Optimal Lifecyle Consumption and Investment with Long Term Disability Risk and Consumption Ratcheting *

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Long term disability poses significant unemployment risk and adversely affects living standard. We propose a lifecycle consumption and investment model in the presence of long term disability risk for an investor who needs to maintain a living standard that is at least a certain fraction of the historically highest level. There exists an optimal wealth-to-historically-highest-consumption threshold ratio above which the investor increases consumption beyond the historically highest level. The long-term disability risk significantly reduces consumption and investment. The inability to borrow against future income magnifies the impact of long term disability and further decreases consumption and investment. Our model generates hump shaped lifecycle consumption and investment patterns that are consistent with empirical evidence and shows the importance of the access to long term disability insurance. The traditional financial advice that one should invest less as one ages is only partially correct.

Journal of Economic Literature Classification Numbers: D11, D91, G11, C61.

Keywords: Unemployment risk, Long term disability, Consumption ratcheting, Portfolio selection, Insurance.

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Abstract

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I. Introduction

According to U.S. Social Security Administration, over one in four of today's 20 year-olds will become disabled before they retire (Fact Sheet February 7, 2013). Over 37 million Americans are classified as disabled; about 12% of the total population. More than 50% of those disabled Americans are in their working years, from 18-64. Most of the disabled individuals have to reduce their work load significantly (see U.S. Census Bureau, American Community Survey, 2011). In this paper, we examine the impact of long term disability (LTD) on lifecycle consumption and investment for an investor who exhibits ratcheting of consumption as proposed by Duesenberry.

Our paper offers the following new results:

- The traditional financial advice that one should invest less as one ages is only partially correct. In particular, the fraction of financial wealth invested in stock should increase as time passes at the beginning of working life.
- The long-term disability risk significantly reduces consumption and investment.
- The inability to borrow against future income magnifies the impact of long term disability and further decreases consumption and investment.
- Our model generates hump shaped lifecycle consumption and investment patterns that are consistent with empirical evidence and shows the importance of the access to long term disability insurance.

These results are derived analytically in a consistent framework that yields rich empirical predictions. We hope that these analysis and extensions will lend themselves to the study of policy questions in insurance, pensions, and retirement.

We solve three cases to isolate the effects on the optimal consumption and investment strategy of LTD and consumption ratcheting with and without borrowing against future income. We derive almost explicit solutions (at least parametrically up to at most some constants) in two of these three models. Except for income and LTD status these cases share common features: a constant mortality rate, different marginal utility per unit of consumption before and after LTD, possibly age dependent income, mandatary retirement, and LTD insurance.[1]

The first case serves as a benchmark, it considers the investor's problem after LTD. This simple model extends Dybvig (1995) in three main aspects. First, we allow the consumption to fall below the historically highest level, as long as it is above a certain fraction, say 75%, of it. Second, the investor has income stream that can be or cannot be borrowed against. Third, there is LTD risk. This model seems intractable in the primal, so we solve it in the dual (i.e., as a function of the marginal utility of wealth) and obtain an explicit parametric solution up to two constants that are easy to determine numerically. We show that there exist two critical wealth-to-historically-high-consumption ratios above which it is optimal to increase the consumption beyond the habit level and the historically high level respectively. Compared to a model without consumption ratcheting, the investor consumes less and invest less conservatively to ensure all future consumption to be above the habit level. Compared to Dybvig (1995), the investor consumes more and invest more because of

[1]In general, labor income may be stochastic before LTD. In this paper, we only consider the labor income riskiness from LTD. As shown in Dybvig and Liu (2007), the effect of the riskiness of labor income is small when labor income is a diffusion process. See Lazear [15] for why mandatory retirement may be optimal. An alternative model of mandatory retirement that allows for early retirement is more complicated because of the extra time dimension, but can be solved using the randomization method employed by Liu and Loewenstein (2002) (see also Panageas and Farhi (2007)). Our simpler assumption of retirement at a fixed date T enables us to solve the model exactly and it is easier to compare with the other models.

