The Peso Problem: Evidence from the S&P 500 Options Market

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Abstract

The concern about possible market crashes has important consequence on asset pricing. Such tail events in a given sample period mat not realize, driving a wedge between the ex-ante risk perceived by investors and the ex-post losses during the given sample. This *Peso problem* has been proposed to explain high equity risk premiums, but its existence has not been rigorously verified. We propose a methodology to verify the existence and to measure the extent of the Peso problem by using information from both asset returns and prices of options written on the asset jointly. Applying the framework to the S&P 500 Index over the period from 1996 to 2013, we find supportive evidence of the existence of Peso problem and document pro-cyclical dynamics of the Peso problem measure.

JEL classification: G10, G12

Keywords: The Peso problem, option-implied risk-neutral distribution, option-implied risk-neutral distribution, stochastic discount factor, empirical distribution

1 Introduction

Tail events are infrequent events with extreme economic outcomes. A case in point in the financial history of the U.S. is the Great Depression, during which the level of aggregate corporate earnings and aggregate stock prices declined dramatically. Such tail events, however, occur infrequently and hence they may fail to materialize in a given sample period. The absence of tail events drives a wedge between ex-ante risk perceived by investors and ex-post realized risk in a give sample, known as the "Peso problem".[1] The Peso problem has profound implications on asset pricing. In particular, it has a potential to resolve the well-known equity premium puzzle, raised by Mehra and Prescott (1985), that the average realized return on U.S. equity appears too high, relative to the risk free rate, to be justified by the observed volatility of the aggregate consumption growth in a Lucas (1978) endowment economy. Rietz (1988) first proposes a Peso problem explanation of the puzzle within Mehra and Prescott (1985)'s framework. The realized high equity premium is attributed to compensation for bearing the risk of a consumption disaster that is expected rationally by investors but is under-represented by the historical consumption data. Extending the Rietz's model, Barro (2006), Barro and Ursúa (2008, 2009) and Nakamura et al. (2010) match the observed U.S. equity premium using international consumption data as benchmark.

While many studies have demonstrated that an assumed Peso problem can resolve the equity premium puzzle, whether there is an actual Peso problem in equity returns and how large the extent of the problem is have not yet been rigorously examined empirically. The difficulty is that conditional distributions perceived by investors in ex-ante are not

[1]Mexican Peso was pegged to the US dollar in the early 1970s. As its economy deteriorated, investors had expected that the peg would go and such expectation was reflected in the large difference between spot and forward exchange rates. Over a very long period, however, devaluation of Peso did not happen until 1976 when Peso was allowed to float and plummeted by 46%. Such tail events that were expected to happen but did not happen until much later are known as Peso events. The term Peso problem was coined by Milton Friedman to describe the situation in which forward looking return distribution is more left skewed than an empirical distribution from the past realizations.

directly observable. We propose a framework to measure the extent of the Peso problem by using information from time series of equity returns and cross-section options written on the equity jointly. From cross-sectional options prices at a given point of time, we first obtain the option-implied conditional distribution of future equity returns at constant maturity under the risk-neutral measure, convert it to the conditional distribution under the physical measure in conjunction with a specification of the stochastic discount factor, and compare it with the conditional physical distribution of equity returns implied by a time series model, estimated on a finite sample of realized returns. The key notion is that the conditional physical distribution implied from options prices captures the ex-ante risk perceived by investors, no matter they realize in the given sample or not, while the equity return-based conditional physical distribution only represents the extent of realized losses in ex-post. To quantify the wedge in the left tail between the two conditional physical distributions, we define a Peso measure as the proportional difference between the conditional probabilities of one month gross return of the asset below a certain threshold, say 0.9, from the option prices-implied physical distribution and from the equity return-implied physical distribution. We apply our method to study the Peso problem for the S&P 500 Index, a broad market index for the U.S. equity market, over the period from 1996 to 2013. Using the S&P 500 Index returns and its European options, we find that there exists a Peso problem over this sample period and the average value of the Peso problem measure is about 1.1. The Peso problem measure is time-varying and exhibits pro-cyclical variations. It is high in economic expansions, increases during the stock market boom, and vanishes after realizations of large stock market downturns. These findings fill the void in the Peso problem literature.

The advantage of our methodology is twofold. First, options prices embed investors' ex-ante perceived risk, which may not materialized in a given sample. Therefore our approach sidesteps the empirical difficulty in measuring the ex-ante perceived risk by investors. The common approach in the literature to quantify the ex-ante perceived probability and magnitude of a peso component in consumption growth or equity return is calibration. For instance, parameters of asset fundamental are calibrated from historical consumption and dividend data by matching model-implied moments with their sample counterparts (e.g., Rietz (1988), Longstaff and Piazzesi (2004), Barro (2006) and Gabaix (2012), among others). However, by the nature of the Peso problem, matching the ex-post (unconditional) sample moments may not adequately capture the ex-ante concerns of tail events. Secondly, options prices reflect market participants' conditional assessment of future asset movements and options markets provide a wide spectrum of options contracts at each given point of time. Consequently we are able to recover the ex-ante perceived distributions of equity returns and calculate the difference between the ex-ante perceived risk by investors and ex-post realized losses at each time point over the entire sample period. To our best knowledge, quantifying the extent of the Peso problem in this way has not yet been done previously.

Recently several papers also employ options to study the Peso problem in equity and currency returns. Using a novel mixed jump-diffusion process jointly calibrated from time series of S&P 500 Index returns and European options, Santa-Clara and Yan (2010) provide model-specific evidence of the Peso problem hypothesis for the S&P 500 Index. Over the sample period from 1996 to 2002, they find substantial time-varying jump risk with the expected percentage jump size -9.8%, though negative jumps with such magnitude never occurred in the sample. Thus, the Peso problem in their paper manifests as the divergence in the expected jump size between the ex-ante perceived distribution and the ex-post empirical distribution of realized jumps. By contrast, we characterize the extent of the Peso problem as the left tail of the cumulative distribution of equity returns, which incorporates the cumulative effect associated with either single or multiple price jumps and we make less parametric assumptions on tail behaviors of asset returns than they do. Burnside et al. (2011) use foreign currency options to study the Peso problem explanation for the high premium earned by currency carry trade strategiesand find evidence in favor of the Peso problem hypothesis. However, they do not quantify the ex-ante perceived risk in currency carry trade strategies, whereas borrow it from the calibration in the disaster risk literature (Nakamura et al. 2010). By contrast, quantifying the ex-ante probability of tail events perceived by investors is the key component of our paper.

Finally, it should be noted that in our analyses we maintain the assumption that the options market is fully integrated with the spot market and there is no options market specific friction that segment the options market from the spot market, otherwise the ex-ante perceived risk implied from options prices will be biased (Pan, 2002).

The rest of the paper is organized as follows. Section 2 provides a brief literature review of the peso problem, focusing on its implication for the equity premium puzzle. Section 3 lays down the framework of how to measure the extent of the peso problem. Section 4 and 5 apply the framework to study the extent of the peso problem of the S&P 500 Index. Section 4 describes the data for empirical analyses. Section 5 discusses the empirical results and the asset pricing implication of the peso problem on equity premium. Finally, Section 6 concludes.

2 A Brief Literature Review

Our paper is closely related to the Peso problem explanation of the equity premium puzzle. In this section, we briefly review the literature.

Rietz (1988) first put forward a simple yet power Peso problem explanation for the equity premium puzzle within the framework of Mehra and Prescott (1985). He attributes high equity premium to compensation for bearing the risk of a consumption disaster s-tate (large contraction in aggregate consumption) that is expected rationally by investors but are under-represented in the historical consumption data. Building upon the Rietz model, Barro (2006), Barro and Ursúa (2008, 2009) and Nakamura et al. (2010) suc-cessfully match the observed U.S. equity premium with reasonable relative risk aversion parameters. To mitigate the concerns of the Peso problem of the historical U.S. aggregate consumption and equity returns data, they allow the ex-ante perceived consumption disaster probability and magnitude to be different from the U.S. sample frequencies and calibrate them from cross-country data over the twentieth century[2]. Other important contributions in the literature include Danthine and Donaldson (1999), Veronesi (2004) and the survivorship bias hypothesis proposed by Brown, Goetzmann and Ross (1995) and Jorion and Goetzmann (1999), which shares the same spirits as the Peso problem hypothesis.

The main critique of the Peso problem explanation for the equity premium puzzle, initialized by Mehra and Prescott (1988), concentrates on whether a disaster model calibrated to the U.S. historical consumption and dividend data can match the observed U.S. equity premium with reasonable risk aversions. Longstaff and Piazzesi (2004) and Julliard and Ghosh (2012) argue that if calibrated to the U.S. historical evidence, disaster models can not match the level of realized equity premium with reasonable risk aversions. Backus, Chernov and Martin (2011) uncover the unconditional probability and distribution of consumption disasters from S&P 500 Index options prices. They extract the index option-implied unconditional distribution of equity returns by Merton (1976)'s jump-diffusion model and link index returns to the aggregate consumption growth in a Lucas economy with a representative agent with CRRA preference. The index option-implied probability of consumption disasters turns out to be substantially lower than that calibrated by Barro and his coauthor from international consumption data.

Our paper differs from this stream of literature in several dimensions. First, taking the prospective of an investor who holds the equity market in equilibrium, we examine

[2]The probability of consumption disasters in Rietz (1988), Longstaff and Piazzesi (2004) and Barro (2006) is constant and all of them assume that the representative agent has CRRA preference. Recent advances in the literature focus on the asset pricing implication of time-varying disaster risk, generalized utility functions of the representative agent, e.g., recursive preference and habit formation, and recoveries after disasters (e.g., Gourio (2008), Du (2011), Gabaix (2012), Wachter (2013) and Martin (2013), among others).

whether there is a Peso problem for the U.S. aggregate equity market. Another reason of studying equity returns directly is that there are measure errors and temporal aggregation problems of the aggregate consumption data (Campbell, 1993 and the reference therein). Second, we examine whether there is a Peso problem by explicitly characterizing the gap between the ex-ante perceived risk and ex-post realized risk, rather than assuming the existence of the Peso problem in the first place, which is very different from Rietz (1988). We infer the ex-ante perceived probability of disasters in U.S. aggregate equity market from the broad market index options (S&P 500 Index options) rather than calibrating from international consumption data (Barro (2006) and Barro and Ursúa (2008, 2009)). More importantly, we use index options prices jointly with equity returns to infer the ex-ante perceived crash risk by investors rather than solely relying on ex-post realized observations (Longstaff and Piazzesi (2004) and Julliard and Ghosh (2012)). Finally our paper also differs from Backus, Chernov and Martin (2011) who also use S&P 500 Index options to quantify the ex-ante perceived unconditional probability of consumption disasters, in that we estimate the perceived conditional probability of market crashes and study the time variation of the extent of the Peso problem.

3 Methodology

3.1 The Framework

In this section, we propose a methodology to quantify the extent of the Peso problem of an asset. Since options prices incorporate investors' ex-ante assessment of future movement of the underlying asset, a Peso phenomenon emerges when large losses have not been found in a given sample period, but the concerns that they may occur are actually reflected in options prices. Naturally, the extent of the Peso problem can be captured by the gap between ex-ante concerns of large market downturns and their ex-post realizations. A novel feature of our framework is that we explicitly characterize the gap. Specifically, we capture the ex-ante perceived risk by the conditional physical distribution (also known as objective or data-generating distribution) of the asset return implied from its European options and measure the extent of realized risks by the conditional physical distribution, obtained from a time series model estimated from past realized asset returns. The gap in the left tail between the two conditional physical distributions reflects the extent of the Peso problem.

Formally, let $R_{t,t+\tau}$ be the gross return on an asset from the end of period t to the end of period $t+\tau$, let a be a number, say 0.85 to represent 15% loss during the period, let $P_t^S(R_{t,t+\tau} \leq a)$ be the conditioning probability of the gross return over the period less than a, obtained from a time series model estimated from past realized returns only, and let $P_t^O(R_{t,t+\tau} \leq a)$ be the conditioning probability obtained from a physical distribution implied from option prices. The Peso problem measure (PPM (a)) is defined as a percentage difference between the two conditional probabilities as follows:

$$\mathrm{PPM}(a) = \log P_t^O(R_{t,t+\tau} \leq a) - \log P_t^S(R_{t,t+\tau} \leq a). \tag{1}$$

The prices of European options written on the asset are used to recover the option-implied conditional risk-neutral distribution of $R_{t,t+\tau}$ and a projected stochastic discount factor is necessary to transform the option-implied conditional risk neutral distribution to option-implied conditional physical distribution and derive $P_t^O(R_{t,t+\tau} \leq a)$.

The remaining subsections are organized as follows. Subsection 3.2 outlines the procedure of recovering option-implied risk-neutral distributions. Subsection 3.3 discusses the construction of a projected stochastic discount factor to transform the option-implied risk-neutral distribution to physical distributions and how to estimate it. Subsection 3.4 explains how the asset return-based conditional physical distribution, $P_t^S(R_{t,t+\tau})$, is constructed.

3.2 Recover Risk-Neutral Densities from Options Prices

In the absence of arbitrage opportunities, the price of a European option is the expected discounted value of its terminal payoff under the risk-neutral probability measure $\tilde{\mathbb{P}}$, where the discounted rate is the prevalent risk-free interest rate over the life of the option. Specifically, let $C(t, T, K)$ denote the time-t price of a European call option with strike price K and maturity $\tau \equiv T - t$, then

$$C(t, T, K) = \tilde{E}_t[(S_T - K)^+ / R^f_{t,t+\tau}] = \int_K^\infty [(S_T - K)/R^f_{t,T}]\tilde{f}_t(S_T)dS_T \qquad (2)$$

where S_T is the value of the underlying asset at option expiry date T, $\tilde{f}_t(S_T)$ is the time-t conditional density function of S_T under the risk-neutral measure, $R^f_{t,t+\tau}$ denotes the (unannualized) gross risk-free interest rate for the period t to $t + \tau$ and $\tilde{E}_t$ indicates the mathematical expectation under the risk-neutral measure conditioning on time-t information. Through out this paper, the risk-free interest rate is assumed to be deterministic.

When there are continuum European call options with strikes from zero to infinity, Breeden and Litzenberger (1978) (hereafter BL) show that a unique conditional risk-neutral density function of S_T can be recovered by twice differentiating of European call option prices with respective to their strikes (See also (Ross, 1976; Banz and Miller, 1978)),

$$R^f_{t,t+\tau} \frac{\partial^2 C(t, T, K)}{\partial K^2} = \tilde{f}_t(S_T)|_{S_T=K} \qquad (3)$$

In practice, however, only a finite number of call options with strikes surrounding the current price of the underlying asset are available. Proper smoothness of call option prices within the region of available strikes is necessary before applying the BL's formula. The literature of recovering option-implied risk-neutral density is vast and has been extended from equity options to options written on other asset classes.[3] Following the literature,

[3]For example, see Jackwerth and Rubinstein (1996), Ait-Sahalia and Lo (1998, 2000), Ait-Sahalia and Duarte (2003), Bliss and Panigirtzoglou (2002, 2004) for equity index options, Li and Zhao (2009) for

we transform call options prices into their Black's (1976) implied volatilities (IV), use local linear regression to smooth the IV, and convert the smoothed IV back to call option prices at pre-specified grids. Finally, a second order numerical derivative is applied to obtain the value of the risk-neutral density at those grids. The implementation details will be discussed in detail in Section 4.

