theorem 1 on page 915 of SW. We repeat this part of their proof along with the equation that they label (A10):¹⁸

$$\inf \left\{ t \in \tilde{\Lambda} : \tilde{U}_t \leq \tilde{K}_t + \tilde{C}_t \right\} = \inf \left\{ t \in \tilde{\Lambda} : \tilde{U}_t \leq \tilde{K}_t(\tilde{n} + \tilde{m}_t)/\tilde{n} \right\} \quad (\text{A10})$$

The above equation holds for all possible paths of \tilde{U}_t if and only if $\tilde{K}_t + \tilde{C}_t = \tilde{K}_t(\tilde{n} + \tilde{m}_t)/\tilde{n}$

¹⁸After a first draft of our paper was written, we became aware of a revised version of Glasserman and Nouri (2012b) that also points out that equation (A10) in Sundaresan and Wang (2015) is incorrect.

for all $t \in \tilde{\Lambda}$, which implies $\tilde{m}_t = \tilde{n}\tilde{C}_t/\tilde{K}_t$ for all $t \in \tilde{\Lambda}$. Therefore, to have a unique equilibrium, the conversion ratio must satisfy $\tilde{m}_t = \tilde{n}\tilde{C}_t/\tilde{K}_t$ for $t \in \tilde{\Lambda}$. \square

The statement “The above equation holds for all possible paths of \tilde{U}_t if and only if $\tilde{K}_t + \tilde{C}_t = \tilde{K}_t(\tilde{n} + \tilde{m}_t)/\tilde{n}$ for all $t \in \tilde{\Lambda}$ ” would be correct if \tilde{C}_t and \tilde{U}_t were independent stochastic processes. However, they are not independent. As a result, equation (A10) simply implies that $\tilde{m}_t = \tilde{n}\tilde{C}_t/\tilde{K}_t$ must hold **only at the conversion time**: it does not impose any restrictions on the CoCo value strictly **before** conversion.

To illustrate that the proof of Theorem 1 of SW is incorrect, let us examine what equation (A10) says in our setting that delivers closed-form expressions for the relevant quantities in SW’s proof. We have

$$\begin{aligned}\tilde{K}_t &= nL, \quad \tilde{n} = n, \quad \tilde{m}_t = m, \quad \tilde{\Lambda} = [0, \infty), \\ \tilde{U}_t &= (n + m)U_t = A_t - \frac{bB}{r},\end{aligned}\tag{37}$$

and when either condition (i) or (ii) of Lemma 2 is satisfied so that from our Theorem 1 there is a unique stock price equilibrium, then

$$\tilde{C}_t = C_t = A_t - \frac{bB}{r} - nS_t .\tag{38}$$

Substituting (37) and (38) into the left-hand side of (A10), one obtains

$$\begin{aligned}\inf \left\{ t \in \tilde{\Lambda} : \tilde{U}_t \leq \tilde{K}_t + \tilde{C}_t \right\} &= \inf \left\{ t : A_t - \frac{bB}{r} \leq nL + A_t - \frac{bB}{r} - nS_t \right\} \\ &= \inf \{ t : S_t \leq L \} .\end{aligned}\tag{39}$$

Substituting (37) into the right-hand side of (A10), one obtains

$$\begin{aligned}\inf \left\{ t \in \tilde{\Lambda} : \tilde{U}_t \leq \tilde{K}_t(\tilde{n} + \tilde{m}_t)/\tilde{n} \right\} &= \inf \left\{ t : A_t - \frac{bB}{r} \leq nL(n + m)/n \right\} \\ &= \inf \left\{ t : A_t \leq L(n + m) + \frac{bB}{r} \right\} \\ &= \inf \{ t : A_t \leq A_{uc} \} .\end{aligned}\tag{40}$$

Consequently, (A10) implies

$$\inf \{ t : S_t \leq L \} = \inf \{ t : A_t \leq A_{uc} \} .\tag{41}$$

which, from our Proposition 1 is a requirement **only at the time of conversion**. Of course (41) “holds for all possible paths of \tilde{U}_t ,” which in terms of our notation is $(n + m)U_t = A_t - \frac{bB}{r}$ for all paths of A_t . But (41) does not imply that $\tilde{m}_t = \tilde{n}\tilde{C}_t/\tilde{K}_t$ for all $t \in \tilde{\Lambda}$, which in our notation is $C_t = mL$ for all times prior to conversion. Rather, it must be that $C_t = mL$ only at the conversion time $\hat{\tau} = \tau_{uc} = \inf \{t \in [0, \infty) : A_t \leq A_{uc}\}$. In general, for the broad set of parameters for which condition (i) or (ii) of Lemma 2 holds, prior to conversion C_t is given by equation (29) which does not need to equal mL .

4 Comparison to Glasserman and Nouri (2012b)

This section reviews Glasserman and Nouri (2012b), hereafter GN, whose main result is that in a continuous-time setting, a unique stock price equilibrium exists when CoCo conversion terms are favorable to CoCo investors. This result contradicts SW’s claim of multiple equilibria when conversion terms are strictly favorable to CoCo investors. Most of GN’s analysis assumes that CoCos have a finite maturity. However, their Section 6.3 briefly considers an infinite horizon model with perpetual CoCos and senior debt, and they verify that for this model there is a unique equilibrium when conversion terms favor CoCo investors. However, unlike our paper, they do not explore the possibility of there being a unique equilibrium when conversion terms favor the initial shareholders. GN also conclude in their Section 5.2 on the basis of a finite-maturity CoCo setting that there is no equilibrium stock price when CoCos are written down. In other words, when in our notation $\alpha < 1$, there is no equilibrium. Of course, this differs from the result in our perpetual maturity setting that there can be unique equilibria when $\alpha < 1$, even when CoCos are completely written off ($\alpha = 0$). What we now want to clarify is that it is the finite- versus perpetual- maturity feature of CoCos which is critical for the possible existence of a unique equilibrium when conversion terms favor the bank’s initial shareholders.

Similar to GN, we start by outlining the general logic for why there will be no stock price equilibrium when conversion terms are favorable to shareholders and CoCos involve a lump sum payment of principal at a finite maturity date. We then illustrate this result for a finite-maturity CoCo valuation model that gives closed-form expressions for candidate stock price and CoCo values. Using analysis similar to that of our previous perpetual CoCo model, we show that there is no stock price equilibrium whenever conversion terms strictly favor the bank’s initial shareholders. The conclusion from this analysis is that

the finite-maturity CoCo's requirement of a lump sum payment of principal is critical for eliminating possible unique equilibria when conversion terms favor the bank's initial shareholders.

4.1 Nonexistence of Equilibrium when CoCo Maturity is Finite and Conversion Favors Shareholders

We start by providing intuition for GN's finding that when conversion terms favor shareholders and CoCos have a finite maturity, there is no equilibrium stock price. Our objective is not to replicate GN, but rather to highlight the effect of CoCo maturity on the existence of equilibrium. We will show that when the principal (face) value of the CoCo is greater than the value of shares given to CoCo investors at conversion, there is a range of asset values for which there is no equilibrium stock price at a time just before maturity. In this range of asset values, a "candidate" stock price would be below the conversion trigger if CoCos were not converted because the shareholders have to make a large CoCo principal payment at maturity. On the other hand, if CoCos were converted, the candidate stock price would be above the conversion trigger since the CoCo conversion payoff is less than its principal value.

Assume that senior debt and CoCos pay B and C , respectively, to their investors at the future maturity date T . All the other assumptions of our previous perpetual maturity model are unchanged. We use an upper bar " $\bar{\cdot}$ " to denote values in the finite maturity setting. The per share values of the bank's stock at maturity are given by \bar{U}_T if CoCos are converted and by \bar{V}_T if CoCos are not converted:

$$\bar{U}_T(A_T) = \frac{1}{n+m} (A_T - B), \quad (42)$$

$$\bar{V}_T(A_T) = \frac{1}{n} (A_T - B - C). \quad (43)$$

Similar to (8) and (9), we can define \bar{A}_{uc} and \bar{A}_{vc} as the asset values such that the post-conversion and the no-conversion banks' per share stock values equal L at maturity; that is, $\bar{U}_T(\bar{A}_{uc}) = L$, and $\bar{V}_T(\bar{A}_{vc}) = L$. Hence, we have

$$\bar{A}_{uc} = L(n+m) + B, \quad (44)$$

$$\bar{A}_{vc} = Ln + B + C. \quad (45)$$

We note that when $mL < C$, i.e., conversion favors shareholders, we have $\bar{A}_{uc} < \bar{A}_{vc}$. In this case for $A_T \in (\bar{A}_{uc}, \bar{A}_{vc})$, the post-conversion payoff per share is above the conversion trigger, i.e., $\bar{U}_T(A_T) > L$, while the no-conversion payoff per share is below the conversion trigger, i.e., $\bar{V}_T(A_T) < L$.