the less stringent consumption restriction. In addition, consumption exhibits two flat segments: one at the habit level and one at the historically high level. The implied consumption paths are less smooth than those in Dybvig (1995) but smoother than in a model without consumption ratcheting. The second case considers the period before LTD but after retirement. Compared to the first case, the investor has one more decision to make: how much LTD insurance to buy. As in the first case, we show that there exist two critical wealth-to-historically-high-consumption ratios above which it is optimal to increase the consumption beyond the habit level and the historically high level respectively. Due to the presence of LTD risk, the investor invests and consumes much less than the case without LTD risk. An increase of the LTD insurance premium significantly hurts the investor and further reduces consumption and investment. The third case considers the period before LTD and before retirement. If the investor can borrow against future income, the the investor's problem is essentially the same as the case, because in both cases the investor simply capitalize all future income and the problems reduce to one that does not have income flow. Because the wage rate is hump shaped against age, the optimal consumption and investment as a fraction of the total wealth (financial wealth plus human capital) is also hump shaped, matching empirical evidence. When borrowing against income is prohibited, the two critical wealth-to-historically-high-consumption ratios above which it is optimal to increase the consumption beyond the habit level and the historically high level respectively become age dependent.

Financial advisors often advise investors to invest more in the stock market when young and to shift gradually into the riskless asset as they age. Three main justifications are provided in the literature. Bodie, Merton, and Samuelson (1992) (BMS) show that if investors can frequently change working-hours, then labor income will be negatively perfectly correlated with the stock market and therefore the young should invest more in the stock market, because they can work longer hours if market goes down. However, working-hours are typically inflexible and consistent with this, an extensive empirical literature shows that labor income has a very low correlation with the stock market (e.g., Heaton and Lucas (1997)). Therefore this working-hour flexibility is unlikely the main justification for the traditional advice. Dybvig and Liu (2007) (DL) show that even though wage rate itself might be uncorrelated with the stock market, human capital can be significantly negatively correlated with the market given voluntary retirement. This retirement flexibility, like working-hour flexibility, can make it optimal for the young to invest more in the stock. In contrast, Jagannathan and Kocherlakota [11] (JK) argue that total capital is human capital (which is bond-like) plus financial capital (whose market risk can be chosen). To keep the overall mix constant, financial capital has to have a high beta on market risk when young (when total wealth consists mostly of human capital) but a more modest beta on market risk when old (when total wealth consists mostly of financial capital).

Our analysis contributes to this literature by providing two more important factors that affect the validity of the traditional advice. Our model implies that consumption ratcheting and LTD risk may be more important for portfolio choice than just age and the relative size of financial wealth and earnings capacity. In contrast to our model, neither BMS nor DL nor JK considers consumption ratcheting or LTD risk, which is arguably one of the most important life-cycle risks. Compared to JK, our analysis suggests that the traditional advice may be valid for an even larger class of labor income distributions, because LTD risk and consumption ratcheting can even make the fraction of total wealth invested in stock decline over time. However, contrary to the traditional advice, the fraction of financial wealth invested in stock should increase as time passes at the beginning of working life.

This paper is also related to the literature on portfolio choice with borrowing constraint against labor income (e.g., He and Paǵe (1993)) and the literature with drawdown constraint on wealth. Different from these literature, we consider the joint impact of borrowing constraint and consumption ratcheting in the presence of LTD.

The rest of the paper is organized as follows. Section II presents the model with LTD risk and consumption ratcheting. We provide theoretical results in Section III. Numerical and graphical analysis is presented in Section IV. Section V closes the paper. All of the proofs are in the Appendix.