Limited strikes of available options result truncation errors to the option-implied conditional risk-neutral densities. Since the information of perceived probabilities of tail events is contained in the left tail of conditional risk-neutral densities, how to deal with the left tail is important for studying the peso problem. A common practice in previous studies is to smoothly paste log-normal tails to option-implied risk-neutral densities (e.g., Bliss and Panigirtzoglou (2004)). The underlying assumption is that the tails of index returns follows normal distribution, which is inconsistent with the well documented evidence of stochastic volatility and jump risks in option pricing literature. The tail-completion method by Figlewski (2009) is highly flexible in the heaviness of the tails attached. The method consists of two steps. First, the values of conditional risk-neutral densities within the region of available strikes are recovered as usual. Next, both left and right tails are completed by smoothly attaching the density in the Generalized Extreme Value (GEV) distribution family. In the following empirical analyses, we apply the method to complete tails of the risk-neutral density recovered from S&P 500 Index options. Appendix A details the implementation procedure and summary statistics on parameters of attached GEV densities.[4] Notice that the Figlewski's approach is not arbitrary since it requires that the GEV densities are smoothly attached and have the same probability in the both tails as the original option-implied risk-neutral densities have.

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interest rate floors and caps, and Kitsul and Wright (2013) for options on consumer price index inflation. Jackwerth (1999, 2004) and Figlewski (2009) provide survey of the literature.

[4]The choice of the GEV distribution family is justified by the Fisher–Tippett–Gnedenko Theorem, which states that the maximum of a sample of i.i.d random variables (after proper re-normalization) can only converge in distribution to one of three types of distributions, all of which belong to the family of the GEV distribution.

3.3 Construction and Estimation of the Projected SDF

The difference between risk-neutral probabilities and physical probabilities is that the former also incorporate risk premium attached to each state. Hansen and Richard (1987) show that the absence of arbitrage opportunities implies the existence of a strictly positive random variable, known as *Stochastic Discount Factor* (hereafter SDF) or pricing kernel, denoted by $M_{t,t+\tau}$, such that for each time t, the price of a tradable asset is given by the expectation of its future payoffs weighted by $M_{t,t+\tau}$, conditioning on the time-t information. We rely on this non-arbitrage condition to transform the conditional risk-neutral distribution to the conditional physical distribution. Since we examine the Peso problem of a particular asset, i.e., the S&P 500 Index in our context, we focus on a specific SDF that can price S&P 500 Index returns and its contingent claims. Specifically we consider the projected SDF, which is the market-wide SDF $M_{t,t+\tau}$ projecting on the space spanned by the index returns, conditioning on time-t information (Ait-Sahalia and Lo, 2000 and Rosenberg and Engle, 2002). Unlike the true SDF, which is defined on the fundamental states of the economy, the projected SDF is defined on the states represented by the asset returns. As Cochrane (2005) points out, the projected SDF has the same pricing implication as the true SDF for contingent claims on the underlying asset.

Formally, let $\tilde{M}_{t,t+\tau}(R_{t,t+\tau}) \equiv E_t(M_{t,t+\tau}|R_{t,t+\tau})$ be the (unobservable) projected SDF where $R_{t,t+\tau}$ is the gross return of the index over the period $[t, t+\tau]$. Let $\tilde{f}_t(R_{t,t+\tau})$ be the time-t conditional density of $R_{t,t+\tau}$ under the risk-neutral measure, which is recovered from time-t options prices with maturity τ. Let $f_t(R_{t,t+\tau})$ be the conditional density of $R_{t,t+\tau}$ under the physical measure $\mathbb{P}$. In the absence of arbitrage opportunities, the relation between $\tilde{f}_t(R_{t,t+\tau})$, $f_t(R_{t,t+\tau})$ and the projected SDF $\tilde{M}_{t,t+\tau}(R_{t,t+\tau})$ is given by,

$$\tilde{f}_t(R_{t,t+\tau}) = f_t(R_{t,t+\tau}) \frac{\tilde{M}_{t,t+\tau}(R_{t,t+\tau})}{E_t[\tilde{M}_{t,t+\tau}(R_{t,t+\tau})]} \tag{4}$$

Given the option-implied conditional risk-neutral density $\tilde{f}_t(R_{t,t+\tau})$ and the projected SDF $\tilde{M}_{t,t+\tau}(R_{t,t+\tau})$, the option-implied conditional physical density $f_t(R_{t,t+\tau})$ is determined by

$$f_t(R_{t,t+\tau}) = \frac{\tilde{f}_t(R_{t,t+\tau})/\tilde{M}_{t,t+\tau}(R_{t,t+\tau})}{\int_0^\infty \tilde{f}_t(R)/\tilde{M}_{t,t+\tau}(R)dR} \tag{5}$$

There are many ways to construct and estimate a projected SDF. One approach is to specify a functional form, which can be derived from more preliminary assumptions about preferences and endowments. Given the functional form, say, $\tilde{M}_{t,t+\tau}(R;\theta)$, we estimate the projected SDF via non-linear least squares with the following two conditions. The first condition takes from the definition of the SDF,

$$E_t[\tilde{M}_{t,t+\tau}(R_{t,t+\tau};\theta)] = 1/R^f_{t,t+\tau} \tag{6}$$

The second condition exploits the property that time-t conditional expected return should be the optimal ex-ante estimate of the ex-post realized return, conditioning on the time-t information. Since the time-t conditional expected return is determined by the conditional physical density $f_t(R_{t,t+\tau})$, which in turn is determined jointly by the conditional risk-neutral density $\tilde{f}_t(R_{t,t+\tau})$ in conjunction with the specified projected SDF. Since $\tilde{f}_t(R_{t,t+\tau})$ summarizes all the available information embedded in options prices in a non-parametrical manner, the estimation of the projected SDF boils down to find the parameter θ such that the time-t option-implied conditional physical density can best predict the next period asset return.

Denote $R^\star_{t,t+\tau}$ as the realized gross return of the S&P 500 Index. The error $\eta_{t+\tau}(\theta)$ of the two least square conditions is simply,

$$\eta_{t+\tau}(\theta) \equiv \begin{bmatrix} R^\star_{t,t+\tau} - E_t[R_{t,t+\tau};\theta] \\ 1/R^f_{t,t+\tau} - E_t[\tilde{M}_{t,t+\tau}(R;\theta)] \end{bmatrix} \tag{7}$$

where the conditional expected gross return $E_t[R_{t,t+\tau};\theta]$ is evaluated numerically via the adaptive Gaussian quadrature method. The nonlinear least square estimates θ by minimizing the summation of weighted errors, where Σ_t is a weighting matrix, which is allowed to be time-varying,

$$\tilde{\theta} = Argmin_\theta \sum_{t=1}^{T} \eta_{t+\tau}(\theta)' \Sigma_t \eta_{t+\tau}(\theta) \tag{8}$$

We use bootstrap method to compute the confidence interval of parameters. Specifically, we re-sample the estimated residual $\eta_{t+\tau}(\tilde{\theta})$ with replacement, add re-sampled residuals back to the fitted value to generate bootstrap sample of $(R^\star_{t,t+\tau}, 1/R^f_{t,t+\tau})$ and re-estimate θ using the new sample. We repeat the procedure 500 times and compute the bootstrap percentile intervals for each parameter.

3.4 Construction and Estimation of the Empirical Distribution

In constructing the conditional physical distributions estimated from past realized asset returns, three related dimensions are to be considered. The first is whether a parametric time series model, for instance a structural stochastic process with latent state variables, should be chosen for returns or pure non-parametric approach is more appropriate. The second is whether a fixed window of past sample should be chosen or all the available past data should be used. The third choice is whether all the data in the chosen sample should be treated equally or more emphasis should be given to more recent data. While all these choices will generate different conditional physical distributions, a valid inference should maintain a reasonable degree of robustness.

We implement various combinations of the choices above and present the results in the paper using a specified return distribution, over all the available data from a starting point, and give exponentially declining weights to the data in the past relative the current time point. Alternatively, our empirical results are qualitatively similar when we use a structural stochastic process to describe the dynamics of the S&P 500 Index returns, a continuous-time square-root stochastic volatility with compound Poisson jumps model (hereafter SVJ). SVJ models are commonly used in the option pricing literature to describe the time series behavior of equity index returns. We adopt the Bates's (2006) specification in which the jump intensity of the compound Poisson process is allowed to depend on the time-varying volatility and estimate the model by the approximate maximum likelihood method in Bates (2006).

4 Data

We apply the our framework in the previous section to study the Peso problem hypothesis of the S&P 500 Index. In this section, we describe data and the procedure of recovering conditional risk-neutral densities from S&P 500 Index options.

4.1 Data and Summary Statistics

Our sample period spans from January 1996 to January 2013, 18 years (205 months) in total. Data on S&P 500 Index options, obtained from the OptionMetrics Ivy DB database, consists of closing quotes, trading volumes and open interests of options written on the S&P 500 Index (symbol SPX). SPX options are European cash-settled options expired on the third Friday of each month and are the most actively traded options listed on the Chicago Board Options Exchange (CBOE). The risk-free interest rates are proxied the U.S. LIBOR rates, obtained from Data Stream. The closing values of the S&P 500 Index and the CBOE VIX index are from the Center for Research in Security Prices (CRSP) and CBOE, respectively. The CBOE VIX index is the expected risk-neutral variance of the S&P 500 Index returns over the subsequent 30 calendar days, calculated by the CBOE using SPX options prices in a model-free manner.[5] In the later empirical analyses,

[5]See Demeterfi et al. (1999), Britten-Jones and Neuberger (2000), Carr and Madan (2001) and Jiang and Tian (2005) for more details about the underlying theory and design of the CBOE VIX index.

we square the VIX index and divided it by 12 since the VIX index is expressed in the annualized volatility unit and we want to translate it into monthly variance unit. The VIX term structure data, downloaded from Travis L. Johnson's homepage, consists of the expected risk-neutral variance of the future S&P 500 Index returns at constant maturities of 2, 3, 6, 9 and 12 months, which are constructed in the exact model-free manner as the CBOE VIX index.[6]

To clean the SPX option data, first, any option contracts with missing values of bid, ask quotes or implied volatility computed by OptionMetrics is excluded. Then two standard filters are applied: (1) options whose bid price is zero or higher than ask price are excluded to avoid market microstructural noises; (2) options whose mid-quotes, i.e., the averages of bids and asks, violate non-arbitrage bounds, and options with maturity greater than one year or less than one week, are dropped. Through our empirical analyses, we take closing mid-quotes as the observed option prices.

To mitigate the concern of non-synchronous trading of index components and reduce the potential errors introduced by estimating the index dividends, we follow Ait-Sahalia and Lo (1998) to compute the option-implied forward price using Put-Call Parity. At each month-t, for every maturity, we identify the at-the-money (ATM) put and call pair[7] as the one which has the minimal absolute difference between their mid quotes. The time-t S&P 500 forward price $F_{t,t+\tau}$ for each maturity τ is then implied from the ATM put and call pair by Put-Call Parity, which should hold if there is no arbitrage opportunity,

$$C(t,T,K) + K/R^f_{t,t+\tau} = P(t,T,K) + F_{t,t+\tau}/R^f_{t,t+\tau}$$

where $R^f_{t,t+\tau}$ is the gross LIBOR rate over the remaining life of the option, interpolated from adjacent LIBOR rates whenever needed. Because out-of-money (OTM) puts are

[6]http://faculty.mccombs.utexas.edu/johnson/data.html

[7]For a put/call pair, we mean that it consist of two options with the same maturity and strike, but one is put and another is call.

very liquid while in-the-money (ITM) calls are often illiquid, we use the implied forward prices and OTM put prices to obtain the associated ITM call options prices by Put-Call Parity. Finally we discard all put options since information embedded in OTM puts is already extracted by Put-Call Parity.

In what follows, we focus on the Peso problem at one-month horizon (or 30 calendar days) and hence fix $\tau = 1$, though our method can be applied to other horizons. Consequently, we keep SPX options observations on the third Wednesday of each month only, because it is the date within a month that is mostly like to have option contracts with maturity of exact 30 calendar days. If at that day, there is no option with maturity of 30 calendar days, two option series with adjacent maturities that straddle 30 calendar days are selected. The final sample for recovery of the risk-neutral density consists of call options with maturity within two months. The average number of single option series in the cross-section with maturity closed to 30 calendar days is around 40 before the year 2004 and more than 100 after that.

Table 1 reports the summary statistics of the number of options contracts and average implied volatilities in the final sample. The total sample size is $24,022$. We divide the entire sample into six moneyness and two maturities bins, creating 12 intersecting groups in all. The cutoff points in maturity dimension corresponding to one month and two month are 39 days and 69 days, respectively. In the two month maturity, there are more options belong to the deep out-of-the-money groups ($K/S \leq 0.94$). Across moneyness, for both maturities, the implied volatilities exhibit smile patterns. On average, the implied volatility at a given moneyness level decreases in maturity.

[Table 1 is here.]

Table 2 reports the summary statistics of monthly observations of the monthly (30 calendar days) gross return on the S&P 500 Index and gross LIBOR rates. The month index return has sample skewness of -0.82 and kurtosis of 4.48, indicating asymmetry and heavy tail behavior. Note that to align with the SPX options data, the monthly return is computed from the third Wednesday in the current month to the 30 calendar days ahead.

[Table 2 is here.]

Figure 1 graphs the monthly observations of the S&P 500 Index level, its 30 day-ahead gross return and the gross U.S. LIBOR rates. The shaded areas are recession periods defined by the National Bureau of Economic Research (NBER). There are three large market downturn in our sample with index returns below -15%: July 2002 (the Crash of Dot-com firms), October 2008 (the default of Lehman Brothers) and August 2011 (the European Sovereign debt crisis), respectively.

[Figure 1 is here.]

Figure 2 examines the SPX options sample on its moneyness coverage. The time series of the upper and lower bounds of option moneyness (strike-to-spot ratio K/S) in the cross-section are plotted. During the normal time (NBER expansion periods), the upper and lower bounds of moneyness fluctuate around 1.1 and 0.75 respectively. By contrast, during recessions, the moneyness spread widened substantially, in response to sharply raised market volatilities. In November 2008, two months after the default of Lehman Brothers, the spread expanded to $[0.37, 1.52]$ and remained widely since then. The black dot line in figure shows the fixed moneyness level $K/S = 0.8$, which translates into the one month index gross return of 0.8. Notice that the moneyness curve lies inside the moneyness coverage region almost all the time. In the later empirical analyses, we evaluate the conditional probability that one month ahead S&P 500 Index gross return less than 0.85 and 0.9, which is well covered by the SPX option moness. Taken together, our option sample provides adequate coverage of one-month ahead S&P 500 Index movement.