We focus on time $T - \Delta$ just before maturity. Because we assume that Δ is very small, coupon payments and discounting between $T - \Delta$ and T are negligible and can be ignored. Furthermore, since A_t is a continuous process, it is highly unlikely that A_T can end up being significantly different from $A_{T-\Delta}$. Thus, if $A_{T-\Delta} > \bar{A}_{vc}$, then both payoffs \bar{U}_T and \bar{V}_T are almost surely to be above L . This means that when $A_{T-\Delta} > \bar{A}_{vc}$, there will be no conversion and the stock price at $T - \Delta$ is well defined and equal to $\frac{1}{n}(A_{T-\Delta} - B - C) > L$. We also note that an equilibrium conversion cannot occur at the asset level $A_{T-\Delta} = \bar{A}_{vc}$ because the post-conversion payoff per share is substantially greater than the conversion trigger, i.e., $\bar{U}_T(\bar{A}_{vc}) > L$. Thus, it is possible for $A_{T-\Delta}$ to be below \bar{A}_{vc} before conversion.

The situation is very different when $A_{T-\Delta} \in (\bar{A}_{uc}, \bar{A}_{vc})$. Because of asset process continuity, $A_T \in (\bar{A}_{uc}, \bar{A}_{vc})$ almost surely. Suppose there is an equilibrium stock price process. If the stock price always remains above L between $T - \Delta$ and T , then there will be no conversion and the payoff per share will be $\bar{V}_T(A_T) < L$, which implies that the stock price should be below L , an inconsistency. If, instead, the stock price falls to L between $T - \Delta$ and T , this will trigger conversion and the payoff per share will be $\bar{U}_T(A_T) > L$, which implies that a stock price equal to or less than L is also inconsistent with the payoff after conversion. Thus, there is no equilibrium stock price and conversion event shortly before maturity when the asset level is between \bar{A}_{uc} and \bar{A}_{vc} .

Nonexistence of a stock price equilibrium shortly before maturity implies that it is impossible to specify the CoCo's conversion outcome, and thus the relative payoffs of the CoCo and stock, for a range of asset values. As a result, there is no equilibrium stock price at any time $t < T$, because the stock payoff is affected by CoCo conversion, or lack thereof, before maturity. Proposition 2 summarizes our findings.

Proposition 2: *When a CoCo has a finite maturity and $C > mL$, there is no equilibrium stock price.*

Our argument shows that the no equilibrium outcome is caused by the lump sum principal payment of C to CoCo investors at maturity. When $C > mL$, CoCo conversion increases the stock's value, which can be inconsistent with the conversion rule. In the

infinite maturity setting, there is no one-time large payment to CoCo investors. Consequently, conversion does not have a discontinuous affect on the stock payoff, which leads (with some exceptions, see Theorem 1) to existence of equilibrium even when conversion favors shareholders.

4.2 A Model of Finite-Maturity CoCos and Stock Price Equilibrium

This section develops a valuation model nearly identical to our previous one but where CoCos have a finite, rather than perpetual, maturity. Its closed-form solutions illustrate why a finite maturity leads to the nonexistence of a stock price equilibrium when $C > mL$. Similar to our previous analysis, we show that the candidate equilibrium stock price is not monotone in the value of the bank's assets whenever $C > mL$ and the CoCo is sufficiently close to maturity.

For simplicity we assume that $b = r$, which implies that the value of senior debt is always equal to its face value B . Then, the post-conversion date t stock price per share is

$$\bar{U}_t(A_t) = \frac{1}{n+m} (A_t - B). \quad (46)$$

The asset value such that the post-conversion per share stock value equals L is then

$$\bar{A}_{uc} = L(n+m) + B. \quad (47)$$

Similar to our arguments in the perpetual maturity case, in equilibrium conversion should happen when the asset level falls to \bar{A}_{uc} for the first time. Conversion at any different asset level would lead to a predictable jump in the stock's value that is inconsistent with equilibrium.

Our model is similar to Leland and Toft (1996) who consider default-risky, finite-maturity debt.¹⁹ Let $f(s, A_t, \bar{A}_{uc})$ be the risk-neutral probability density of the first passage time of A_t to \bar{A}_{uc} at date $t+s$, and let $F(s, A_t, \bar{A}_{uc})$ be corresponding cumulative risk-neutral probability distribution. Also define $q \equiv T - t$ as the CoCo's time until

¹⁹Valuing CoCos involves similar mathematics to default-risky debt. Although CoCos do not explicitly default, when conversion terms are unfavorable to CoCo investors they absorb losses relative to their unconverted values.

maturity. Then the only possible candidate equilibrium CoCo value at date t is

$$\begin{aligned}\bar{C}(A_t, \bar{A}_{uc}, q) &= \int_0^q e^{-rs} cC[1 - F(s, A_t, \bar{A}_{uc})] ds + e^{-rq} C[1 - F(q, A_t, \bar{A}_{uc})] \\ &\quad + \int_0^q e^{-rs} mLf(s, A_t, \bar{A}_{uc}) ds\end{aligned}\quad (48)$$

The first term in equation (48) represents the discounted risk-neutral expected value of the coupon flow, which is paid at s periods in the future with probability $(1 - F(s, A_t, \bar{A}_{uc}))$. The second term represents the risk-neutral expected discounted value of repayment of principal, and the third term represents the risk-neutral expected discounted value of the shares given to CoCo investors at conversion if conversion occurs.

Integrating the first term by parts yields

$$\bar{C}(A_t, \bar{A}_{uc}, q) = \frac{cC}{r} + e^{-rq} \left[C - \frac{cC}{r} \right] [1 - F(q, A_t, \bar{A}_{uc})] + \left[mL - \frac{cC}{r} \right] G(q, A_t, \bar{A}_{uc}), \quad (49)$$

where

$$G(q, A_t, \bar{A}_{uc}) \equiv \int_0^q e^{-rs} f(s, A_t, \bar{A}_{uc}) ds. \quad (50)$$

One can verify that

$$\bar{C}(\bar{A}_{uc}, \bar{A}_{uc}, q) = mL. \quad (51)$$

We also note that

$$G(q, A_t, \bar{A}_{uc}) < F(q, A_t, \bar{A}_{uc}), \quad (52)$$

since $r > 0$.

According to Harrison (1990) and Rubinstein and Reiner (1991),

$$F(q, A_t, \bar{A}_{uc}) = \Phi[x_{1t}(q)] + \left(\frac{A_t}{\bar{A}_{uc}} \right)^{-2a} \Phi[x_{2t}(q)], \quad (53)$$

$$G(q, A_t, \bar{A}_{uc}) = \left(\frac{A_t}{\bar{A}_{uc}} \right)^{-a+z} \Phi[y_{1t}(q)] + \left(\frac{A_t}{\bar{A}_{uc}} \right)^{-a-z} \Phi[y_{2t}(q)], \quad (54)$$

where

$$\begin{aligned}
y_{1t}(q) &= \frac{-h_t - z\sigma^2 q}{\sigma\sqrt{q}}, & y_{2t}(q) &= \frac{-h_t + z\sigma^2 q}{\sigma\sqrt{q}}, \\
x_{1t}(q) &= \frac{-h_t - a\sigma^2 q}{\sigma\sqrt{q}}, & x_{2t}(q) &= \frac{-h_t + a\sigma^2 q}{\sigma\sqrt{q}}, \\
h_t &= \ln\left(\frac{A_t}{\bar{A}_{uc}}\right), & a &= \frac{\mu - \frac{1}{2}\sigma^2}{\sigma^2}, & z &= \frac{\sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2r\sigma^2}}{\sigma^2},
\end{aligned}$$

and $\Phi(\cdot)$ is the cumulative standard normal distribution.

We note that $a + z = \gamma$, and as $q \rightarrow \infty$,

$$\bar{C}(A_t, \bar{A}_{uc}, q) \rightarrow \frac{cC}{r} + \left[mL - \frac{cC}{r} \right] \left(\frac{A_t}{\bar{A}_{uc}} \right)^{-\gamma}, \quad (55)$$

which is the same as equation (29). Thus, this model incorporates our previous perpetual maturity model as a special case.

If $\bar{C}(A_t, \bar{A}_{uc}, q)$ is the equilibrium CoCo value, then the pre-conversion per-share stock value has to be equal to the asset value minus the value of the straight debt and CoCos:

$$\bar{S}(A_t, \bar{A}_{uc}, q) = \frac{1}{n}(A_t - B - \bar{C}(A_t, \bar{A}_{uc}, q)). \quad (56)$$

We split our analysis into three cases based on the CoCos' conversion terms.