II. The model

We consider an investor with an infinite horizon who maximizes his expected utility from intertemporal consumption. The investor can invest in two financial assets. The first asset ("bond") is riskless, growing at a continuously compounded, constant rate r. The second one is risky ("stock"), which can be viewed as an index or a portfolio of stocks, whose price S_t evolves continuously as

$$\frac{dS_t}{S_t} = \mu dt + \sigma d\mathcal{B}_t, \tag{1}$$

where $\mu > r, \sigma > 0$ are constants and $\{\mathcal{B}_t; t \geq 0\}$ is a one-dimensional Brownian motion.

Different from the standard literature, we assume that the investor requires his living standard (measured by consumption rate) never falls below a certain fraction $\alpha \in [0, 1]$ of the highest living standard he has ever enjoyed. Specifically, let $\{c\}$

denote the adapted consumption process. Then the investor requires

$$c_t \geq \alpha M_t, \tag{2}$$

where

$$M_t = \sup_{0 \leq s \leq t} c_s.$$

In addition, the investor is subject to a long term disability (LTD) risk. The LTD shock occurs at the first jump time τ of an independent Poisson process with a constant intensity of $\lambda \geq 0$. Before the LTD shock, the investor works to earn a deterministic wage rate of $w(t)$ with $w(t) = w_R \equiv \iota w(T)$ for any $t > T$, where T is the full retirement age T and $\iota \leq 1$ represents the ratio of the after-retirement income to the income at retirement. After the LTD shock, the investor loses his job and may need to pay medical expenses at a constant rate of $e \geq 0$. In addition, the marginal utility per unit of consumption may also be different as a result of the LTD. To hedge against the LTD risk, the investor can continuously purchase and vary LTD insurance at a constant premium of $p > 0$ per dollar of insurance payment at the LTD shock.[2] Let θ_t be the dollar amount invested in the stock, I_t be the dollar coverage of the LTD insurance, and W_t be the financial wealth of the investor at t. Then the investor's budget constraint before the shock is

$$dW_t = rW_t dt + \theta_t(\mu - r)dt + \theta_t \sigma d\mathcal{B}_t - c_t dt - pI_t\, dt + w(t)\, dt \tag{3}$$

and the investor's budget constraint after the shock is

$$dW_t = rW_t dt + \theta_t(\mu - r)dt + \theta_t \sigma d\mathcal{B}_t - c_t dt - m(\tau)\, dt, \tag{4}$$

[2]One alternative form of LTD insurance payment is to periodic payment of a certain fraction of the wage rate before LTD. This alternative form can be dealt with using the same approach and yields the same qualitative results.

with

$$W_\tau = W_{\tau-} + I_{\tau-},$$

where

$$m(\tau) \equiv e - \frac{\tau \wedge T - t_0}{T - t_0} \iota\, w(\tau \wedge T)$$

denotes the medical expense net of income after LTD, which is proportional to the years of working before LTD occurs and capped by w_R and t_0 is the starting working age.

The investor has constant relative risk aversion (CRRA) preferences before and after the shock. We assume that the utility function before the shock is

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}$$

and becomes

$$u^D(c) = \frac{(k\,c)^{1-\gamma}}{1-\gamma}$$

after the shock, where γ is the risk aversion coefficient and $k > 0$ measures the change in the utility from one unit of consumption after disabled.

To prevent arbitrage such as a doubling strategy, we impose the following wealth constraint (Dybvig and Huang (1986)):

$$W_t \geq -K, \tag{5}$$

where $K \geq 0$. Setting K to be the present value of all future income means that the investor is allowed to borrow against his labor income. On the other hand, setting $K = 0$ prohibits borrowing against any future income. We will examine the impact of the no-borrowing constraint.

The investor's problem is then

$$\max_{\{c,\theta,I\}} \mathrm{E}\left[\int_0^\tau e^{-\rho t} u(c_t) dt + \int_\tau^\infty e^{-\rho t} u_D(c_t) dt\right]$$

subject to (3), (4), (2), and (5).