[Figure 2 is here.]

4.2 SPX Option-Implied Risk-Neutral Density

We recover SPX option-implied conditional risk-neutral densities with constant maturity of 30 calendar days. First, for the third Wednesday of each month, call options with maturity being exact 30 calendar days are selected and their prices are converted into Black's implied volatilities via Black (1976)'s option pricing formula with option-implied forward prices as the underlying. Next, the region of moneyness (the strike-spot ratio K/S) of available options are divided into fine grids. We treat the Black's implied volatility (IV) for a given maturity τ as a univariate function of moneyness, denoted as $\sigma^{imp}(K/S, \tau)$, and use the local linear regression to estimate the IVs at the specified moneyness grids[8].

As a non-parametric regression, the local linear regression runs a linear weighted least square regression at each point of interest (Fan and Gijbels, 1996). To estimate $\sigma^{imp}(x_0, \tau)$, the IV at the moneyness level x_0 for the given maturity τ ($\tau = 1$, i.e., one-month in our context), local linear regression solves the following weighted least square problem,

$$min_{\beta_0(x_0),\beta_1(x_0)} \sum_{i=1}^{N} (\sigma^{imp}(X_i, \tau) - \beta_0(x_0) - \beta_1(x_0)(X_i - x_0))^2 G(\frac{X_i - x_0}{h})/h, X_i \equiv \frac{K_i}{S}$$

where N is the number of options with maturity τ in the cross-section, $G(\cdot)$ is a kernel function that controls the weight assigned to each observation in the neighborhood around $\sigma^{imp}(x_0, \tau)$, and h, the so-called "bandwidth", controls the size of the neighborhood. The estimated $\sigma^{imp}(x_0, \tau)$ is simply the intercept $\hat{\beta}_0(x_0)$. We use the Gaussian kernel $G(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ and select the bandwidth h via the leave-one-out cross-validation. Finally we convert estimated constant-maturity IVs back to call options prices and take second order numerical derivative at the specified moneyness grids. Beyond the region of

[8]When there are no options with maturity being exact 30 calendar days, two adjacent option series with maturity that straddle 30 calendar days are selected. The Black's implied volatility of the two option series are smoothed and linearly interpolated across maturity to obtain the implied volatility at maturity of 30 calendar days.

available strikes, both tails of the implied conditional risk-neutral density are completed by attaching the GEV density via Figlewski's method.

Figure 3 plots the option-implied conditional risk-neutral densities on four selected sample dates, 2007/02/21, 1996/04/17, 2008/07/16 and 2008/11/19, respectively. The panel A and D are two dates which represent the least and most volatile market condition, measured by the CBOE VIX index over the sample period. Panel B and C are the two dates when the CBOE VIX index is at the 30% and 75% sample percentile, respectively. Visual examination of the figure indicates that the shape and scale of the conditional risk-neutral densities vary significantly over market conditions. When the expected future volatility raises, risk-neutral densities become more widely dispersed with the probability mass moving disproportionately towards the left tail. With the semi-parametric recovery method, the rich dynamics of the conditional risk-neutral densities, especially of their left tail behavior, are readily captured. In addition, we compute the 30 days VIX index based on the SPX option-implied distributions, which is found to be highly correlated with the CBOE VIX index with the correlation coefficient of more than 0.99.

[Figure 3 is here.]

5 Empirical Results

The section is organized as follows. Section 5.1 discusses the specification of the parametric form of the projected SDF and estimation results. In Section 5.2, we transform the option-implied conditional risk-neutral density into conditional physical density by the estimated projected SDF and investigate the time series variation of the extent of the Peso problem for the S&P 500 Index. Section 5.3 examines the implication of the Peso problem for equity risk premium.

5.1 Projected SDF Specification and Estimation

We use conditional projected SDFs to conduct measure transformation and motivate the functional form of the conditional projected SDF from general equilibrium models. Compared to unconditional (projected) SDFs used to transforms option-implied risk-neutral distribution (Bliss and Panigirtzoglou, 2004), Kang and Kim, 2006, Liu, Shackleton, Taylor and Xu, 2007 and Linn, Shive and Shumway, 2014), conditional (projected) SDFs are both theoretically appealing and empirical supported. Theoretically, dynamic general equilibrium models show that state variables which determine the investment opportunity set faced by investors affect the representative agent' marginal utility and hence enter into the SDF (e.g., Merton (1973), Lucas (1978), Cox, Ingersoll, and Ross (1985) and Campbell (1993)). Empirically, mounting evidence from literature on index option pricing and cross-section stock return show that aggregate stock market variance is stochastic and carry a negative risk premium (Bakshi and Kapadia (2003), Adrian and Rosenberg (2008), Bollerslev, Tauchen and Zhou (2009), Carr and Wu (2009) and Drechsler and Yaron (2010) among others).

We derive the conditional projected SDF in an infinite horizon Lucas endowment economy in which there is a representative agent who receives the (exogenously given) aggregate consumption stream C_t at discrete-time $t = 1, 2, ...$ Her time-t lifetime utility U_t over the consumption stream is described by Epstein and Zin (1989) recursive preference,

$$U_t = [(1-\delta)C_t^{1-\frac{1}{\psi}} + \delta(E_t U_{t+1}^{1-\gamma})^{\frac{1-\frac{1}{\psi}}{1-\gamma}}]^{\frac{1}{1-\frac{1}{\psi}}} \tag{9}$$

where δ is her subjective discount factor, γ is the coefficient of relative risk aversion (RRA) and ψ refers to the elasticity of inter-temporal substitution (EIS). In the Epstein-Zin recursive utility, relative risk aversion is separated from elasticity of inter-temporal substitution, while in the CRRA preference, one is the reciprocal of the other.

Epstein and Zin (1989) have shown that the logarithm of the one-period SDF $m_{t,t+1} \equiv logM_{t,t+1}$ satisfies,

$$m_{t,t+1} = \theta log\delta - \frac{\theta}{\psi} g_{t+1} + (\theta - 1) r_{t+1} \tag{10}$$

where $\theta \equiv \frac{1-\gamma}{1-\frac{1}{\psi}}$ and $r_{t+1} \equiv logR_{t+1}$ is the log return on the aggregate portfolio that delivers the aggregate consumption stream $\{C_{t+s}\}_{s=1}^{\infty}$ as its dividend. We further assume that the aggregate consumption stream is the dividend derived from the aggregate market portfolio, therefore R_{t+1} is the return on aggregate market portfolio. The assumption can be relaxed, however. The projected one period SDF in the economy, which is the true SDF projected on the aggregate equity return space conditional on time-t information, is given by,

$$\tilde{M}_{t,t+1} \equiv E_t[M_{t+1}|R_{t+1}] = e^{\theta log\delta + (\theta-1) r_{t+1}} \underbrace{E_t[e^{\frac{\theta}{\psi} g_{t+1}}|R_{t+1}]}_{\text{projection}} \tag{11}$$

To work out the analytical form of the conditional projected SDF, we complete the model by assuming the aggregate consumption dynamics. In fact, different conditional projected SDFs correspond to different assumption of consumption dynamics. Here we take the stylized general equilibrium model by Bollerslev, Tauchen and Zhou (2009) (hereafter BTZ), and derive its implied projected SDF. BTZ extend the long-run risk model of Bansal and Yaron (2004) by allowing the volatility of the aggregate consumption growth to be driven by two volatility factors. With Epstein-Zin recursive preference, the agent cares about the inter-temporal risk, which is the time-varying stochastic volatility of consumption growth in the model. Consequently, shocks to the two consumption volatility states are allowed to be priced and carry negative risk premium if the agent prefers early resolution of uncertainty. The authors demonstrate that the variance risk premium, defined as the difference between the expected risk-neutral variance of returns to aggregate wealth and its expectation under the physical measure, accounts for a major fraction of the equity premium level and also its dynamics. Empirically, they find that the variance risk premium is one of the best short-run return predictor.

In BTZ, the one period consumption growth $log(C_{t+1}/C_t)$ has a constant mean μ_g and stochastic volatility $\sigma^2_{g,t}$. The consumption volatility $\sigma^2_{g,t}$ and the volatility of the consumption volatility q_t take the discrete-time analogy of the continuous time square-root process as follows,

$$\begin{aligned} log(C_{t+1}/C_t) \equiv g_{t+1} &= \mu_g + \sigma_{g,t} z_{g,t+1} \\ \sigma^2_{g,t+1} &= a_\sigma + \rho_\sigma \sigma^2_{g,t} + \sqrt{q_t} z_{\sigma,t+1} \\ q_{t+1} &= a_q + \rho_q q_t + \phi_q \sqrt{q_t} z_{q,t+1} \end{aligned} \tag{12}$$

where $a_\sigma > 0$, $a_q > 0$, $|\rho_\sigma| < 1$ and $|\rho_q| < 1$, $\phi_q > 0$, and $(z_{g,t+1}, z_{\sigma,t+1}, z_{q,t+1})$ follows i.i.d standard multivariate normal distribution. The representative agent is endowed with the Epstein-Zin recursive utility. As the usual assumption in the long-run risk literature, $\gamma > 1$ and $\psi > 1$ and hence $\theta < 0$. The agent prefers early resolution of uncertainty. Under the preference, both volatility factors are priced and carry negative premium. We state the model implied one-period projected SDF in the following proposition. Appendix B provides the derivation.

Proposition 1 (Projected SDF in BTZ) *The projected one period SDF in BTZ, $\tilde{M}_{t,t+1}(R_{t,t+1}) \equiv E_t[\tilde{M}_{t,t+1}|R_{t,t+1}] = E[\tilde{M}_{t,t+1}|R_{t,t+1}, \sigma_{g,t}, q_t]$ takes the following form,*

$$\tilde{M}_{t,t+1} = exp[(\gamma_0 + \gamma_1 \eta_t) log R_{t,t+1} + (1 + \gamma_0 + \gamma_1 \eta_t)\zeta_0 + (\xi_0 + \xi_1 \eta_t)\sigma^2_{g,t} + (\varphi_0 + \varphi_1 \eta_t) q^2_t] \tag{13}$$

where $\gamma_0 = -\gamma < -1$ and $\gamma_1 = \frac{\theta}{\psi} < 0$. η_t is the one period variance risk premium over the linear combination of the two volatility state variables.

We name this projected SDF as the "conditional slope" projected SDF as it features for a state-dependent coefficient of market return that is an affine function of η_t. The two factor consumption volatility structure is key. When there is only one consumption volatility state variable, that is, q_t degenerates into a constant, η_t degenerates into a constant and the projected SDF is simply

Corollary 2 (Projected SDF in BTZ with one state variable)

$$\tilde{M}_{t,t+1} = exp[\tilde{\gamma_0} log R_{t,t+1} + \tilde{\zeta_0} + \tilde{\xi_0}\sigma^2_{g,t}] \tag{14}$$

where $\tilde{\gamma_0} < -1$.

The equation (13) serves as our projected SDF specification. However, it involves state variables and the return on the aggregate market portfolio, which is not observable by econometricians. To tackle this issue, first, following a large literature, we regard the S&P 500 Index return as a reasonable proxy for the return of the aggregate wealth (Campbell et al. 2012 and reference therein). Second, we show in the Appendix B that both the squared CBOE VIX and 12 month VIX are affine functions of two volatility state variables $\sigma^2_{g,t}$ and q_t. Consequently, in the projected SDF, the two state variables $\sigma^2_{g,t}$ and q_t can be substituted by $VIX_t^2/12$ and $VIX12_t^2/12$.[9]

Theoretically, η_t is the one period variance risk premium VRP_t divided by the linear combination of the two volatility state variables. The denominator of η_t is proxied by a linear combination of the squared CBOE VIX and the 12 month VIX, given the affine relation between VIX indices and volatility state variables. The nominator of η_t, VRP_t is calculated by the difference between the squared CBOE VIX index and the expected physical variance $\sigma^2_{r,t}$ (Bollerslev, Tauchen and Zhou (2009), Drechsler and Yaron (2010)). To have a proxy for the latter, we find that statistically an EGARCH(2,1) model (Nelson, 1991) with return innovations from the family of *Generalized Hyperbolic Distribution* (Barndorff-Nielsen, 1977) captures the persistence and leverage effect of variance dynamics of daily S&P 500 Index returns well. We then forecast expected 30 days physical

[9]Since the squared VIX indices are annualized risk-neutral expected variance, we divide them by 12 to translate them into monthly unit.

variance by the EGARCH model, fitted to past 20 years daily index returns and compute the one period variance risk premium[10]. Finally, in our empirical implementation, the conditional projected SDF takes the following functional form,

$$\tilde{M}_{t,t+1} = exp[(\gamma_0+\gamma_1\eta_t)logR_{t,t+1}+(1+\gamma_0+\gamma_1\eta_t)\zeta_0+(\xi_0+\xi_1\eta_t)\frac{VIX_t^2}{12}+(\varphi_0+\varphi_1\eta_t)\frac{VIX12_t^2}{12}] \tag{15}$$

where η_t is specified as,

$$\eta_t = \frac{VRP_t}{VIX_t^2/12+\kappa_0 VIX12_t^2/12} \tag{16}$$

Panel B of Table 2 reports the summary statistics of variables used in the projected SDF, including the monthly squared VIX indices, $VIX_t^2/12$, $VIX12_t^2/12$, the estimated 30 days ahead expected physical variance $\sigma_{r,t}^2$ by the EGARCH model and the associated variance risk premium VRP_t. Two prominent features emerge. First, the sample average of both squared monthly VIX indices is the closed to each other, around 0.0047 and is higher than the average expected physical variance, 0.0029. The average of one period variance premium, calculated as the gap between the squared one month VIX and the expected physical variance, is 0.0018 and accounts for 38% percent of the average squared CBOE VIX. Second, the short term VIX is less persistent than the long term given their first order auto-correlation coefficients, 0.76 and 0.87, respectively. Also, the short term VIX has higher skewness (5.05 Vs 2.36) and kurtosis (37.89 Vs 13.56). These difference imply that the short term VIX was hit by large positive yet less persistent shocks and reverted back to its normal level quickly, as is evident from Panel A of Figure 3. The one month expected physical variance $\sigma_{r,t}^2$ has similar persistence, skewness as well as kurtosis

[10]BTZ use the past 30 days realized variance, estimated from intra-day 5 minutes index returns to proxy the expected physical variance. The resulted variance risk premium takes some extreme negative values, especially during the 2008 Great Recession. By contrast, the estimated variance premium by the EGARCH model takes only three negative values, all of which are closed to zero. To be aligned with the model, we trim the variance risk premium at zero. We entertain various methods to compute the variance premia and find the empirical results are robust to different proxies.

as the square short term VIX index. Panel B of Figure 3 plots the dynamics of VRP_t, the one period variance risk premium. Compared with VIX indices shown in Panel A, VRP_t features for similar counter-cyclical dynamics while is less persistent with the first order auto-correlation coefficient only 0.57.