Case 1: CoCo conversion value exceeds its principal and its coupon value in perpetuity:

$$mL \geq \max\left\{C, \frac{cC}{r}\right\}, \quad (57)$$

that is, conversion always benefits CoCo investors.

When $A_t > \bar{A}_{uc} = L(n + m) + B$, substituting (49) into (56) yields

$$\begin{aligned}
\bar{S}(A_t, \bar{A}_{uc}, q) &> \frac{1}{n} [\bar{A}_{uc} - B - \bar{C}(A_t, \bar{A}_{uc}, q)] \\
&= L + \frac{1}{n} [mL - \bar{C}(A_t, \bar{A}_{uc}, q)] \\
&= L + \frac{1}{n} \left[\left(mL - \frac{cC}{r} \right) (1 - G(q, A_t, \bar{A}_{uc})) - e^{-rq} \left(C - \frac{cC}{r} \right) (1 - F(q, A_t, \bar{A}_{uc})) \right].
\end{aligned} \quad (58)$$

Because of (52) and (57), the above inequality implies that $\bar{S}(A_t, \bar{A}_{uc}, q) > L$ for any $q \geq 0$ and any $A_t > \bar{A}_{uc}$, i.e., the stock price remains above the conversion trigger as long as the asset level remains above \bar{A}_{uc} . Thus, when condition (57) is satisfied, $\bar{S}(A_t, \bar{A}_{uc}, q)$ is the unique equilibrium price prior to conversion.

Case 2: CoCo conversion value is less than its principal:

$$mL < C,$$

that is, CoCo investors receive less than the principal at conversion. Here, we replicate the result of Section 4.1, but use a different approach.

Since in equilibrium, $\bar{S}(A_t, \bar{A}_{uc}, q)$ must be greater than L for all $A_t > \bar{A}_{uc}$, this requires that $\left. \frac{\partial \bar{S}(A_t, \bar{A}_{uc}, q)}{\partial A_t} \right|_{A_t=\bar{A}_{uc}} \geq 0$. Taking the derivative yields

$$\begin{aligned} \left. \frac{\partial \bar{S}(A_t, \bar{A}_{uc}, q)}{\partial A_t} \right|_{A_t=\bar{A}_{uc}} &= \frac{1}{n} \left[1 - \left. \frac{\partial \bar{C}(A_t, \bar{A}_{uc}, q)}{\partial A_t} \right|_{A_t=\bar{A}_{uc}} \right] \\ &= \frac{1}{n} \left[1 + e^{-rq} \left[C - \frac{cC}{r} \right] \left. \frac{\partial F(q)}{\partial A_t} \right|_{A_t=\bar{A}_{uc}} \right. \\ &\quad \left. - \left[mL - \frac{cC}{r} \right] \left. \frac{\partial G(q)}{\partial A_t} \right|_{A_t=\bar{A}_{uc}} \right]. \end{aligned} \quad (59)$$

If $A_t = \bar{A}_{uc}$, then $h_t = 0$, $\Phi[x_{2t}(q)] = 1 - \Phi[x_{1t}(q)] = \Phi[a\sigma\sqrt{q}]$, and $\Phi[y_{2t}(q)] = 1 - \Phi[y_{1t}(q)] = \Phi[z\sigma\sqrt{q}]$. As a result, we have

$$\left. \frac{\partial F(q)}{\partial A_t} \right|_{A_t=\bar{A}_{uc}} = -\frac{1}{\bar{A}_{uc}} \left(2a\Phi(a\sigma\sqrt{q}) + 2\frac{\phi(a\sigma\sqrt{q})}{\sigma\sqrt{q}} \right), \quad (60)$$

$$\left. \frac{\partial G(q)}{\partial A_t} \right|_{A_t=\bar{A}_{uc}} = -\frac{1}{\bar{A}_{uc}} \left((a-z) + 2z\Phi(z\sigma\sqrt{q}) + 2\frac{\phi(z\sigma\sqrt{q})}{\sigma\sqrt{q}} \right), \quad (61)$$

where $\phi(\cdot)$ denotes the standard normal density function. Importantly, note that as $q \rightarrow 0$,

$$\left. \frac{\partial F(q)}{\partial A_t} \right|_{A_t=\bar{A}_{uc}} \rightarrow -\infty, \quad (62)$$

$$\left. \frac{\partial G(q)}{\partial A_t} \right|_{A_t=\bar{A}_{uc}} \rightarrow -\infty. \quad (63)$$

Equation (59) can be rewritten as follows:

$$\begin{aligned}
n \frac{\partial \bar{S}(A_t, \bar{A}_{uc}, q)}{\partial A_t} \Big|_{A_t = \bar{A}_{uc}} &= 1 + (C - mL) \frac{\partial F(q)}{\partial A_t} \Big|_{A_t = \bar{A}_{uc}} \\
&- (1 - e^{-rq}) \left(C - \frac{cC}{r} \right) \frac{\partial F(q)}{\partial A_t} \Big|_{A_t = \bar{A}_{uc}} \\
&- \left(\frac{cC}{r} - mL \right) \left(\frac{\partial F(q)}{\partial A_t} \Big|_{A_t = \bar{A}_{uc}} - \frac{\partial G(q)}{\partial A_t} \Big|_{A_t = \bar{A}_{uc}} \right).
\end{aligned} \tag{64}$$

As $q \rightarrow 0$, the second term converges to $-\infty$ when $mL < C$. The last two terms in (64) converge to zero since $\frac{\partial F(q)}{\partial A_t} \Big|_{A_t = \bar{A}_{uc}}$ and $\frac{\partial G(q)}{\partial A_t} \Big|_{A_t = \bar{A}_{uc}}$ are of the order of magnitude of $\frac{1}{\sqrt{q}}$, while $(1 - e^{-rq}) \approx rq \ll \sqrt{q}$ and $\left(\frac{\partial F(q)}{\partial A_t} \Big|_{A_t = \bar{A}_{uc}} - \frac{\partial G(q)}{\partial A_t} \Big|_{A_t = \bar{A}_{uc}} \right) \approx \frac{(z^2 - a^2)\sigma q}{2\sqrt{2\pi}\sqrt{q}} \rightarrow 0$. Thus, if $mL < C$, then

$$\lim_{q \rightarrow 0} \left\{ \frac{\partial \bar{S}(A, \bar{A}_{uc}, q)}{\partial A_t} \Big|_{A_t = \bar{A}_{uc}} \right\} = -\infty. \tag{65}$$

Because \bar{S} is non-monotonic in the bank's assets at a time sufficiently close to maturity, it cannot be an equilibrium stock price at any time prior to maturity. Thus, there is no equilibrium stock price process.

Case 3: CoCo conversion value exceeds its principal but is less than its coupon value in perpetuity:

$$C \leq mL \leq \frac{cC}{r}.$$

It is easy to see that when a CoCo has a long maturity and its coupon rate exceeds the risk-free rate, there may be no equilibrium stock price even though the CoCo's conversion value exceeds its principal. This follows because as $q \rightarrow \infty$, (55) shows that the model becomes the perpetual maturity model that was considered in Section 2. Therefore, according to Theorem 1, there is no equilibrium when condition (ii) in Lemma 2 is not satisfied.

The following theorem summarizes this section's findings.

Theorem 3: *When a CoCo has a finite maturity and*

(i) *if $mL \geq \max\{C, \frac{cC}{r}\}$, then there exists a unique equilibrium in which the CoCo's conversion occurs when the bank's asset level drops to \bar{A}_{uc} for the first time and where the equilibrium stock prices per share before and after conversion are given by (56) and (46),*

respectively;

(ii) if $mL < C$, then there is no equilibrium stock price. Moreover,

$$\lim_{q \rightarrow 0} \left\{ \frac{\partial \bar{S}(A_t, \bar{A}_{uc}, q)}{\partial A_t} \Big|_{A_t = \bar{A}_{uc}} \right\} = -\infty; \quad (66)$$

(iii) if $C \leq mL < \frac{cC}{r}$, then there is no equilibrium stock price if the CoCo's maturity is sufficiently long and condition (ii) of Lemma 2 is not satisfied.

Theorem 3 states that the absence of a stock price equilibrium is possible only when conversion terms benefit shareholders by having a conversion value that is less than the principal payment ($mL < C$) or less than the CoCo's unconverted perpetuity value ($mL < \frac{cC}{r}$). In the former case, there is never an equilibrium stock price process because conversion shortly before maturity creates a large one-time value transfer to shareholders that instantaneously moves the stock price above the conversion trigger. In the latter case, conversion has a weaker effect on equity, since the value transfer is spread over time and, as a result, both equilibrium and non-equilibrium outcomes are possible depending on the parameters. In contrast, a conversion that benefits CoCo investors ($mL > C$ and $mL > \frac{cC}{r}$) negatively affects the stock's value, consistent with the equilibrium requirement that the stock price remains at or below the trigger level immediately after conversion.