III. Analytical results

A. After the LTD Shock

In this section, we first solve the investor's problem after the LTD shock at τ. In this period, the investor's problem becomes

$$\max_{\{c_t,\theta_t\}} \mathrm{E}\left[\int_0^\infty e^{-\rho t}\frac{(k\,c_t)^{1-\gamma}}{1-\gamma}dt\right] \tag{6}$$

subject to

$$dW_t = rW_t dt + \theta_t(\mu - r)dt + \theta_t\sigma d\mathcal{B}_t - c_t dt - m(\tau)\,dt, \tag{7}$$

$$c_t \geq \alpha M_t, \tag{8}$$

subject to

$$W_t \geq \frac{m(\tau)}{r}, \tag{9}$$

where $-\frac{m(\tau)}{r}$ is equal to the present value of income net of medical expense,

$$M_t = \max\left(M_{0-}, \sup_{0\leq s\leq t} c_s\right).$$

Let

$$\eta_D \equiv \frac{k^{-b}\gamma^2}{\rho - (1-\gamma)(r + \frac{\kappa^2}{2\gamma})}, \tag{10}$$

where

$$b = 1 - 1/\gamma, \quad \kappa = \frac{\mu - r}{\sigma}. \tag{11}$$

Let

$$\hat{z}_t \equiv \hat{z}_0 e^{-(r-\rho+\frac{1}{2}\kappa^2)t-\kappa\mathcal{B}_t}$$

be the state price density and $\tilde{V}^D(\hat{z}, M, \tau)$ be the dual value function. Then $\tilde{V}^D(\hat{z}, M, \tau)$ satisfies the following HJB equation:

$$\frac{1}{2}\kappa^2\hat{z}^2\tilde{V}^D_{\hat{z}\hat{z}}(\hat{z}, M, \tau) - (r-\rho)\hat{z}\tilde{V}^D_{\hat{z}}(\hat{z}, M, \tau) - \rho V^D(\hat{z}, M, \tau) + \max_{\alpha M \le c \le M}[u^D(c) - c\hat{z}] - m(\tau)\hat{z} = 0. \tag{12}$$

Due to homogeneity, we can write $\tilde{V}^D(\hat{z}, M, \tau)$ in the following form:

$$\tilde{V}^D(\hat{z}, M, \tau) = M^{1-\gamma}\varphi_D(y, \tau) - \frac{m(\tau)}{r}\hat{z}, \quad y = \frac{z_t}{M}, z_t = \hat{z}^{-1/\gamma}, \tag{13}$$

for some function φ_D. The optimal consumption can be shown to be

$$c_t^* = \begin{cases} \frac{z_t}{\bar{y}_D} & \text{if } \frac{z_t}{M_{t-}} \ge \bar{y}_D \\ M_{t-} & \text{if } \bar{y}_D > \frac{z_t}{M_{t-}} \ge k^b, \\ k^{-b}z_t & \text{if } k^b > \frac{z_t}{M_{t-}} \ge \alpha k^b, \\ \alpha M_{t-} & \text{if } \frac{z_t}{M_{t-}} < \alpha k^b. \end{cases} \tag{14}$$

Then the HJB equation can be simplified to

$$\begin{aligned} \frac{1}{2}\kappa^2 y^2\varphi''_D(y, \tau) + \left((r-\rho)\gamma + \frac{1}{2}(1+\gamma)\kappa^2\right) y\varphi'_D(y, \tau) - \gamma^2\rho\varphi_D(y, \tau) \\ + \frac{\gamma^2}{1-\gamma}(f_D(y)^{1-\gamma} - f_D(y)y^{-\gamma}(1-\gamma)) = 0, \end{aligned} \tag{15}$$

where

$$f_D(y) = \begin{cases} \frac{y}{\bar{y}_D} & \text{if } y \ge \bar{y}_D \\ 1 & \text{if } \bar{y}_D > y \ge k^b, \\ k^{-b}y & \text{if } k^b > y \ge \alpha k^b, \\ \alpha & \text{if } y < \alpha k^b. \end{cases} \tag{16}$$

For the existence of a solution, the following assumption is necessary and sufficient:[3]

Assumption 1 $\eta_D > 0$.