We estimate the projected SDF via nonlinear least squares discussed in Section 3.3. To improve estimation efficiency, for the first pricing error in equation (7), we scale it by the square root of the EGARCH forecasted 30 days ahead expected variance. Panel A of Table 3 reports the estimate. Consistent with the model restriction on preference parameters, the estimated γ_1 is -1.8847 and the estimated γ_0 is less than -1. Thus conditioning on the two state variables VIX_t^2 and $VIX18_t^2$, the shape of the projected SDF is monotonically decreasing function of S&P 500 Index returns. Our estimated projected SDF is consistent with the economic restriction on the SDF in a representative agent model and it is also in line with the findings by Linn, Shive and Shumway (2014) who argue that lack of using conditioning information in estimation of the projected SDF can cause the estimated projected SDF, as a function of the index return, to exhibit non-monotonic patterns.

[Table 3 is here.]

Figure 5 plots the absolute value of $\gamma_0 + \gamma_1\eta_t$, the coefficient of market return in the projected SDF. The coefficient exhibits large time series variation. It shot up during two recent NBER recessions and in particular, reached the all-time high during the 2011 European Sovereign debt crisis. Due to the time-varying $\gamma_0 + \gamma_1\eta_t$, the slope of the projected SDF and consequently the model-implied expected equity risk premia vary over business cycles. As will be shown in the Section 5.3, the increasing stock market volatility (quantity of risk) and the steepened slope of the projected SDF (price of risk) during high market volatility regime generate large counter-cyclical equity risk premia, which peaked in the 2008 Great Recession and consequently caused the realized index returns to be substantially negative.

[Figure 5 are here]

5.2 Evidence of the Peso Problem Hypothesis

Given the estimated conditional projected SDF $\tilde{M}(R_{t,t+1};\tilde{\theta})$, we transform the time-$t$ option-implied conditional risk-neutral density of S&P 500 index returns into the option-implied conditional physical density and calculate the probability of one month index gross return below 0.85 and 0.9 under the option-implied distribution, i.e., $P_t^O(R_{t,t+1} \leq 0.85)$ and $P_t^O(R_{t,t+1} \leq 0.9)$. Alternatively, we construct index-return based conditional physical distribution, obtained from a time series model estimated from past realized index returns and compute the probabilities under the index-return based distribution $P_t^S(R_{t,t+1} \leq 0.85)$ and $P_t^S(R_{t,t+1} \leq 0.9)$. To measure the extent of the Peso problem, we compute our Peso problem measure (PPM (a)), defined in equation (1), which are the percentage difference in conditional probability of one month index gross return below 0.85 and 0.9 under the two conditional physical distributions, $PPM(a) \equiv log(P_t^O(R_{t,t+1} \leq a)) - log(P_t^S(R_{t,t+1} \leq a))$ for $a = 0.85$ or $a = 0.9$.

To construct the index-return based physical distribution, we model the dynamics of S&P 500 Index parsimoniously without introducing looking-ahead biases. A method in spirit of the filtered historical simulation method is adopted by us (Barone-Adesi, Giannopoulos and Vosper (1999) and Faias and Santa-Clara (2011)). First we standardize monthly index log returns $r_{t,t+1}$ by the expected volatility of 30 days ahead index returns $\sqrt{\sigma_{r,t}^2}$ forecasted by the EGARCH (2,1) model to remove the ARCH effect of the monthly index returns. Next at each time-t, a specified return distribution is fitted to all past standardized log returns $\{\frac{r_{t-j,t-j+1}}{\sqrt{\sigma_{r,t-j}^2}}\}_{j=1}$ since the beginning of year 1980 by maximum likelihood. When maximizing the likelihood, we assign exponentially declining weights to historical data points to give more emphasis to more recent data.[11] We choose the Variance

[11] Suppose at time-t the past standardized returns are $\{\tilde{r}_{t-j-1,t-j}\}_{j=1}$, then each $\tilde{r}_{t-j-1,t-j}$ is assigned a weight $w_j \equiv \omega^j(1-\omega)$ where $\omega = 0.99$. We maximize the weighted likelihood function to estimate the

Gamma (VG) distribution, a special case of the family of the Generalized Hyperbolic distribution, as our candidate return distribution which nests the normal distribution as a limiting case. The Variance Gamma process, introduced by Madan and Seneta (1990) and extended by Madan, Carr and Chang (1998) to allow for asymmetry, is a pure jump Lévy process with infinite activity rate, meaning that the number of price jumps in any given time interval is infinite. The increments of the VG process follow independent VG distribution. The benefit of using pure jump processes with infinite activity rate to model asset returns is that they are highly flexible in terms of capturing asset prices variation to various extent.[12] Finally, the index-return based $P_t^S(R_{t,t+1} \leq a)$ is given by $F_t^{VG}(\frac{loga}{\sqrt{\sigma_{r,t}^2}})$ where F_t^{VG} is the cumulative distribution function of the fitted VG distribution.

Figure 6 displays the dynamics of $P_t^S(R_{t,t+1} \leq a)$ and $P_t^O(R_{t,t+1} \leq a)$ for $a = 0.85$ and 0.9. In both panels, the conditional probabilities under option-implied physical distributions (blue solid line) $P_t^O(R_{t,t+1} \leq a)$ almost always dominate the conditional probability under index-return based distribution $P_t^S(R_{t,t+1} \leq a)$ (black dash dotted line). However, there are two exceptions. The first is the default of Lehman Brothers on October 2008, accompanied by the most negative monthly realized S&P 500 Index return during our sample period, -18.67%. The second one occurred during the time period from July 2011 to August 2011 when investors became increasingly worried about the resolution of the European Sovereign debt crisis[13]. The global stock markets, especially the European sector, were hit hard and the realized SPX index return -15.3% was the second most negative one during the sample period. For comparison, both panels of Figure 6 also display

distribution parameters. The results are the same if we use a rolling window approach where we fix a window size and use standard maximum likelihood method to estimate the return distribution parameters using past standardized returns that fall into the fixed window.

[12]Either frequently small changes or tail events like market crashes in 2001-2002 can be captured as price jumps easily. Carr, Geman, Madan and Yor (2002) provide evidence that SPX index returns are better described as pure jump processes with infinite activity rate.

[13]Portugal's long-term government bond ratings was downgraded by Moody's to junk status on July 5, 2001. The government bond yields of Italy and Spain raised to 6% in the early of August before the European Central Bank's intervention. See `http://en.wikipedia.org/wiki/2000s_European_sovereign_debt_crisis_timeline` for the timeline of European debt crisis.

the conditional probability of one month index gross return below 0.85 and 0.9 under the option-implied risk-neutral distribution (pink dash line). The gap between the pink dash line and the blue solid line, which presents the risk premium adjustment under the conditional projected SDF, was substantial during the large market downturns and was particular extreme in the 2008, leading to large counter-cyclical equity risk premium.

Figure 6 points to the time-varying wedge between the two conditional probabilities. To visualize their relative gap, we show our Peso problem measure PPM(0.85) (PPM(0.9)) in the panel A (panel B) of Figure 7. Both PPM are pro-cyclical. The time average of PPM(0.85) is 1.13 in recessions and 2.57 in expansions, while the time averages of PPM(0.9) are 0.57 and 1.21, respectively. Pro-cyclicality of PPM reveals that during normal time, perceived probability of stock market downturn is persistently higher than the extent of realization of market crash. By contrast, during hard times when the economy is already in recession, e.g., the year 2001 or when the future macroeconomic outlook is poor, like the situation in July and August of 2011, PPM almost vanishes. It even became negative, after realizations of large stock market downturns, e.g., October 2008 and August 2011. This behavior of PPM supports the Peso problem hypothesis since according to the hypothesis, the gap between ex-ante perceived risk and ex-post realized risk will narrow and vanish if large losses expected by investors have eventually occurred. Compared our findings with Santa-Clara and Yan (2010), as our sample period spans much longer time (1996-2013) which covers not only the stock market boom from 2003 to 2007, but also large market downturns like the 2008 Great Recession and 2011 European Sovereign debt crisis, we do find that the large wedge between ex-ante perceived risk and ex-post realized risk, which was substantial during stock market booms, vanished in those crisis periods. Thus we provide fresh evidence on the time-varying extent of the Peso problem of the S&P 500 index.

5.3 The Equity Risk Premium Implied From SPX Options

We examine the asset pricing implication of the estimated conditional slope projected SDF by focusing on its ability to explain the realized risk premium of the SPX index, defined as the difference between one month S&P 500 Index realized return and the one month U.S. LIBOR rates. Specifically, given the estimated projected SDF, we compute the option-implied expected monthly return on the S&P 500 Index $E_t[R_{t,t+1}] - 1$, the monthly risk-free interest rates $1/E_t[M_{t,t+1}] - 1$ and the option-implied risk premium, defined as their difference. The first column of Table 4 presents their time series mean for the entire sample. The statistics of the (annualized) monthly realized return on the S&P 500 Index $R_{t,t+1} - 1$, one-month LIBOR rates $exp(r^f_{t,t+1}) - 1$ and their difference are shown in the row next to the option-implied. To make our interpretation easy, in what follows, all the variables are annualized.

[Table 4 is here]

The average monthly realized return and one month LIBOR rate is 7.53% and 3.16% per year over the sample period, respectively, which translate into the annual risk premium of 4.37% per year. The option-implied ex-ante expected return is 10.8% per year and the implied risk-free interest rate is 2.86%. The associated annual risk premium is 7.94%, which is 80% higher than the realized risk premium. Santa-Clara and Yan (2010) find that the unconditional ex-ante risk premium of the S&P 500 Index inferred from SPX options is 70% higher than the premium corresponding to the realized risk, the magnitude of which is similar to ours.

Figure 8 plots the dynamics of the realized and expected index returns. The expected return inferred from SPX options are strong counter-cyclical and quite persistent with the first order auto-correlation coefficient AC(1) of 0.7. By contrast, the realized returns consistent of many transitory shocks and its AC(1) is only 0.0046. The dynamics of the two return series also differs in their volatility. The time series standard deviation of the expected return is 9.2%, comparable to 8.9% reported in Santa-Clara and Yan (2010), while the standard deviation of the realized return is 57.88%.

[Figure 4 is here]

The next two column of Table 4 reports the time series mean for sub-sample periods categorized as recession and expansion defined by the NBER. The option-implied ex-ante risk premium is strong counter-cyclical. It is substantially higher in recessions 19.59% than in expansion 6.22%. By contrast, the realized risk premium is proc-cyclical, -8.29% in recession and 6.2% in expansion. The different cyclical behavior of expected risk premia and realized risk premia is consistent with the long-run risk model of Bansal and Yaron (2004). The heightened macroeconomic volatility during recession periods raises discount rates and consequently increase the equity risk premia. Since the macroeconomic volatility is persistent, heightened volatility generates negative and persistent discount rate shocks, leading to consecutively negative realized return. Finally we notice that there is difficult for the SDF-implied risk free rate to match the LIBOR rate during recession periods. The average model-implied risk free rate is too low in recession, 0.01%, compared to the average one month LIBOR of 2.38%. Alternatively speaking, the projected SDF specification in conjunction with the option-implied distribution generates excessive pre-cautionary saving effect and drives the model-implied interest rate too low.

Overall, we conclude that the conditional projected SDF with forward looking distribution recovered from SPX options generates sizable ex-ante risk premia of the S&P 500 Index that is more than 80% higher than realized risk premia. More importantly, the preference parameters of the Epstein-Zin representative agent are reasonable with RRA around 1.2 and EIS greater than 1. Since the conditional slope projected SDF features for a time-varying coefficient of market return $\gamma + \gamma_1 \eta_t$, if we take the overall magnitude of $\gamma + \gamma_1 \eta_t$ into consideration, its time series average is 1.96, which is comparable to the RRA of 1.92 estimated by Santa-Clara and Yan (2010) in a setting where the representative agent has CRRA preference and is only half of the RRA estimated by Bliss and Panigirtzoglou (2004).

6 Concluding Remarks

It has been well established that endowment economy models in a representative agent framework with a peso component can generate sizable equity premium. However, the empirical evidence of the existence of the Peso problem for the U.S. equity market is limited. In this paper, we document that there is Peso problem for the S&P 500 Index, a broad market index of U.S. equity market and the extent of the Peso problem is pro-cyclical.

We first propose with a framework to quantify the extent of the Peso problem by using information from time series of asset returns and cross-section options prices jointly. From cross-sectional options prices at a given point of time, we obtain the option-implied conditional risk-neutral density of equity returns with constant maturity, convert it to the conditional physical density in conjunction with a specification of the Stochastic Discount Factor (SDF), and compare it with the conditional physical density of equity returns implied by a time series model estimated on a finite sample of realized returns. We define *Peso Probability* as the relative gap in the conditional probability of large negative equity returns between the option-implied physical distribution and the time series model implied physical distribution.

Empirically, we apply our method to investigate the Peso problem hypothesis for the S&P 500 Index, over the period from 1996 to 2013. Motivated by the general equilibrium long-run risk model, we specify a projected SDF, derived from Bollerslev, Tauchen and Zhou (2009). Using the S&P 500 Index returns and index options, we find that there is a Peso problem for the S&P 500 Index over the sample period and document that the extent of the Peso problem is pro-cyclical. In particular, it was high during stock market booms while vanished after realization of large stock market downturn. These finding complement the Peso problem literature. Finally we examine the SDF-implied risk premia and compare it with the realized risk premia. With the ex-ante distribution of S&P 500 Index return recovered from SPX options, the projected SDF generates sizable annualized expected risk premia of 7.94% which is 80% higher than the realized risk premia. The expected risk premia exhibits strong counter-cyclical dynamics which peaked during high economic uncertainty states, consistent with long run risk literature.