Theorem 3 does not consider the case in which $C \leq mL < \frac{cC}{r}$ and condition (ii) of Lemma 2 is satisfied. Intuition suggests that there should be a unique equilibrium in this case, since there is no value transfer to shareholders at maturity and condition (ii) of Lemma 2 ensures that the effect of reducing the coupon burden ($mL < \frac{cC}{r}$) is insufficient to cause stock price non-monotonicity in the perpetual maturity case. Proving this analytically is quite challenging. However, our various numerical calculations using a variety of parameters are consistent with the following conjecture.

Conjecture 1: *If $C \leq mL < \frac{cC}{r}$ and condition (ii) of Lemma 1 is satisfied, then there is a unique equilibrium stock price.*

5 Conclusion

CoCos can enhance a bank's safety and soundness by pre-committing it to convert debt to equity at the onset of financial stress, thereby solving the debt overhang problem.

Given the inadequate response of regulatory capital ratios and a history of forbearance by regulators, only market-based triggers are likely to be adequate for converting CoCos in a timely fashion. But prior research on CoCos has cast doubt on the viability of basing triggers on a market value, such as the bank's stock price, particularly if conversion terms are unfavorable to CoCo investors.

In contrast, we have shown that when CoCos have a perpetual maturity, which characterizes the majority of actual CoCos, there are a wide variety of realistic conditions under which a unique stock price equilibrium exists. Situations that lack a stock price equilibrium seem to require that the bank have unrealistically low asset risk. Moreover, the existence of a unique stock price becomes more likely when there are direct costs of bankruptcy or CoCos are callable by the issuing bank. By developing CoCo valuation models that lead to closed-form solutions when CoCo maturity is perpetual or finite, we are able to clearly illustrate the critical role that maturity plays on the existence of a stock price equilibrium.

By showing that an ill-defined stock price is unlikely when CoCos are perpetuities, our results have practical implications for CoCo design. Fortunately, Basel III capital regulations provide incentives to issue CoCos with perpetual maturities. A natural next step is that regulation encourage banks to choose market-based triggers rather than regulatory capital ones. Doing so would restore the original vision of CoCos as instruments that preserve banks as going concerns and reduce the likelihood of financial crises.

Appendix: Model Extensions

This appendix considers two extensions of the basic perpetual-maturity CoCo model. The first considers direct costs of bankruptcy. The second allows the issuing bank to call CoCos at their par (principal or face) value. As will be shown, these extensions increase the likelihood that a unique bank stock price equilibrium exists.

Direct Costs of Bankruptcy

Consider a modification of the model that incorporates bankruptcy costs. Suppose that at the time that regulators close the bank, a proportion ω of the bank's assets must be paid in the form of direct costs to resolve the bank's failure. If there is no change in the model's other assumptions, then when the bank's assets first fall to the level bB/r and it is closed by regulators, the senior debtholders would receive $(1 - \omega)bB/r$ while $\omega bB/r$ of the asset value would be paid in direct costs of the bankruptcy. Given the coupon (b) and principal (B) of the senior debt, Theorem 1 continues to hold. Thus, when either condition (i) or (ii) of Lemma 2 is satisfied, the equilibrium stock price continues to be given by equation (18) prior to conversion and equation (6) following conversion. Moreover, equation (29) remains the value of the CoCo prior to conversion. The only change in interpretation is that bB/r is no longer the value of senior debt but the sum of the value of senior debt plus the present value of bankruptcy costs. When $A_t \geq bB/r$, the present value of bankruptcy costs is

$$\omega \frac{bB}{r} \left(\frac{A_t}{bB/r} \right)^{-\gamma} \quad (\text{A.1})$$

so that the value of senior debt equals

$$\frac{bB}{r} - \omega \frac{bB}{r} \left(\frac{A_t}{bB/r} \right)^{-\gamma} = \frac{bB}{r} \left[1 - \omega \left(\frac{A_t}{bB/r} \right)^{-\gamma} \right]. \quad (\text{A.2})$$

While we have taken the coupon on senior debt as given, the existence of bankruptcy costs that creates losses for senior debtholders will tend to raise the coupon rate set at the debt's issuance date. For example, suppose that senior debt is issued at date $t = 0$ and its coupon is set such that its new-issue market value equals its par value B . Then if the bank's total assets equals A_0 following the debt issuance (or equals $A_0 - B$ prior to

issuance), then the coupon rate, b , must satisfy:

$$\begin{aligned} B &= \frac{bB}{r} \left[1 - \omega \left(\frac{A_0}{bB/r} \right)^{-\gamma} \right] \\ &= \frac{bB}{r} - \omega \left(\frac{bB}{r} \right)^{1+\gamma} A_0^{-\gamma}. \end{aligned} \quad (\text{A.3})$$

Differentiating (A.3) shows that $\partial b/\partial \omega > 0$ when $(1 + \gamma) \omega \left(\frac{A_0}{bB/r} \right)^{-\gamma} < 1$, which holds for sufficiently low costs of bankruptcy, ω . Thus, while senior debtholders absorb losses at the time of bankruptcy, they are compensated prior to bankruptcy in the form of a higher coupon flow.

Interestingly, the higher equilibrium coupon rate, b , needed to compensate senior debtholders for the costs of bankruptcy increases the likelihood of a unique stock price equilibrium. This is seen from inequality (23) that follows Lemma 2. Specifically, when conversion terms favor initial shareholders so that $mL < cC/r$, a unique stock price equilibrium exists when $L \geq (\gamma cC - bB) / [r(n + (1 + \gamma)m)]$, which is more likely the larger is b .

An alternative response to the presence of bankruptcy costs might come from bank regulators. To prevent senior debt investors from suffering losses, suppose that the regulator closes the bank when its assets are just sufficient to pay the default-free value of this debt plus the direct costs of bankruptcy. Thus, assume that regulators close the bank when its assets first reach the value $bB / [(1 - \omega)r]$, which we define as $A_{b\delta}$.

In this case the “post-conversion” bank’s stock price reflects the value of the bank’s assets less the default-free value of the senior debt and less the present value of bankruptcy costs:

$$\widehat{U}_t = \frac{1}{n + m} \left(A_t - \frac{bB}{r} - \omega A_{b\delta} \left(\frac{A_t}{A_{b\delta}} \right)^{-\gamma} \right). \quad (\text{A.4})$$

Note that

$$\frac{\partial \widehat{U}_t}{\partial A_t} = \frac{1}{n + m} \left(1 + \gamma \omega \left(\frac{A_t}{A_{b\delta}} \right)^{-(1+\gamma)} \right) > 0 \quad (\text{A.5})$$

$$\frac{\partial^2 \widehat{U}_t}{\partial A_t^2} = -\frac{\gamma(1 + \gamma)\omega}{n + m} \left(\frac{A_t}{A_{b\delta}} \right)^{-(2+\gamma)} < 0 \quad (\text{A.6})$$

so that \widehat{U}_t is an increasing, concave function of A_t .

Similar to the derivation of the equilibrium stock price when direct costs of bankruptcy were zero, consider a candidate per share stock price for a bank that issues CoCos. First define A_{buc} as the asset value such that the post-conversion bank's per share stock price equals L ; that is, $\widehat{U}_t(A_{buc}) = L$. From (A.4), A_{buc} satisfies

$$\begin{aligned} A_{buc} \left(1 - \omega \left(\frac{A_{buc}}{A_{b\delta}} \right)^{-(1+\gamma)} \right) &= L(n+m) + \frac{bB}{r} \\ &= L(n+m) + (1-\omega)A_{b\delta}. \end{aligned} \quad (\text{A.7})$$

Second, define $\tau_{buc} = \inf \{t \in [0, \infty) : A_t \leq A_{buc}\}$. If conversion happens at τ_{buc} , which we later argue is the time of conversion for a unique equilibrium, then the share price for $t < \tau_{buc}$ has to be

$$\begin{aligned} \widehat{S}_t(A_t) &= \frac{1}{n} E_t^Q \left[\int_t^{\tau_{buc}} e^{-r(s-t)} (a_s - bB - cC) ds \right] + \frac{1}{n+m} E_t^Q \left[\int_{\tau_{buc}}^{\tau_{b\delta}} e^{-r(s-t)} (a_s - bB) ds \right] \\ &= \frac{1}{n} \left\{ E_t^Q \left[\int_t^{\tau_{b\delta}} e^{-r(s-t)} (a_s - bB) ds \right] - E_t^Q \left[\int_t^{\tau_{buc}} e^{-r(s-t)} cC ds \right] \right. \\ &\quad \left. - \frac{m}{n+m} E_t^Q \left[\int_{\tau_{buc}}^{\tau_{b\delta}} e^{-r(s-t)} (a_s - bB) ds \right] \right\} \\ &= \frac{1}{n} \left\{ E_t^Q \left[\int_t^{\tau_{b\delta}} e^{-r(s-t)} (a_s - bB) ds \right] - E_t^Q \left[\int_t^{\tau_{buc}} e^{-r(s-t)} cC ds \right] - E_t^Q \left[e^{-r(\tau_{buc}-t)} mL \right] \right\}. \end{aligned} \quad (\text{A.8})$$