[3]This assumption is also the condition for the corresponding Merton problem to have a solution. If $\eta_D < 0$, then the investor can achieve infinite utility by delaying consumption. If relative risk aversion $\gamma > 1$, η_D is always positive, but for $1 > \gamma > 0$, whether η_D is positive depends on the other parameters.

When borrowing against income is allowed or when $m(\tau) \geq 0$, we have the following boundary conditions:

$$(1-\gamma)\varphi_D(\bar{y}_D,\tau) = \bar{y}_D\varphi'_D(\bar{y}_D,\tau) \tag{17}$$

$$-\gamma\varphi'_D(\bar{y}_D,\tau) = \bar{y}_D\varphi''_D(\bar{y}_D,\tau) \tag{18}$$

$$\lim_{y\downarrow 0}\left(\varphi_D(y,\tau) + \frac{1}{\gamma}y\varphi'_D(y,\tau)\right) = \frac{(k\alpha)^{1-\gamma}}{\rho(1-\gamma)}, \tag{19}$$

where (17) and (18) follow from the smooth pasting conditions at $\bar{y}_D$, and (19) follows because at $y = 0$ (which is equivalent to $\tilde{V}_M = 0$ and $\tilde{V}_{MM} = 0$ at $\bar{y}_D$), the investor can only consume αM forever and thus $V^D(W,M,\tau) = M^{1-\gamma}\left(\varphi_D(y,\tau) + \frac{1}{\gamma}y\varphi'_D(y,\tau)\right) = \frac{(k\alpha M)^{1-\gamma}}{\rho(1-\gamma)}$.

If $m(\tau) < 0$, i.e., income is greater than the medical expense, and the investor cannot borrow against future net income, then we have the same boundary conditions at $\bar{y}_D$, but the lower boundary conditions are different. In this case, there exist a lower boundary $\underline{y}_D$ at which the financial wealth W_t is 0 and at $y = \underline{y}_D$, the investor cannot invest any amount in stock, otherwise any loss would make the investor violate the consumption constraint. Therefore, we have the following boundary conditions:[4]

$$\frac{1}{\gamma}\underline{y}_D^{1+\gamma}\varphi'_D(\underline{y}_D,\tau)M_{t-} + \frac{m(\tau)}{r} = \frac{\alpha M_{t-}}{r} \tag{20}$$

$$(1+\gamma)\varphi'_D(\underline{y}_D,\tau) + \underline{y}_D\varphi''_D(\underline{y}_D,\tau) = 0. \tag{21}$$

We can then solve the HJB equation (15) subject to the above boundary conditions. In general, solving a PDE with time varying free boundaries is difficult. On the other hand, a special structure of the HJB equation (15) helps simplify the solution: the HJB equation does not have any term that involves a derivative with respect to

[4]With the transformation of $\hat{W}_t = W_t - \frac{\alpha M_{t-}}{r}$, the argument of He and Pagé (1993) implies the boundary conditions at the lower boundary $\underline{y}_D$.

M. So one can fix an M and solve an ODE with free boundaries, then pick another M and solve again. Thus the problem reduces to a series of ODEs with free boundaries. Even better, we next show that there exists an explicit solution up to one constant ($\bar{y}_D$) for the case without borrowing constraint and up to two constants ($\bar{y}_D$ and $\underline{y}_D$) for the case with borrowing constraint.