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Table 1: **Summary Statistics of the Option Contracts and Implied Volatilities from the SPX Options**

Panel A: Number of Option Contracts			
	TTM		
Moneyness(K/F)	$1m$	$2m$	Subtotal
$K/F \leq 0.94$	6801	4761	11562
$0.94 < K/F \leq 0.97$	1238	739	1977
$0.97 < K/F \leq 1.00$	1346	856	2202
$1.00 < K/F \leq 1.03$	1325	827	2152
$1.03 < K/F \leq 1.06$	1196	732	1928
$1.06 < K/F$	2375	1826	4201
Subtotal	14281	9741	24022

Panel B: Average implied volatility			
	TTM		
	$1m$	$2m$	Subtotal
$K/F \leq 0.94$	0.3714	0.3370	0.3573
$0.94 < K/F \leq 0.97$	0.2197	0.2143	0.2177
$0.97 < K/F \leq 1.00$	0.1950	0.1934	0.1944
$1.00 < K/F \leq 1.03$	0.1750	0.1760	0.1754
$1.03 < K/F \leq 1.06$	0.1618	0.1651	0.1631
$1.06 < K/F$	0.2139	0.1912	0.2041
Subtotal	0.2797	0.2612	0.2722

Table 2: **Summary Statistics**

Data sources: Data on S&P 500 Index is from CRSP. U.S. LIBOR rates come from Data Stream. The one month VIX index and VIX term structure are obtained from the CBOE and homepage of Travis L. Johnson. The sample period spans from January 1996 to January 2013, 205 months in total. In Panel A, $R_{t,t+1} \equiv S_{t+1}/S_t$ is the 30 days gross return on the S&P 500 Index. $R^f_{t,t+1}$ is the 30 days gross LIBOR rates. In Panel B, $VIX_t^2/12$ and $VIX18_t^2/12$ are squared 1 month and 12 month VIX indices, respectively. $\sigma^2_{r,t}$ is the expected physical variance of 30 days ahead S&P 500 Index log return, forecasted by the EGARCH (2,1) model. VRP_t is the 30 days variance risk premium, calculated by the difference between $VIX_t^2/12$ and $\sigma^2_{r,t}$. η_t is the ratio of VRP_t over the linear combination of the squared 1 month and 12 month VIX indices: $\eta_t \equiv \frac{VRP_t}{VIX_t^2+\kappa_0 VIX12_t^2}$. The table reports the sample mean, median, standard deviation, minimum, maximum, first order autocorrelation $AC(1)$, skewness and kurtosis of the variables.

Panel A SPX Gross Return and Gross LIBOR Rates

Variable	*Mean*	*Median*	*StdD*	*AC(1)*	*Skew*	*Kurt*	*Min*	*Max*
$R_{t,t+1}$	1.0063	1.0104	0.0482	0.0046	-0.8226	4.4843	0.8134	1.1077
$R^f_{t,t+1}$	1.0025	1.0025	0.0018	0.9831	-0.061	1.3698	1.0002	1.0053

Panel B Conditioning Variables in SDF

$VIX_t^2/12$	0.0047	0.0035	0.0050	0.7585	5.0582	37.8865	0.0008	0.046
$VIX12_t^2/12$	0.0047	0.0042	0.0028	0.8666	2.3638	13.555	0.0013	0.0229
$\sigma^2_{r,t}$	0.0029	0.0021	0.0032	0.7763	5.2422	39.6188	0.0004	0.0297
VRP_t	0.0018	0.0013	0.0021	0.5808	3.867	27.2709	0	0.0192
η_t	0.4592	0.4722	0.2397	0.5726	-0.062	2.6102	0	1.0806

Table 3: **Estimates of Conditional Slope Projected SDF Specification**

The table reports the estimated one period conditional slope projected SDF $\tilde{M}_{t,t+1}$ by nonlinear least squares. The sample for estimation is the monthly gross return on the S&P 500 Index and monthly gross U.S. LIBOR rates from January 1996 to January 2013. The parametric form of $\tilde{M}_{t,t+1}$ is given by,

$$\tilde{M}_{t,t+1} = exp[(\gamma_0 + \gamma_1\eta_t)logR_{t,t+1} + (1 + \gamma_0 + \gamma_1\eta_t)\zeta_0 + (\xi_0 + \xi_1\eta_t)\frac{VIX_t^2}{12} + (\varphi_0 + \varphi_1\eta_t)\frac{VIX12_t^2}{12}]$$

η_t is the ratio of VRP_t over the linear combination of the squared 1 month and 12 month VIX indices: $\eta_t \equiv \frac{VRP_t}{VIX_t^2+\kappa_0 VIX12_t^2}$. VRP_t is the 30 days variance risk premium, calculated by the difference between $VIX_t^2/12$ and $\sigma_{r,t}^2$ and $\sigma_{r,t}^2 \equiv Var_t^{\mathbb{P}}(logR_{t,t+1})$ is the time-t expected physical variance of 30 days ahead S&P 500 Index log return. *, ** and *** indicate statistical significance at 10%, 5% and 1% level, respectively, based on the bootstrap percentile interval.

Conditional-Slope Projected SDF

Parameter	γ_0	γ_1	ζ_0	ξ_0	ξ_1	φ_0	φ_1	κ_0
Point Est	-1.1056^{***}	-1.8847^{**}	-0.0027	-0.0350	2.0861	-0.0234	-1.2389	-0.1651
ObjFun	198.316							

Table 4: **Risk Premium Implied from SPX Options**

Table 4 reports the (time series) average of the (annualized) option-implied monthly expected return on the S&P 500 Index, $E_t[R_{t,t+1} - 1]$, (annualized) option-implied monthly risk-free interest rates $1/E_t[M_{t,t+1}] - 1$, as well as the option-implied risk premium, defined as their difference, under the estimated conditional slope projected SDF. The average of monthly realized return on the S&P 500 Index $R_{t,t+1} - 1$, one month U.S. LIBOR rates $exp(r^f_{t,t+1}) - 1$ and their difference are shown next to option-implied. The sample period is from January 1996 to January 2013. The Mean(Recession) and Mean(Expansion) are sub-sample mean of periods of recession and expansion defined by the NBER.

	Mean	Mean(Recession)	Mean(Expansion)
$E_t[R_{t,t+1} - 1]$	0.108	0.196	0.0953
$R_{t,t+1}$	0.0753	-0.0591	0.0948
$1/E_t[M_{t,t+1}] - 1$	0.0286	0.0001	0.0331
$exp(r^f_{t,t+1}) - 1$	0.0316	0.0238	0.0328
$E_t[R_{t,t+1}] - 1/E_t[M_{t,t+1}]$	0.0794	0.1959	0.0622
$R_{t,t+1} - exp(r^f_{t,t+1})$	0.0437	-0.0829	0.062

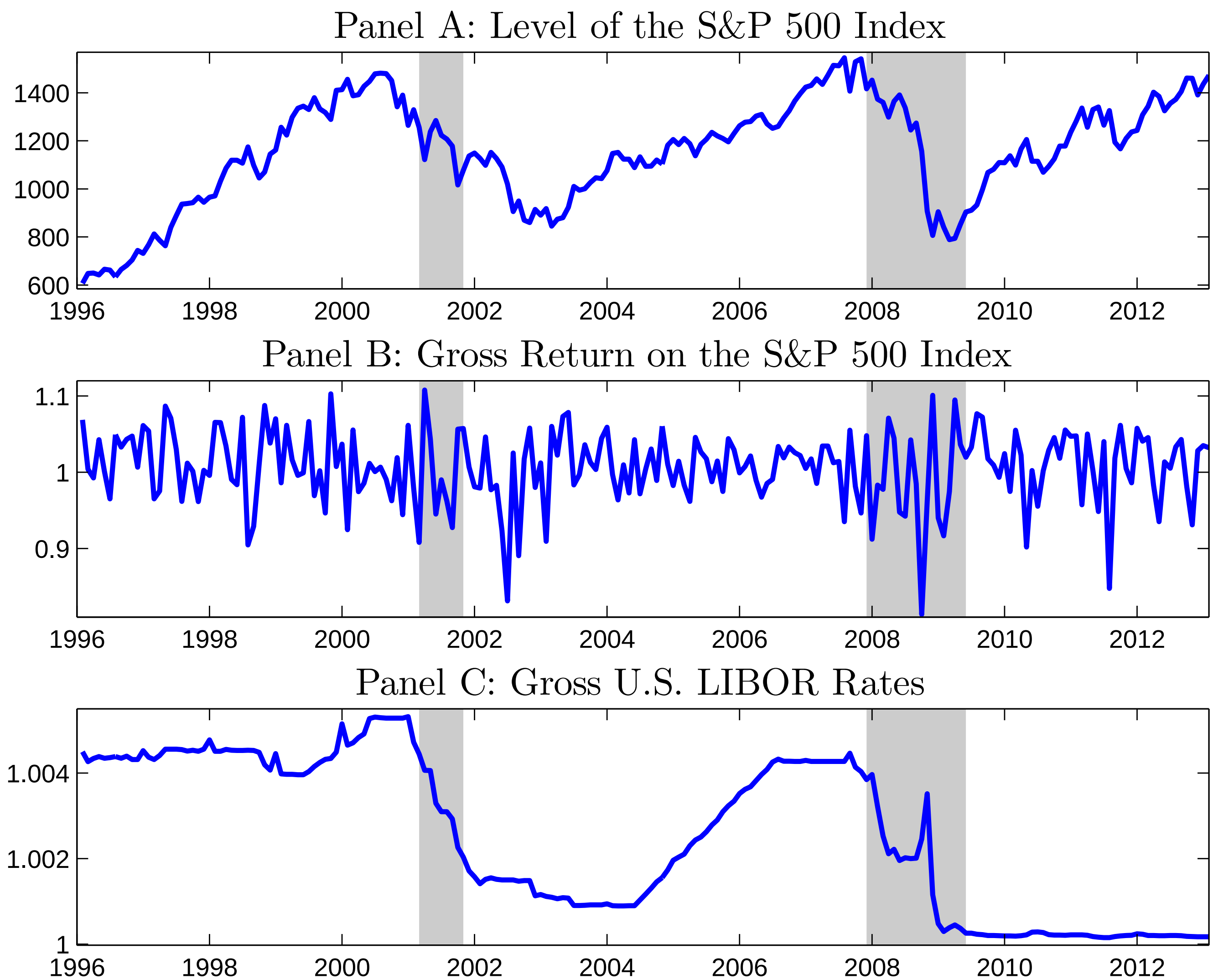

Figure 1: **S&P 500 Index and U.S. LIBOR Rates**

Figure 1 plots the monthly observations of the closing price of the S&P 500 Index, its 30 calendar day-ahead gross return and the gross U.S. LIBOR rates. The sample period is from January 1996 to January 2013, 205 months in total. The shaded areas are the recession periods defined by the NBER.

Spread of Moneyss (K/S)

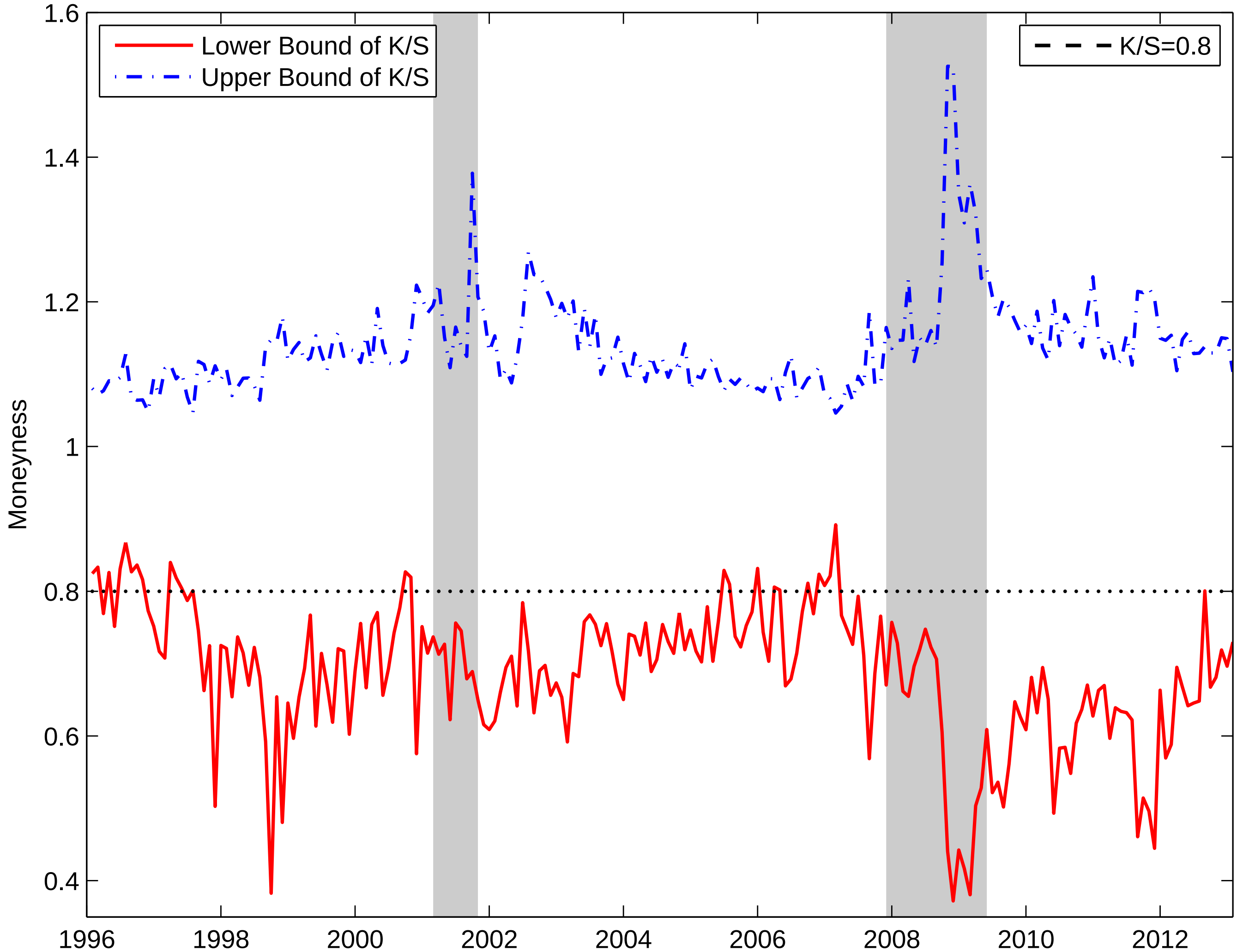

Figure 2: **The Moneyness Coverage of SPX Options**

The figure plots the monthly observation of the SPX option moneyness spread (strike-to-spot ratio K/S) in the cross-section used in the risk-neutral density recovery. The red solid line (blue dash-dot line) represents the minimum (maximum) of option moneyness. The black dot line shows the fixed moneyness level at 0.8. The shaded areas are the recession periods defined by the NBER.