The logic for (A.8) turns out to be nearly identical to the no bankruptcy case. Evaluating the three terms in the last line, one obtains

$$\widehat{S}_t(A_t) = \frac{1}{n} \left\{ A_t - \frac{bB}{r} - \omega A_{b\delta} \left(\frac{A_t}{A_{b\delta}} \right)^{-\gamma} - \frac{cC}{r} \left[1 - \left(\frac{A_t}{A_{buc}} \right)^{-\gamma} \right] - mL \left(\frac{A_t}{A_{buc}} \right)^{-\gamma} \right\}. \quad (\text{A.9})$$

Note that we can confirm that $\widehat{S}_t(A_{buc}) = L$:

$$\begin{aligned} \widehat{S}_t(A_{buc}) &= \frac{1}{n} \left\{ A_{buc} - \frac{bB}{r} - \omega A_{b\delta} \left(\frac{A_{buc}}{A_{b\delta}} \right)^{-\gamma} - \frac{cC}{r} \left[1 - \left(\frac{A_{buc}}{A_{buc}} \right)^{-\gamma} \right] - mL \left(\frac{A_{buc}}{A_{buc}} \right)^{-\gamma} \right\} \\ &= \frac{1}{n} \left\{ A_{buc} \left(1 - \omega \left(\frac{A_{buc}}{A_{b\delta}} \right)^{-(1+\gamma)} \right) - \frac{bB}{r} - mL \right\} \end{aligned} \quad (\text{A.10})$$

and using (A.7), (A.10) becomes

$$\widehat{S}_t(A_{buc}) = \frac{1}{n} \left\{ L(n+m) + \frac{bB}{r} - \frac{bB}{r} - mL \right\} = L. \quad (\text{A.11})$$

Also, from (A.9) when $t \leq \tau_{buc}$

$$\begin{aligned} \frac{\partial \widehat{S}_t}{\partial A_t} &= \frac{1}{n} \left\{ 1 + \gamma \left[\omega \left(\frac{A_{b\delta}}{A_t} \right)^{(1+\gamma)} + \frac{1}{A_{buc}} \left(mL - \frac{cC}{r} \right) \left(\frac{A_{buc}}{A_t} \right)^{(1+\gamma)} \right] \right\} \\ &= \frac{1}{n} \left\{ 1 + \gamma A_t^{-(1+\gamma)} \left[\omega A_{b\delta}^{1+\gamma} + \left(mL - \frac{cC}{r} \right) A_{buc}^\gamma \right] \right\} \end{aligned} \quad (\text{A.12})$$

From (A.12) one sees that when $mL \geq \frac{cC}{r} - \omega A_{b\delta} \left(\frac{A_{b\delta}}{A_{buc}} \right)^\gamma$, then $\frac{\partial S_t}{\partial A_t}$ is unambiguously positive. This case includes all situations where conversion terms favor CoCo investors as well as some situations where conversion terms favor the bank's initial shareholders. For the opposite case where $mL < \frac{cC}{r} - \omega A_{b\delta} \left(\frac{A_{b\delta}}{A_{buc}} \right)^\gamma$, one also sees from (A.12) that $\frac{\partial \widehat{S}_t}{\partial A_t}$ is increasing in A_t . Consequently, for this case \widehat{S}_t is increasing in A_t for all $A_t \geq A_{buc}$ if and only if $\left. \frac{\partial \widehat{S}_t}{\partial A_t} \right|_{A_t=A_{buc}} \geq 0$. This derivative is

$$\left. \frac{\partial \widehat{S}_t}{\partial A_t} \right|_{A_t=A_{buc}} = \frac{1}{n} \left\{ 1 + \gamma \left[\omega \left(\frac{A_{b\delta}}{A_{buc}} \right)^{(1+\gamma)} + \frac{1}{A_{buc}} \left(mL - \frac{cC}{r} \right) \right] \right\}. \quad (\text{A.13})$$

Since (A.7) implies

$$\omega \left(\frac{A_{b\delta}}{A_{buc}} \right)^{(1+\gamma)} = 1 - \frac{L(n+m) + \frac{bB}{r}}{A_{buc}}, \quad (\text{A.14})$$

substituting (A.14) into (A.13) gives the result

$$\begin{aligned} \left. \frac{\partial \widehat{S}_t}{\partial A_t} \right|_{A_t=A_{buc}} &= \frac{1}{n} \left\{ 1 + \gamma \left[1 - \frac{L(n+m) + \frac{bB}{r}}{A_{buc}} + \frac{1}{A_{buc}} \left(mL - \frac{cC}{r} \right) \right] \right\} \\ &= \frac{1}{n} \left\{ 1 + \gamma \left[1 - \frac{Ln + \frac{bB+cC}{r}}{A_{buc}} \right] \right\}. \end{aligned} \quad (\text{A.15})$$

Now note that for the case of zero bankruptcy costs, (25) can be rewritten as

$$\left. \frac{\partial S_t}{\partial A_t} \right|_{A_t=A_{uc}} = \frac{1}{n} \left\{ 1 + \gamma \left[1 - \frac{Ln + \frac{bB+cC}{r}}{A_{uc}} \right] \right\}. \quad (\text{A.16})$$

Furthermore, equation (A.7) shows that $A_{buc} > L(n + m) + \frac{bB}{r} = A_{uc}$. Therefore, comparing (A.15) to (A.16) shows that given the model parameters $r, \mu, \sigma, b, B, c,$ and C , $\frac{\partial \widehat{S}_t}{\partial A_t} \Big|_{A_t=A_{buc}} \geq 0$ whenever $\frac{\partial S_t}{\partial A_t} \Big|_{A_t=A_{buc}} \geq 0$. Moreover, due to $A_{buc} > A_{uc}$ when $\omega > 0$, there will be some sets of the model parameters where $\frac{\partial \widehat{S}_t}{\partial A_t} \Big|_{A_t=A_{buc}} \geq 0$ when $\frac{\partial S_t}{\partial A_t} \Big|_{A_t=A_{buc}} < 0$.

Since Theorem 1 establishes that a necessary and sufficient condition for a unique stock price equilibrium to exist is that the candidate stock price is strictly increasing for asset values greater than that for which the post-conversion stock price equals L , the derivation above shows that the parameter space under which a unique stock price equilibrium exists is greater when $\omega > 0$ versus when $\omega = 0$.

Callable CoCos

Consider another extension of the basic model where perpetual-maturity CoCos can be redeemed by their issuer. As before, CoCos automatically convert to m additional shares when the bank's per share stock price falls to L . In addition, the bank has the right, but not the obligation, to buy back CoCos at their principal (par) value, C , at any time prior to conversion. As a tie-breaking rule, we assume that when the stock price drops to L , the bank has the right to call CoCos before they are converted.²⁰ In addition, because calling bonds inherently requires a lump sum redemption payment, we assume that the bank's asset value is reduced by this payment.²¹ Note that since we assume a constant default-free rate, r , we are not modeling a stochastic interest rate motive for calling CoCos. Rather, the bank's call option has value because the CoCos' value can vary with changes in the bank's underlying asset value.

As in Section 2.2, the stock price process is adapted to the Brownian motion, and dividends are paid continuously. Hence, the stock price process must be continuous in time.²² As a consequence, CoCo investors' payoff at conversion must equal mL . Let τ_c and τ_r be stopping times at which CoCos are converted and redeemed, respectively. Then,

²⁰This rule is without loss of generality, since the bank can always call CoCos when the stock price is equal to $L + \varepsilon$, where ε is an arbitrarily small positive amount.

²¹The results are unchanged if the bank funds the payment with any combination of new senior debt, new CoCos, or new shareholders' equity. As long as these new securities are issued at their fair price and the original senior debt is default free (due to the regulator closing the bank when assets equal the value of total senior debt), issuance of these securities does not change the post-redemption stock price relative to the case where redemption occurs through asset liquidation.