Define

$$\beta_m = \frac{-\left(r-\rho+\kappa^2/2\right)-\sqrt{\left(r-\rho+\kappa^2/2\right)^2+2\,\kappa^2\rho}}{\kappa^2}\gamma < -\gamma;$$

$$\beta_p = \frac{-\left(r-\rho+\kappa^2/2\right)+\sqrt{\left(r-\rho+\kappa^2/2\right)^2+2\,\kappa^2\rho}}{\kappa^2}\gamma;$$

$$\varphi_D(y,\tau) = \begin{cases} A_0 \frac{y^{1-\gamma}}{1-\gamma} & \text{if } \; y \geq \bar{y}_D \\ \frac{k^{1-\gamma}}{\rho(1-\gamma)} - \frac{1}{r} y^{-\gamma} + A_1 y^{\beta_m} + A_2 y^{\beta_p} & \text{if } \; \bar{y}_D > y \geq k^b, \\ \eta_D \frac{y^{1-\gamma}}{1-\gamma} + B_1 y^{\beta_m} + B_2 y^{\beta_p} & \text{if } \; k^b > y \geq \alpha k^b, \\ \frac{(k\alpha)^{1-\gamma}}{\rho(1-\gamma)} - \frac{\alpha}{r} y^{-\gamma} + C_1 y^{\beta_m} + C_2 y^{\beta_p} & \text{if } \; y < \alpha k^b, \end{cases} \tag{22}$$

where A_0, A_1, A_2, B_1, B_2, C_1, C_2, and $\bar{y}_D > k^b$ are constants to be determined.

Given this explicit form of φ_D, we can then solve for A_0, A_1, A_2, B_1, B_2, C_1, C_2, $\bar{y}_D$, and $\underline{y}_D$ (with borrowing constraints) subject to the above boundary conditions and C^1 conditions across across αk^b and k^b.[5] Since these smooth pasting conditions are linear in A_0, A_1, A_2, B_1, B_2, C_1, C_2, one can solve explicitly these constants in terms of $\bar{y}_D$ and then numerically solve one algebraic equation for $\bar{y}_D$ in the absence of borrowing constraints. Note that in the absence of borrowing constraint, neither the HJB equation nor the boundary conditions depend on τ. This implies that without borrowing constraint, $\varphi_D(y,\tau)$ is independent of τ. In addition, it can be easily verified that the boundary condition (19) reduces to $C_1 = 0$. With borrowing constraints,

[5]The matching of the values and the first derivatives across αk^b and k^b ensures that $\varphi_D(y,\tau)$ is also C^2 across these points because of the continuity of the HJB equation across these points.

the system of equations can be reduced to two nonlinear equations in $\underline{y}_D$ and $\bar{y}_D$ to be numerically solved and $\varphi_D(y,\tau)$ depends on τ.

Let the indicator function B equal to 1 if borrowing against income is allowed and 0 otherwise. Then the solution to the investor's problem (6) can be written in terms of the dual variable z_t and M_t.

Theorem 1 *Suppose*

$$W_0 \geq \frac{\alpha M_{0-} + m(\tau)B + m(\tau)^+(1-B)}{r}. \tag{23}$$

Then there exists a unique solution to A_0, A_1, A_2, B_1, B_2, C_1, C_2, $\bar{y}_D$, and $\underline{y}_D$ such that $\varphi_D(y,\tau)$ as defined in (22) is C^2 across αk^b and k^b and satisfies the boundary conditions (17) *to* (19) *for the case without borrowing constraint, and the boundary conditions* (17), (18), (20), (21) *for the case with borrowing constraint, the optimal consumption policy is as in (14), which implies that*

$$M_t^* = \max\left(M_{0-}, \frac{1}{\bar{y}_D} \sup_{0\leq s\leq t} z_t\right), \tag{24}$$

the optimal wealth is

$$W_t^* = \frac{1}{\gamma} y_t^{1+\gamma} \varphi_D'\left(y_t,\tau\right) M_{t-} + \frac{m(\tau)}{r}, \tag{25}$$

the optimal trading strategy is

$$\frac{\theta_t^*}{W_t^* - \frac{m(\tau)}{r}} = \frac{\mu - r}{\gamma\sigma^2} - \frac{\mu - r}{\gamma\sigma^2}\left(\psi_D\left(y_t,\tau\right) - \gamma\right), \tag{26}$$