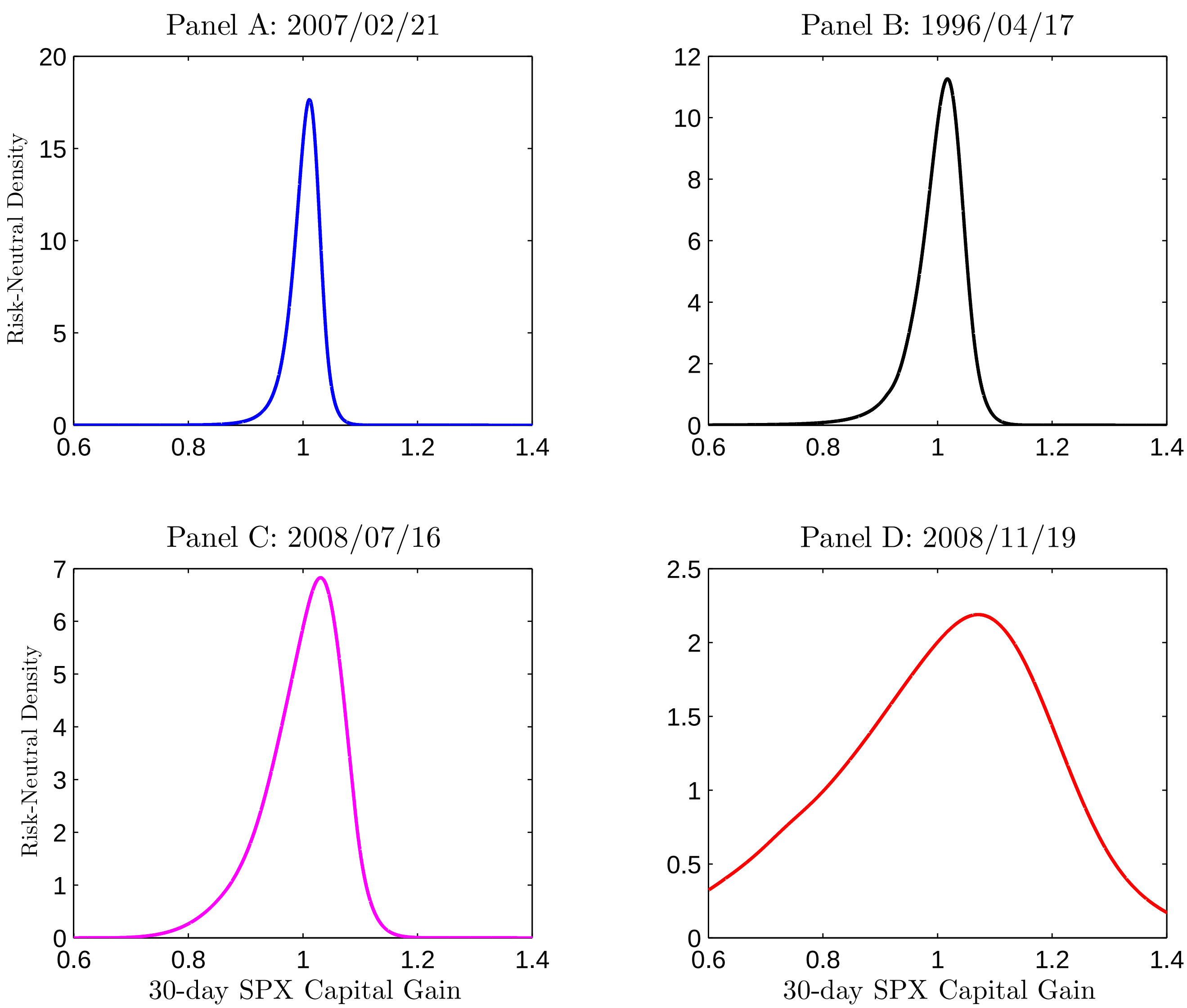

Figure 3: **Dynamics of SPX Option-Implied Risk-Neutral Densities**

Figure 3 plots the SPX option-implied conditional risk-neutral densities on four selected sample dates, 2007/02/21, 1996/04/17, 2008/07/16 and 2008/12/17, respectively. The panel A and D are two dates which represent the least and most volatile market condition, measured by the CBOE VIX index over the sample period. Panel B and C are the two dates when the CBOE VIX index is at the 30% and 75% sample percentile, respectively.

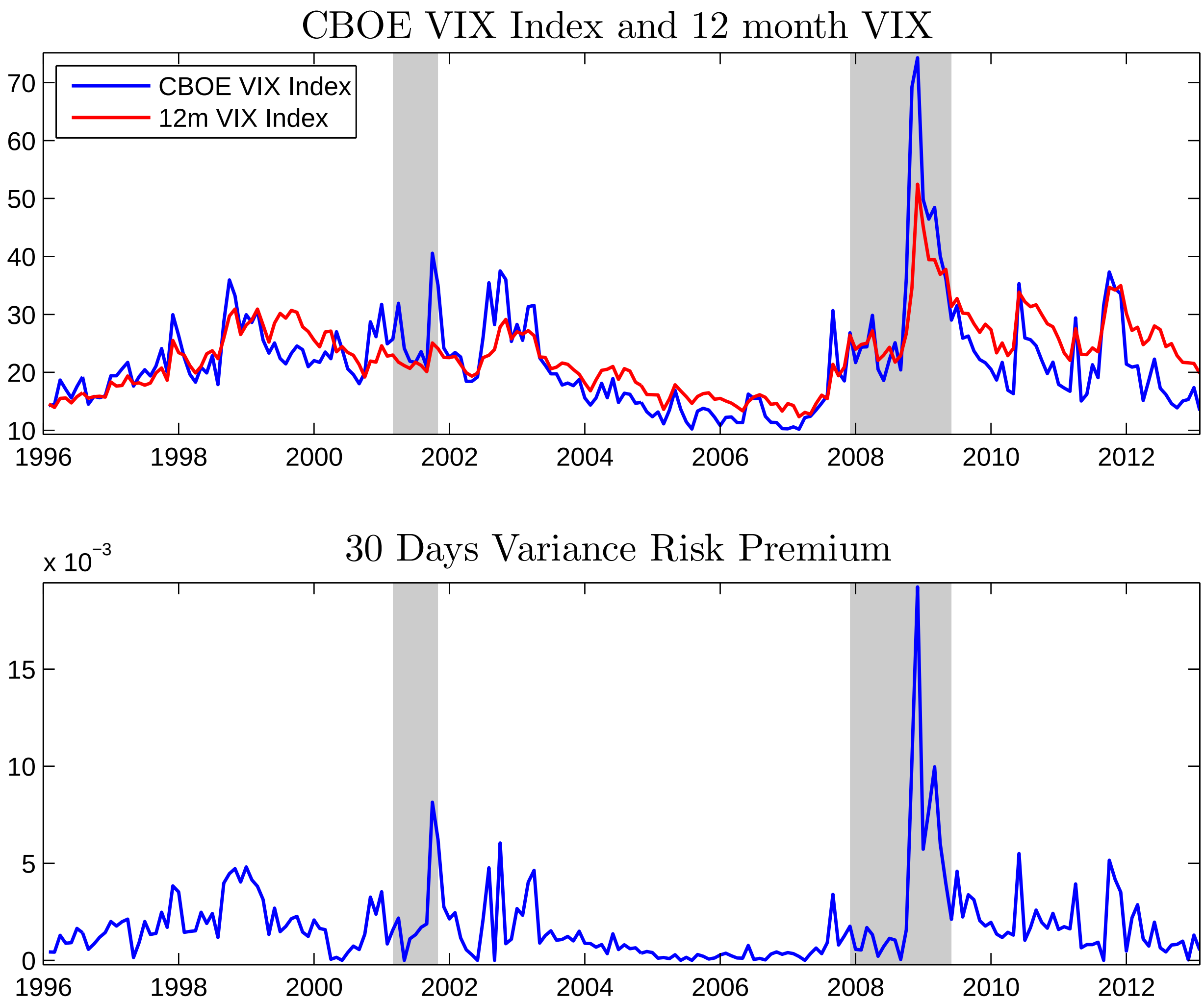

Figure 4: **VIX Indices and Variance Risk Premium**

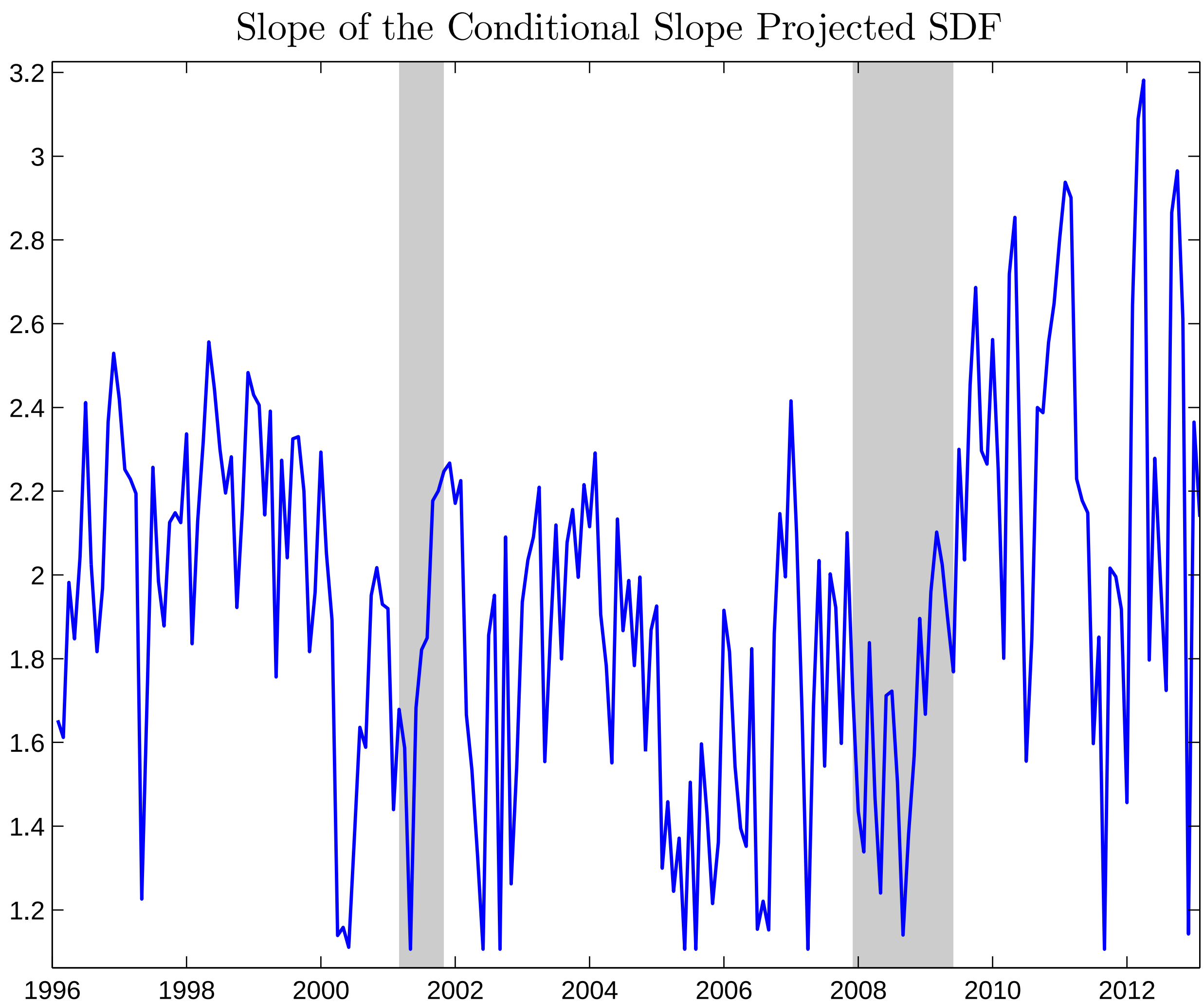

Figure 5: **The Slope of the Conditional Slope Projected SDF**

Figure 5 plots the absolute value of $\gamma_0 + \gamma_1 \eta_t$, the coefficient of market return in the conditional projected SDF, at the estimated values. η_t is the ratio of VRP_t over the linear combination of the squared 1 month and 12 month VIX indices: $\eta_t \equiv \frac{VRP_t}{VIX_t^2 + \kappa_0 VIX12_t^2}$. VRP_t is the 30 days variance risk premium, calculated by the difference between $VIX_t^2/12$ and $\sigma_{r,t}^2$ and $\sigma_{r,t}^2 \equiv Var_t^{\mathbb{P}}(logR_{t,t+1})$ is the time-t expected physical variance of 30 days ahead S&P 500 Index log return.

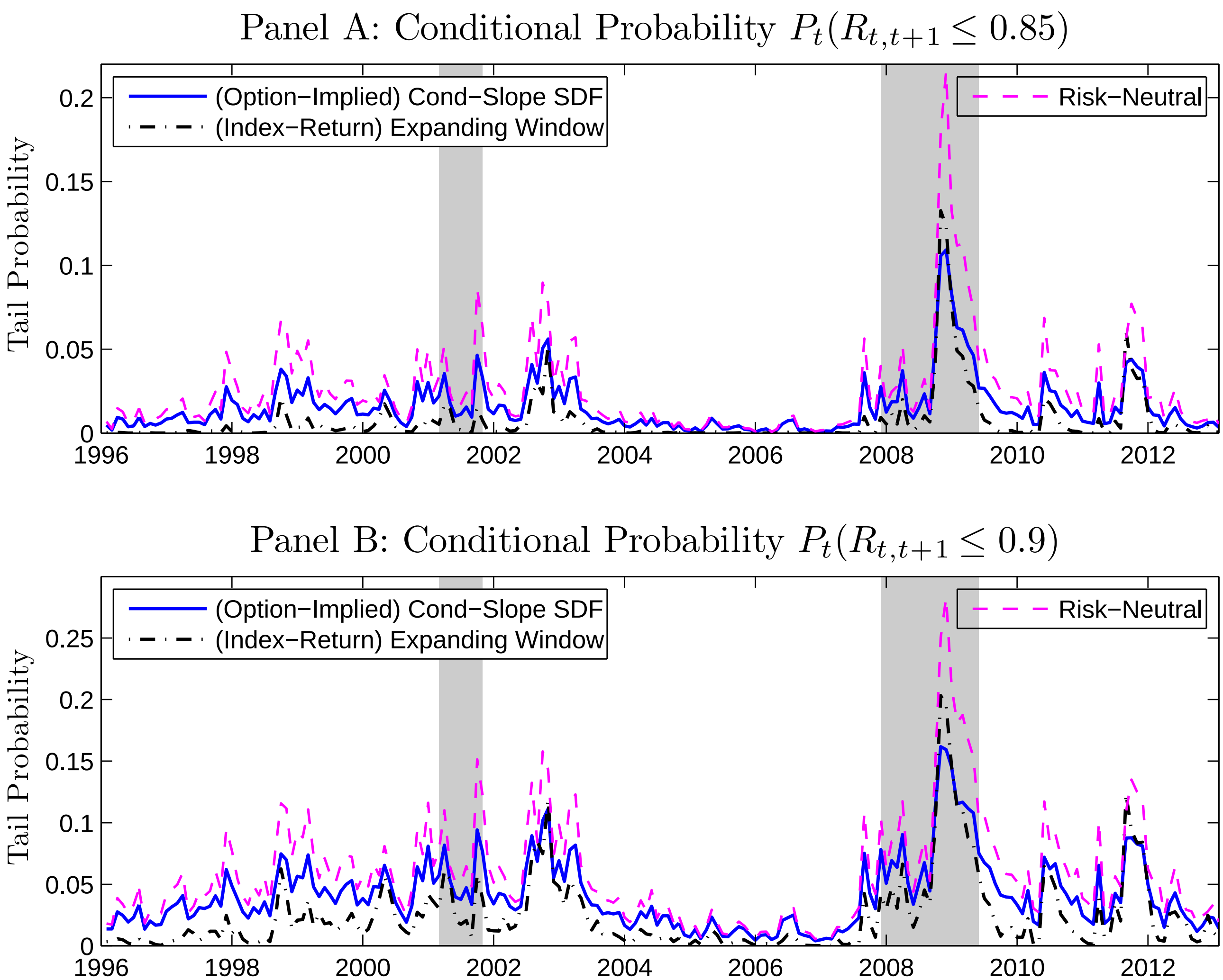

Figure 6: **Dynamics of the Conditional Tail Probability**

$P_t(R_{t,t+1} \le 0.85)$ and $P_t(R_{t,t+1} \le 0.9)$ are the conditional probabilities of one month ahead S&P 500 gross return less than 0.85 and 0.9, respectively. In Panel A (Panel B), the blue solid curve is the conditional probabilities $P_t^O(R_{t,t+1} \le 0.85)$ ($P_t^O(R_{t,t+1} \le 0.9)$) under the option-implied conditional physical distribution. The black dash-dotted curve is the conditional probabilities $P_t^S(R_{t,t+1} \le 0.85)$ ($P_t^S(R_{t,t+1} \le 0.9)$) under the index-return based conditional physical distribution. In both panels, for comparison, the conditional probabilities under the SPX option-implied conditional risk-neutral distribution (red dash line) are also plotted.