²²It is straightforward to show that Lemma 1 also holds in the setting with callable CoCos.

the value of the callable CoCo, \dot{C}_t , is given by

$$\dot{C}_t = E_t^Q \left[\int_t^{\min\{\tau_c, \tau_r\}} e^{-r(s-t)} cC ds + e^{-r(\tau_c-t)} \mathbf{1}_{\{\tau_c < \tau_r\}} mL + e^{-r(\tau_r-t)} \mathbf{1}_{\{\tau_c \geq \tau_r\}} C \right] \quad (\text{A.17})$$

Since the senior debt is risk-free, the bank's per share stock price, \dot{S}_t , must be

$$\dot{S}_t = \frac{1}{n} \left(A_t - \frac{bB}{r} - \dot{C}_t \right). \quad (\text{A.18})$$

The bank chooses to redeem its CoCos at a time τ_r that maximizes the value of shareholders' equity or, equivalently, minimizes the CoCo value. We consider four cases resulting from different values of the parameters C , mL , c , and r :

Case 1: $C \leq mL$ and $c > r$

Because the CoCo conversion value mL is greater than the face value C , CoCos will never be converted since CoCos can be redeemed at C when the stock price drops to L . However, since $c > r$, it is optimal to redeem the CoCos immediately. Thus, in this case there exists a unique equilibrium in which the CoCo value equals C and the stock value equals $\frac{1}{n} \left(A_t - \frac{bB}{r} - C \right)$.

Case 2: $C \leq mL$ and $c \leq r$

Since $c \leq r$, CoCo redemption is not optimal prior to the stock price falling to L . However, when the stock price hits L , CoCos are redeemed to avoid conversion since $C \leq mL$. This will happen when the bank's assets fall to a level $A_t = A_r$ such that the pre-redemption stock price equals L . Specifically:

$$\frac{1}{n} \left(A_r - \frac{bB}{r} - C \right) = L, \quad (\text{A.19})$$

which gives

$$A_r = nL + \frac{bB}{r} + C. \quad (\text{A.20})$$

Note that $A_r < A_{uc}$ since $C < mL$. The intuition is that, ceteris paribus, the stock price is more valuable compared to the case of a non-callable CoCo since redemption at the lower value C substitutes for conversion at the higher value mL , which means that the stock price first equals L at a lower bank asset value. Indeed, this callable CoCo's payoff is equivalent to a noncallable CoCo that converts to $m_r \equiv \frac{C}{L}$ additional shares when the bank's per share stock price falls to L . Consequently, there always exists a unique

stock price equilibrium since condition (i) of Lemma 2 is satisfied. According to (29), the callable CoCo value equals

$$\dot{C}_t = \frac{cC}{r} \left[1 - \left(\frac{A_t}{A_r} \right)^{-\gamma} \right] + C \left(\frac{A_t}{A_r} \right)^{-\gamma}. \quad (\text{A.21})$$

Case 3: $C > mL$ and $c \leq r$

CoCo redemption is never optimal since the coupon rate is below the risk-free rate and the conversion value is less than the face value. As a result, this callable CoCo is equivalent to the noncallable CoCo with the same conversion parameters. Thus, the results of Theorem 1 on the existence and uniqueness of a stock price equilibrium apply so that if condition (i) or (ii) of Lemma 2 holds, there is a unique equilibrium stock price and the CoCo value equals equation (29).

Case 4: $C > mL$ and $c > r$

In this case, shareholders prefer conversion to redemption since $C > mL$. Therefore, the bank will not call CoCos if conversion is likely in the near future because the stock price is close to the conversion trigger L . However, when conversion is not likely because the stock price is high, the bank will call CoCos due to their high coupon rate $c > r$. If there exists an equilibrium, CoCo payoffs can be described by two asset boundaries: the conversion boundary A_{uc} and the redemption boundary A_r . Due to the continuity of the stock price, we know from Proposition 1 that conversion must occur when the bank's asset value falls to A_{uc} given by (8). CoCos will be called when the asset value reaches the redemption boundary A_r which is determined by that asset value which maximizes the value of shareholders' equity. Thus, we can state the following proposition.

Proposition 3: *If $C > mL$, $c > r$, and an equilibrium exists, the value of the callable CoCo is given by*

$$\dot{C}_t = \frac{cC}{r} + \left(mL - \frac{cC}{r} \right) \left(\frac{A_t}{A_{uc}} \right)^{-\gamma} \frac{1 - \left(\frac{A_r}{A_t} \right)^{-(\gamma-\gamma_1)}}{1 - \left(\frac{A_r}{A_{uc}} \right)^{-(\gamma-\gamma_1)}} + \left(C - \frac{cC}{r} \right) \left(\frac{A_r}{A_t} \right)^{\gamma_1} \frac{1 - \left(\frac{A_t}{A_{uc}} \right)^{-(\gamma-\gamma_1)}}{1 - \left(\frac{A_r}{A_{uc}} \right)^{-(\gamma-\gamma_1)}}, \quad (\text{A.22})$$

where

$$\gamma_1 \equiv \frac{1}{\sigma^2} \left[\mu - \frac{1}{2}\sigma^2 - \sqrt{\left(\mu - \frac{1}{2}\sigma^2\right)^2 + 2r\sigma^2} \right], \quad (\text{A.23})$$

and the optimal redemption boundary A_r solves the following equation:

$$\gamma \left(\frac{A_r}{A_{uc}} \right)^{\gamma_1} - \gamma_1 \left(\frac{A_r}{A_{uc}} \right)^{\gamma} = (\gamma - \gamma_1) \frac{\frac{cC}{r} - mL}{\frac{cC}{r} - C}. \quad (\text{A.24})$$

Proof: The value of the CoCo is a function $\dot{C}(A_t)$ of the bank's asset value that satisfies the following ordinary differential equation:

$$\frac{1}{2}\sigma^2 A^2 \dot{C}''(A) + \mu A \dot{C}'(A) + cC = r \dot{C}'(A)$$

with the boundary conditions

$$\begin{aligned} \dot{C}(A_{uc}) &= mL, \\ \dot{C}(A_r) &= C. \end{aligned}$$

This equation has the general solution

$$\dot{C}(A) = \frac{cC}{r} + K_1 \left(\frac{A}{A_{uc}} \right)^{-\gamma} + K_2 \left(\frac{A}{A_r} \right)^{-\gamma_1}. \quad (\text{A.25})$$

where the constants K_1 and K_2 are determined by the boundary conditions:

$$\begin{aligned} mL &= \frac{cC}{r} + K_1 + K_2 \left(\frac{A_{uc}}{A_r} \right)^{-\gamma_1}, \\ C &= \frac{cC}{r} + K_1 \left(\frac{A_r}{A_{uc}} \right)^{-\gamma} + K_2. \end{aligned}$$

Solving the system of the linear equations gives

$$\begin{aligned} K_1 &= \frac{\left(mL - \frac{cC}{r}\right) + \left(\frac{cC}{r} - C\right) \left(\frac{A_r}{A_{uc}}\right)^{\gamma_1}}{1 - \left(\frac{A_r}{A_{uc}}\right)^{-(\gamma-\gamma_1)}} \\ K_2 &= -\frac{\left(\frac{cC}{r} - C\right) + \left(mL - \frac{cC}{r}\right) \left(\frac{A_r}{A_{uc}}\right)^{-\gamma}}{1 - \left(\frac{A_r}{A_{uc}}\right)^{-(\gamma-\gamma_1)}} \end{aligned}$$

Substituting K_1 and K_2 into (A.25) and rearranging terms yields (A.22).

The optimal redemption boundary A_r can be determined from the smooth-pasting condition $\dot{C}'(A_r) = 0$.²³ Differentiating (A.22) gives

$$\left. \frac{\partial \dot{C}}{\partial A} \right|_{A=A_r} = \frac{-(\gamma - \gamma_1) \left(mL - \frac{cC}{r} \right) \left(\frac{A_r}{A_{uc}} \right)^{-\gamma} + \left(C - \frac{cC}{r} \right) \left(-\gamma_1 + \gamma \left(\frac{A_r}{A_{uc}} \right)^{-(\gamma - \gamma_1)} \right)}{1 - \left(\frac{A_r}{A_{uc}} \right)^{-(\gamma - \gamma_1)}} = 0$$

and equation (A.24) follows.

Note that at the conversion boundary A_{uc} , the bank's per share stock price equals L no matter whether the CoCo is callable or not. However, compared to the non-callable case, when the CoCo is callable the stock price must be higher before conversion since it reflects the shareholders' valuable redemption option. Thus, if condition (ii) of Lemma 2 holds, the stock price will always remain above the conversion trigger as long as the asset level stays above the conversion boundary A_{uc} . Hence, a unique stock price equilibrium exists whenever either condition of Lemma 2 is satisfied and the callable CoCo and the stock before conversion are given by (A.22) and (A.18), respectively.

Theorem 4 recaps our analysis of callable CoCos.

Theorem 4: *When CoCos are callable, there exists a unique stock price equilibrium either when $C \leq mL$ or when $C > mL$ and either condition (i) or (ii) in Lemma 2 is satisfied.*

Note that Theorem 4 gives sufficient, though not necessary, conditions for a unique equilibrium. It is possible that for Case 4, the higher stock value due to the redemption option could lead to monotonicity of the stock price for asset values exceeding A_{uc} even when condition (ii) of Lemma 2 is not satisfied. Hence, compared to the case of a non-callable CoCo, the set of parameters for which a unique stock price equilibrium exists expands when CoCos are callable.

²³Equation (A.24) can also be obtained from the first-order condition $\frac{\partial \dot{C}}{\partial A_r} = 0$.

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Figure 1 Conversion Terms that Favor CoCo Investors

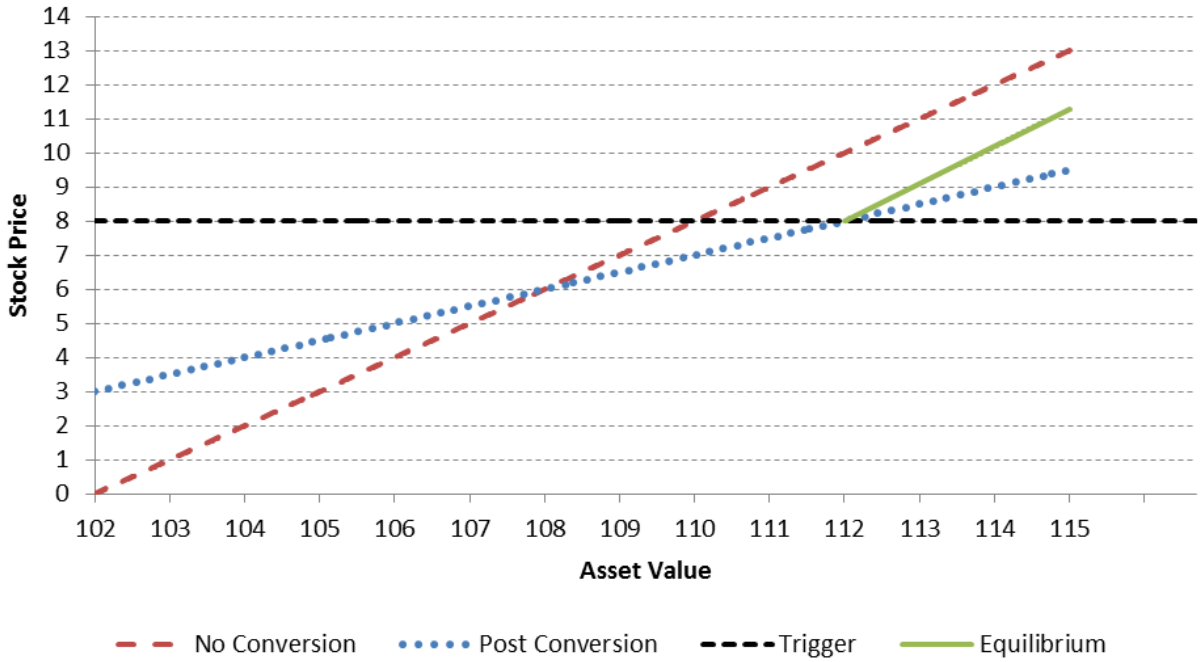


Figure 2 Conversion Terms that Favor Initial Shareholders

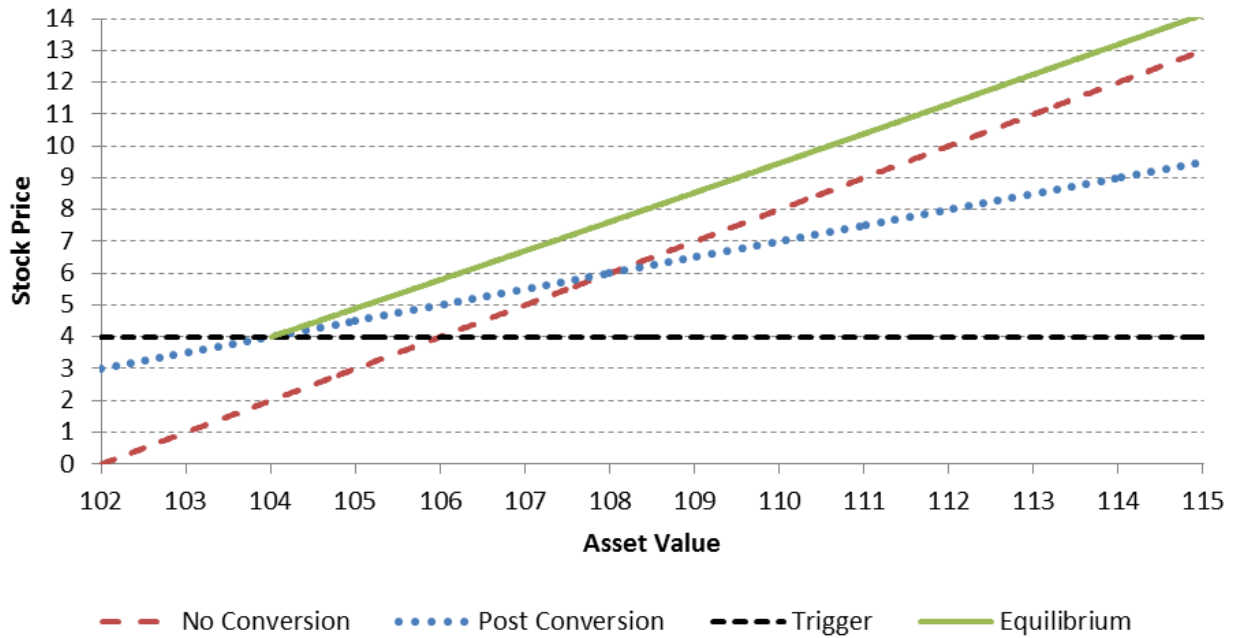


Figure 3 Conversion Terms that Favor Initial Shareholders and Low Asset Return Volatility