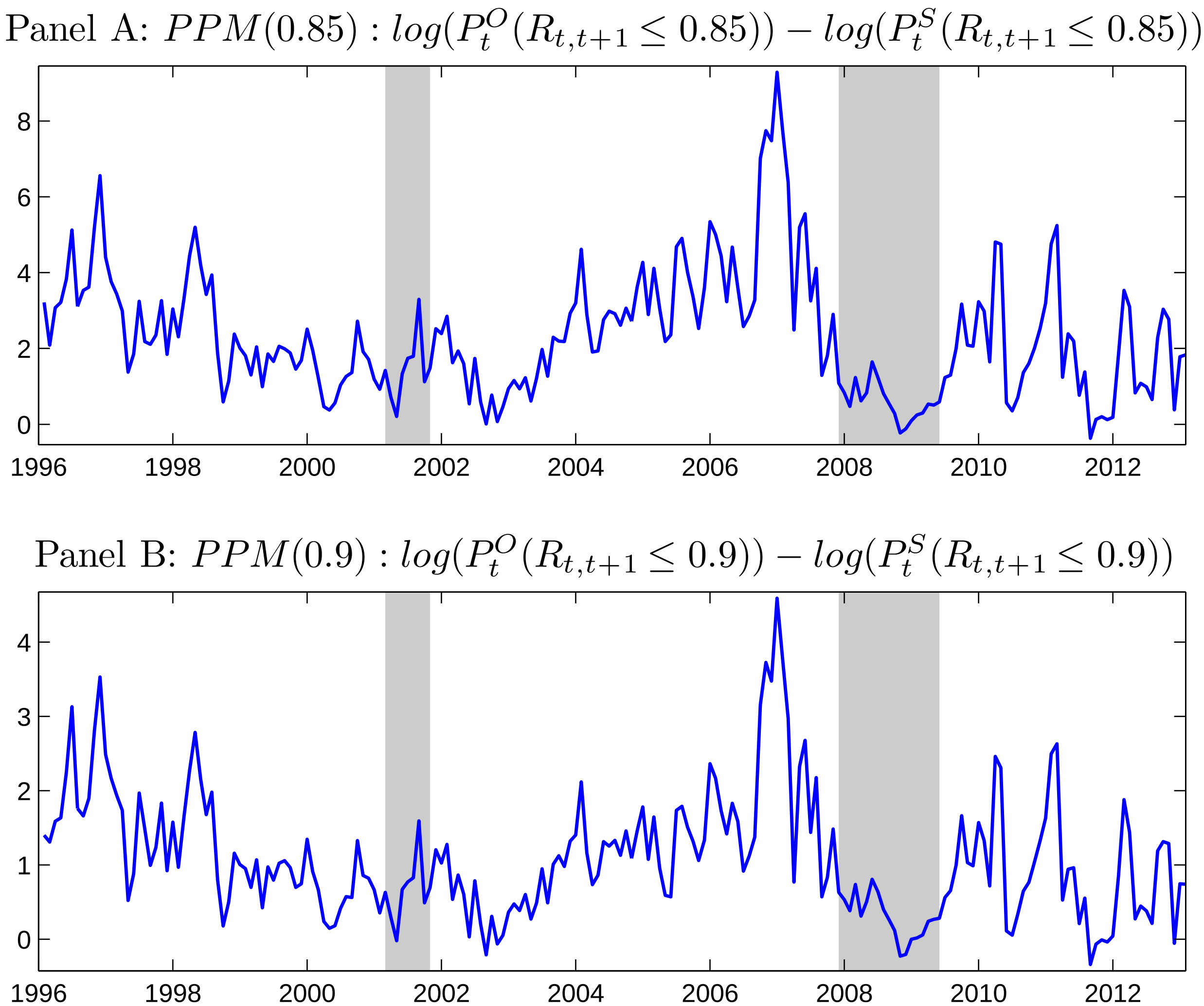

Figure 7: **Peso Problem Measures PPM**(0.85) **and PPM**(0.9)

The peso problem measure (PPM (a)) for $a = 0.85$ or $a = 0.9$, defined as $PPM(a) \equiv log(P_t^O(R_{t,t+1} \leq a)) - log(P_t^S(R_{t,t+1} \leq a))$ are plotted. PPM measures the percentage difference in conditional probability of one month index gross return below 0.85 and 0.9 under the option-implied conditional physical distribution and the index-return based conditional physical distribution.

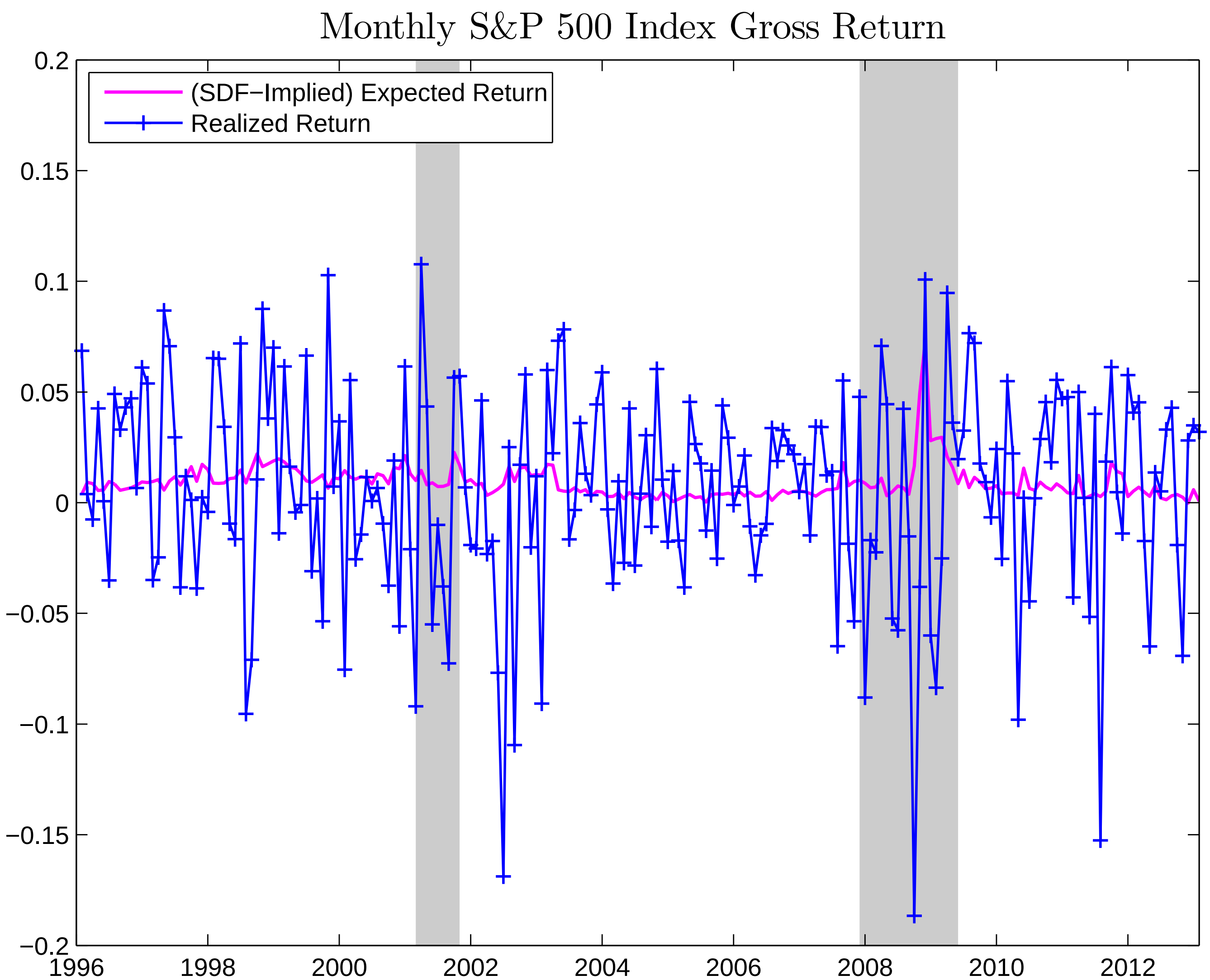

Figure 8: **Expected Return V.s Realized Return of the S&P 500 Index**

Appendix A Complete the Tails of SPX Option-Implied Risk-Neutral Density

In this section, we detail the method of completing the left tail of the option-implied conditional risk-neutral density by the Generalized Extreme Value (GEV) density function, which is first proposed by Figlewski (2009). The method of completing the right tail of the risk-neutral density is similar.

Let $G(x;\xi)$ ($g(x;\xi)$) be the cumulative distribution function (the probability density function) of the GEV distribution family with the shape parameter ξ. $G(x;\xi)$ is given by,

$$G(x;\xi) \equiv exp[-(1+\xi x)^{-1/\xi}]$$

with the probability density function $g(x;\xi)$ being,

$$g(x;\xi) \equiv exp[-(1+\xi x)^{-1/\xi}](1+\xi x)^{-1/\xi-1}$$

Let $f_{EMP}(x)$ be the value of the option-implied conditional risk-neutral density at x. To attach the GEV density to the left tail of the risk-neutral density, two connection points $X_{\alpha_{0L}}$ and $X_{\alpha_{1L}}$ are chosen such that $X_{\alpha_{0L}}$ and $X_{\alpha_{1L}}$ are strike prices at which the cumulative risk-neutral probability recovered from SPX options equal α_{0L} and α_{1L} respectively, that is, they are α_{0L}-th and α_{1L}-th percentiles under the risk-neutral measure.[14]

Following Figlewski (2009), $X(\alpha_{1L})$ is typically chosen to be the second smallest strike price or the 2%-percentile under the risk-neutral measure. $X(\alpha_{1L})$ is chosen to make $\alpha_{0L} = \alpha_{1L} + 0.03$. As shown in the panel B of the Figure 2, in the cross-section, the moneyness coverage of SPX options increases over time, therefore the two connection points are adjusted to be 1%-percentile and 3%-percentile such that we can use more original information from SPX options directly. After the connection points are chosen,

[14] $F(K_n)$, the cumulative risk-neutral distribution at strike K_n is recovered from SPX options by,

$$F(K_n) \approx e^{r^f_{t,t+\tau}} \frac{C(t,T,K_{n+1}) - C(t,T,K_{n-1})}{2\triangle K} + 1$$

Table 5: **Parameters of Attached GEV Density Functions**

Left tail	Mean	Median	StdD	Min	Max
μ	-1192.2	-1199.2	243.87	-2128.9	-567.61
σ	57.67	37.783	63.243	0.0002	584.67
η	0.091	0.087	0.275	-0.492	2.012
Right tail					
μ	1168.4	1197.1	237.49	120.79	1571
σ	30.761	26.682	32.351	0.001	432.97
η	-0.045	-0.059	0.183	-0.419	1.911

the shape parameter ξ with the location-scale parameters (μ, σ) are determined by the following three conditions, which is solved numerically:

$$\begin{aligned} G(\tfrac{-X(\alpha_{0L})-\mu}{\sigma}) &= 1 - \alpha_{0L} \\ g(\tfrac{-X(\alpha_{0L})-\mu}{\sigma})/\sigma &= f_{EMP}(X(\alpha_{0L})) \\ g(\tfrac{-X(\alpha_{1L})-\mu}{\sigma})/\sigma &= f_{EMP}(X(\alpha_{1L})) \end{aligned} \tag{17}$$

In the above equation, all variables are formed in the way: $\frac{-X-\mu}{\sigma}$. This is because we intend to attach the GEV density function to the left tail of the conditional risk-neutral density, therefore we need a negative sign in front of X.

Similarly, to complete the right tail of the conditional risk-neutral density, two connection data points $X(\alpha_{0R})$ and $X(\alpha_{1R})$ are chosen. Typically, $X(\alpha_{1R})$ is 95%th-percentile (or the second largest strike price) and $X(\alpha_{1R})$ is chosen to make $\alpha_{0R} = \alpha_{1R} + 0.03$. We also adjust the two connection data points similarly. Then the parameter vector (ξ, μ, σ) is determined by the following system,

$$\begin{aligned} G(\tfrac{X(\alpha_{0R})-\mu}{\sigma}) &= \alpha_{0R} \\ g(\tfrac{X(\alpha_{0R})-\mu}{\sigma})/\sigma &= f_{EMP}(X(\alpha_{0R})) \\ g(\tfrac{X(\alpha_{1R})-\mu}{\sigma})/\sigma &= f_{EMP}(X(\alpha_{1R})) \end{aligned} \tag{18}$$

Table 5 reports the summary statistics of parameters for the attached GEV density functions. Figure 9 provides a concrete example of how to attach the GEV density to both left and right tails of the conditional risk-neutral density recovered from SPX options on a particular date in the sample.