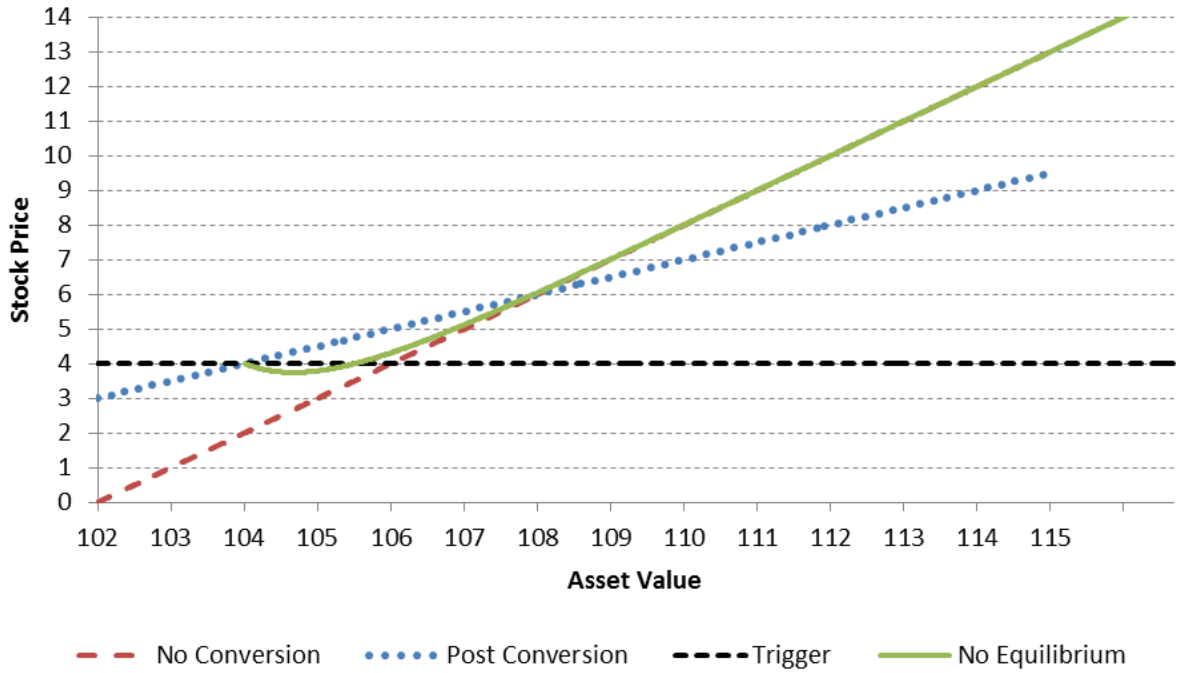


Figure 4 High Trigger with 50% Write Down

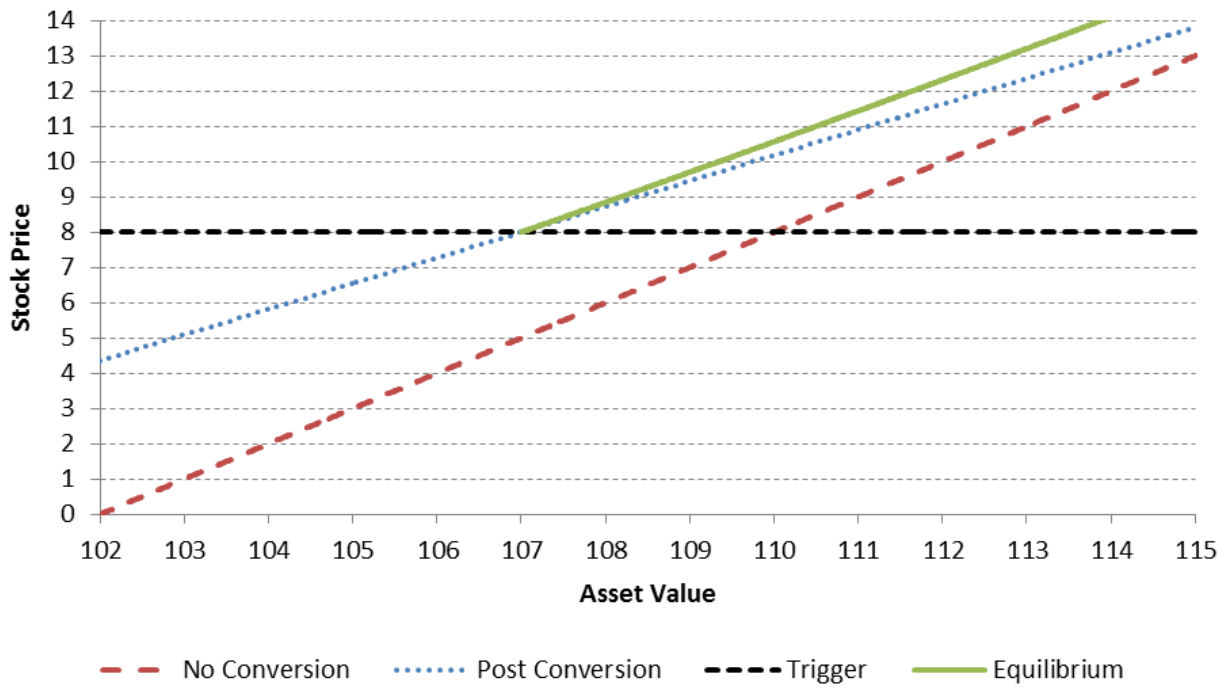


Figure 5 Low Trigger with 50% Write Down

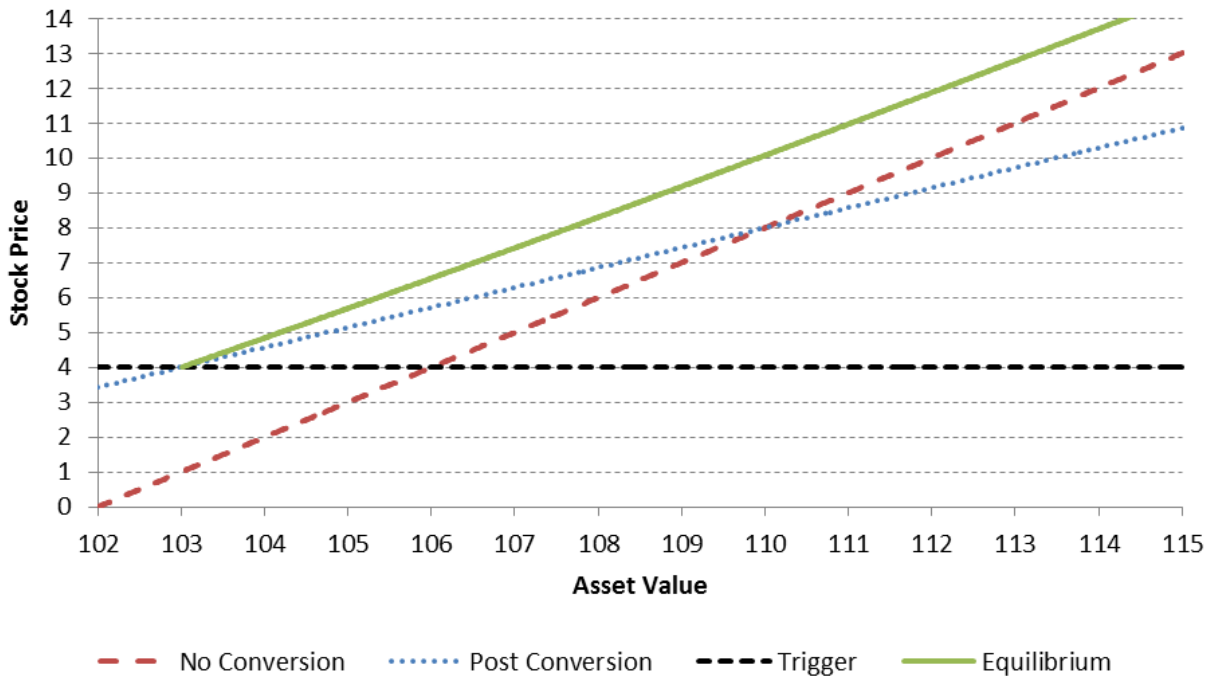


Figure 6 Low Trigger with 100% Write Down

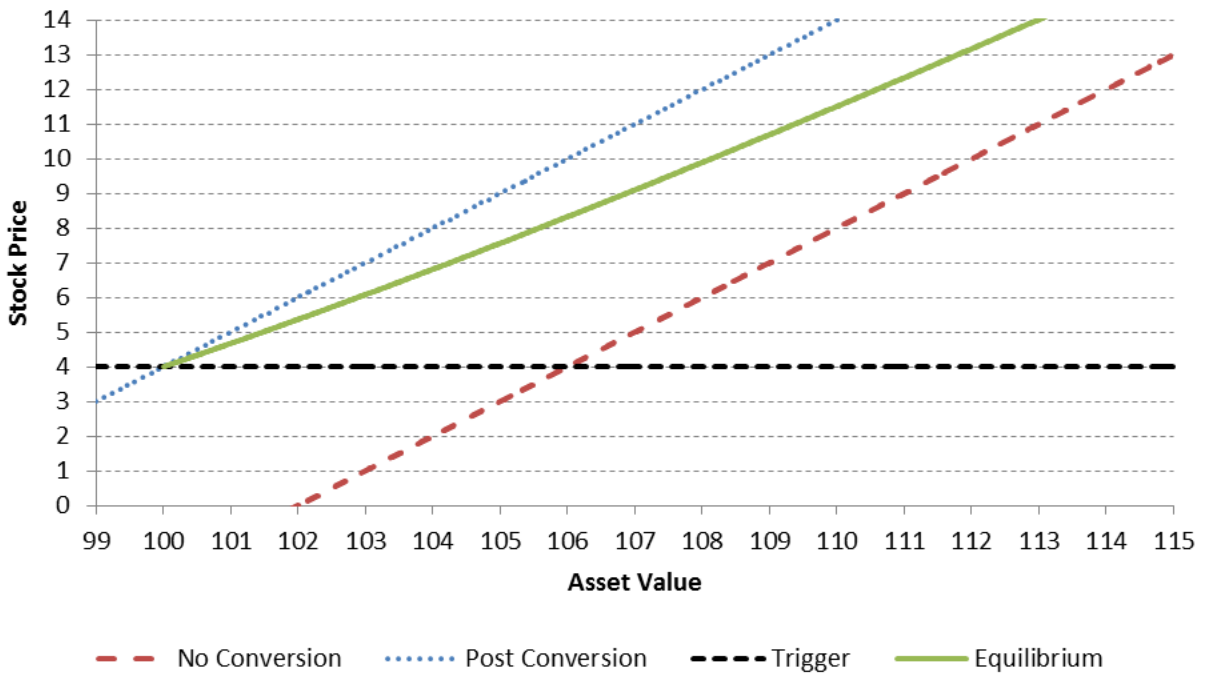


Figure 7 Low Trigger with 100% Write Down and Low Asset Return Volatility