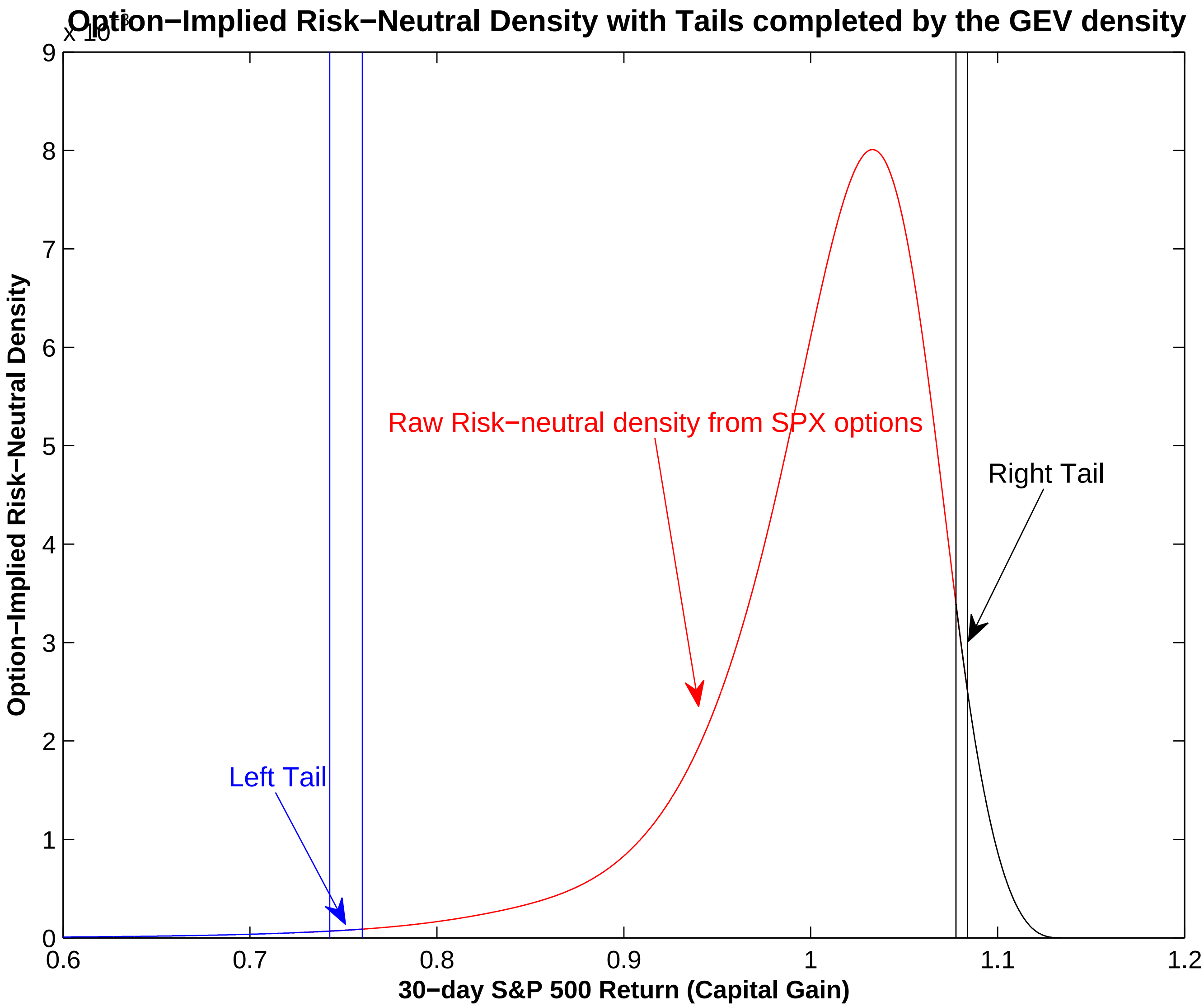

Figure 9: **SPX Risk-Neutral Density with Tails completed by the GEV density function**

Appendix B Projected SDF under BTZ (2009)

In this section, we decribe the model setting of Bollerslev, Tauchen and Zhou (2009) and derive the implied projected SDF. BTZ consider an endowment long-run risk model where the consumption growth $log(C_{t+1}/C_t)$ has a constant mean and time-varying stochastic volatility. The consumption volatility $\sigma_{g,t}^2$ and the volatility of the consumption volatility q_t take the discrete-time square-root process as follows,

$$
\begin{aligned}
log(C_{t+1}/C_t) \equiv g_{t+1} &= \mu_g + \sigma_{g,t} z_{g,t+1} \\
\sigma_{g,t+1}^2 &= a_\sigma + \rho_\sigma \sigma_{g,t}^2 + \sqrt{q_t} z_{\sigma,t+1} \\
q_{t+1} &= a_q + \rho_q q_t + \phi_q \sqrt{q_t} z_{q,t+1}
\end{aligned}
\tag{19}
$$

where $a_\sigma > 0$, $a_q > 0$, $|\rho_\sigma| < 1$ and $|\rho_q| < 1$, $\phi_q > 0$, and $(z_{g,t+1}, z_{\sigma,t+1}, z_{q,t+1})$ follows i.i.d standard multivariate normal distribution.

The representative agent is endowed with Epstein-Zin recursive utility. The logarithm of the one-period SDF m_{t+1} is

$$
m_{t+1} = \theta log\delta - \frac{\theta}{\psi} g_{t+1} + (\theta - 1) r_{t+1} \tag{20}
$$

where δ is the time preference, γ is the risk aversion of the representative agent, ψ refers to the elasticity of intertemporal substitution and $\theta \equiv \frac{1-\gamma}{1-\frac{1}{\psi}}$. Like the usual long-run risk literature, γ and ψ are both assumed to be greater than 1 and hence $\theta < 0$.

In this closed economy, consumption equals dividend and r_{t+1} is the return from t to $t+1$ of the total wealth that delivers the aggregate consumption stream $\{C_{t+i}\}_{i=1}^{\infty}$ as its dividend in each period. To solve the model, conjecture that the log wealth-consumption ratio is an affine function of the two state variables,

$$
w_t = A_0 + A_\sigma \sigma_{g,t}^2 + A_q q_t \tag{21}
$$

Using the Campbell-Shiller (1988) return decomposition for r_{t+1}

$$
r_{t+1} = k_0 + k_1 w_{t+1} - w_t + g_{t+1} \tag{22}
$$

By exploiting the conditional normality of m_{t+1} and r_{t+1}, the Euler equation can be solved analytically,

$$
E_t[e^{m_{t+1} + r_{t+1}}] = E_t[e^{\theta log\delta - \frac{\theta}{\psi} g_{t+1} + \theta r_{t+1}}] = 1 \tag{23}
$$

and get the coefficients A_0, A_σ and A_q, where A_σ and A_q are both negative under the parameter restrictions $\gamma > 1$ and $\psi > 1$.

Plugging (21) and the solution of A_0, A_σ and A_q into (22), BTZ show that the return dynamics can be written as,

$$r_{t+1} = -log\delta + \frac{\mu_g}{\psi} - \frac{(1-\gamma)^2}{2\theta}\sigma_{g,t}^2 + (k_1\rho_q - 1)A_q q_t + k_1\sqrt{q_t}(A_\sigma z_{\sigma,t+1} + A_q\phi_q z_{q,t+1}) + \sigma_{g,t}z_{g,t+1} \quad (24)$$

The log SDF m_{t+1} can be written as a function of r_{t+1} and shocks to state variables,

$$m_{t+1} = -\gamma r_{t+1} + (1-\gamma)(log\delta - \frac{\mu_g}{\psi}) - \frac{(1-\gamma)^2}{2\psi}\sigma_{g,t}^2 + \frac{\theta}{\psi}(k_1\rho_q - 1)A_q q_t + \frac{\theta}{\psi}k_1\sqrt{q_t}(A_\sigma z_{\sigma,t+1} + A_q\phi_q z_{q,t+1}) \quad (25)$$

B.1 Projected SDF

The projected SDF $\tilde{M}_{t+1} = E[e^{m_{t+1}}|r_{t+1}, \sigma_{g,t}, q_t]$ can be calculated by integrating out the last term in (25), which is unknown at time-t,

$$\begin{aligned} E[e^{m_{t+1}}|r_{t+1}, \sigma_{g,t}, q_t] = exp(-\gamma r_{t+1} + (1-\gamma)(log\delta - \tfrac{\mu_g}{\psi}) - \tfrac{(1-\gamma)^2}{2\psi}\sigma_{g,t}^2 + \tfrac{\theta}{\psi}(k_1\rho_q - 1)A_q q_t) \\ E[e^{\frac{\theta}{\psi}k_1\sqrt{q_t}(A_\sigma z_{\sigma,t+1} + A_q\phi_q z_{q,t+1})}|r_{t+1}, \sigma_{g,t}, q_t] \end{aligned} \quad (26)$$

Denote $y_{t+1} = k_1\sqrt{q_t}(A_\sigma z_{\sigma,t+1} + A_q\phi_q z_{q,t+1})$ and its variance as $\sigma_{y,t}^2 \equiv k_1^2(A_\sigma^2 + A_q^2\phi_q^2)q_t$. y_{t+1} follows the standard normal distribution, conditional on q_t

$$y_{t+1}|\{q_t\} \sim N(0, \sigma_{y,t}^2)$$

In addition, given the joint normality of $(z_{g,t+1}, z_{\sigma,t+1}, z_{q,t+1})$, y_{t+1} and r_{t+1} are jointly conditional normal. Hence the conditional distribution of y_{t+1} conditioning on $\{r_{t+1}, \sigma_{g,t}, q_t\}$ still follows the normal distribution. Specifically,

$$y_{t+1}|\{r_{t+1}, \sigma_{g,t}, q_t\} \sim N((r_{t+1} + log\delta - \frac{\mu_g}{\psi} + \frac{(1-\gamma)^2}{2\theta}\sigma_{g,t}^2 - (k_1\rho_q - 1)A_q q_t)\frac{\sigma_{y,t}^2}{\sigma_{y,t}^2 + \sigma_{g,t}^2}, \sigma_{g,t}^2\frac{\sigma_{y,t}^2}{\sigma_{y,t}^2 + \sigma_{g,t}^2})$$

Define $\eta_t \equiv \frac{\sigma^2_{y,t}}{\sigma^2_{y,t}+\sigma^2_{g,t}}$, the remaining expectation in (26) is given by

$$E[e^{\frac{\theta}{\psi}k_1\sqrt{q_t}(A_\sigma z_{\sigma,t+1}+A_q\phi_q z_{q,t+1})}|r_{t+1},\sigma_{g,t},q_t] =$$
$$exp[\frac{\theta}{\psi}(r_{t+1}+log\delta-\frac{\mu_g}{\psi}+\frac{(1-\gamma)^2}{2\theta}\sigma^2_{g,t}-(k_1\rho_q-1)A_q q_t)\eta_t+\frac{1}{2}(\frac{\theta}{\psi})^2\sigma^2_{g,t}\eta_t]$$

Thus the projected SDF $\tilde{M}_{t+1} = E[e^{m_{t+1}}|r_{t+1},\sigma_{g,t},q_t]$, in log form is, is,

$$\begin{aligned}log\tilde{M}_{t+1} = logE[e^{m_{t+1}}|r_{t+1},\sigma_{g,t},q_t] = (-\gamma+\frac{\theta}{\psi}\eta_t)r_{t+1}+ \\ (1-\gamma+\frac{\theta}{\psi}\eta_t)(log\delta-\frac{\mu_g}{\psi})+((1-\gamma)^2\eta_t-(1-\gamma)^2+\frac{\theta^2}{\psi}\eta_t)\frac{\sigma^2_{g,t}}{2\psi}+\frac{\theta}{\psi}(k_1\rho_q-1)A_q q_t(1-\eta_t)\end{aligned} \tag{27}$$

Due to the two factor consumption volatility dynamics, the most salient feature of the projected SDF is that coefficients of market return r_{t+1} and two state variables $(\sigma_{g,t}, q_t)$ is time-varying. Specifically it depends on

$$\eta_t = \frac{\sigma^2_{y,t}}{\sigma^2_{y,t}+\sigma^2_{g,t}} = \frac{k_1^2(A_\sigma^2+A_q^2\phi_q^2)q_t}{k_1^2(A_\sigma^2+A_q^2\phi_q^2)q_t+\sigma^2_{g,t}} \tag{28}$$

To relate η_t to the market return dynamics, we compute the conditional expected variance under the both physical and risk-neutral measure. From the return dynamics (24),

$$\sigma^2_{r,t} = var_t(r_{t+1}) = \sigma^2_{g,t}+k_1^2(A_\sigma^2+A_q^2\phi_q^2)q_t = \sigma^2_{g,t}+\sigma^2_{y,t} \tag{29}$$

The one-step ahead expected physical variance is then

$$E_t[\sigma^2_{r,t+1}] = E_t[\sigma^2_{g,t+1}+k_1^2(A_\sigma^2+A_q^2\phi_q^2)q_{t+1}] = a_\sigma+\rho_\sigma\sigma^2_{g,t}+k_1^2(A_\sigma^2+A_q^2\phi_q^2)(a_q+\rho_q q_t) \tag{30}$$

In discrete-time, $VIX_t^2 = E_t^Q[\sigma^2_{r,t+1}]$, i.e., the time-t expectation of the one-step ahead expected variance under the risk-neutral measure. BTZ show that

$$E_t^Q[\sigma^2_{r,t+1}] = E_t[\sigma^2_{r,t+1}]+(\theta-1)k_1[A_\sigma+A_q\phi_q^2k_1^2(A_\sigma^2+A_q^2\phi_q^2)]q_t \tag{31}$$

The one period variance risk premium is defined as $E_t^Q[\sigma^2_{r,t+1}]-E_t[\sigma^2_{r,t+1}]$, which turns out to be an affine function of q_t only,

$$VRP_t \equiv E_t^Q[\sigma^2_{r,t+1}]-E_t[\sigma^2_{r,t+1}] = (\theta-1)k_1[A_\sigma+A_q\phi_q^2k_1^2(A_\sigma^2+A_q^2\phi_q^2)]q_t \tag{32}$$

Since $\theta<0$, $A_\sigma<0$, and $A_q<0$, VRP_t is strictly positive.

The nominator of η_t is then proportional to the variance risk premium while its denominator is a linear combination of the two volatility state variables $\sigma_{g,t}$ and q_t. From the equation (31), we know that the squared VIX index is an affine function of the two state variables. By the affine structure of the model, the squared 12 month VIX index is also an affine function of state variables. Therefore, the denominator of η_t can be expressed by a linear combination of the two squared VIX indices.

η_t would then be proportional to the following term,

$$\eta_t \propto \frac{VRP_t}{VIX1_t^2/12 + \kappa_0 VIX12_t^2/12} \tag{33}$$

Finally, we replace the two state variables $\sigma_{g,t}$ and q_t in the projected SDF in (27) by observable market variables, e.g., the SPX return, the CBOE VIX index and the 12 month VIX index. We obtain the reduced-form expression of the one period projected SDF, with $\gamma_0 < -1$ and $\gamma_1 < 0$,

$$\tilde{M}_{t,t+1} = exp[(\gamma_0+\gamma_1\eta_t)logR_{t,t+1}+(1+\gamma_0+\gamma_1\eta_t)\zeta_0+(\xi_0+\xi_1\eta_t)VIX_t^2/12+(\varphi_0+\varphi_1\eta_t)VIX12_t^2/12] \tag{34}$$