where

$$\psi_D(y,\tau) \equiv -\frac{y\varphi_D''\left(y,\tau\right)}{\varphi_D'\left(y,\tau\right)}$$

represents the risk aversion coefficient of φ_D, and furthermore, the value function can be written in the following explicit parametric form

$$V^D(W,M,\tau) = M^{1-\gamma}\left(\varphi_D(y,\tau) + \frac{1}{\gamma} y\varphi_D'(y,\tau)\right), \tag{27}$$

with

$$W=\frac{1}{\gamma}y^{1+\gamma}\varphi_D'\left(y,\tau\right)M+\frac{m(\tau)}{r}. \tag{28}$$

Theorem 1 implies that there are four regions for z_t/M_{t-}: $(0,\alpha k^b)$, $[\alpha k^b,k^b)$, $[k^b,\bar{y}_D)$, and $[\bar{y}_D,\infty)$, across which the optimal consumption and optimal investment strategy may differ. It can be shown that these four regions correspond to four regions for the ratio π_t of the adjusted wealth $W_t-m(\tau)/r$ to the historical high M_t, where

$$\pi_t^D=\frac{W_t-m(\tau)/r}{M_{t-}}.$$

specifically, let

$$r_{D1}^*=\frac{1}{\gamma}(\alpha k^b)^{1+\gamma}\varphi_D'\left(\alpha k^b,\tau\right),\ r_{D2}^*=\frac{1}{\gamma}k^{b(1+\gamma)}\varphi_D'\left(k^b,\tau\right),\ r_{D3}^*=\frac{1}{\gamma}\bar{y}_D^{1+\gamma}\varphi_D'\left(\bar{y}_D,\tau\right).$$

Then the four regions for z_t/M_{t-} correspond to the following four regions for π_t: $(\alpha/r,r_{D1}^*)$, $[r_{D1}^*,r_{D2}^*)$, $[r_{D2}^*,r_{D3}^*)$, and $[r_{D3}^*,\infty)$ respectively. Theorem 1 implies that the optimal consumption policy can be equivalently stated as follows, using the adjusted wealth to the historical high ratio to define the regions.

$$c_t^*=\begin{cases}\frac{W_t^*-m(\tau)/r}{r_{D3}^*} & \text{if } \pi_t\geq r_{D3}^*\\ M_{t-} & \text{if } r_{D3}^*>\pi_t\geq r_{D2}^*,\\ k^{-b}z_t & \text{if } r_{D2}^*>\pi_t\geq r_{D1}^*,\\ \alpha M_{t-} & \text{if } r_{D1}^*>\pi_t\geq \alpha/r.\end{cases} \tag{29}$$

The above description of the optimal consumption policy suggests that when the wealth is low the investor consumes the minimum αM_{t-}. When wealth increases beyond a certain level, the investor increases his consumption until it reaches the historical high M_{t-}. When wealth increases further, the investor keeps consuming at the level of the historical high without increasing it. If wealth is above a high threshold, the investor increases his consumption above the historical high such that the wealth to the new high ratio is equal to r^*_{D3}. Dybvig (1992) is a special case where $\alpha = k = 1$. In this special case, $r^*_{D1} = r^*_{D2}$ and thus the second region disappears and the solution reduces to that in Dybvig (1992).

For the optimal trading strategy without borrowing constraint, it can be shown that $\lim_{y\downarrow 0} \psi_D(y,\tau) = 1+\gamma$ and $\forall y \geq \bar{y}_D$, $\psi_D(y,\tau) = \gamma$. Since as $y \downarrow 0$, wealth W^*_t goes to $(\alpha M_{t-} + m(\tau))/r$, Theorem 1 implies that as W^*_t goes to $(\alpha M_{t-} + m(\tau))/r$, the optimal amount invested in the stock goes to zero. Intuitively, this is the only feasible strategy to ensure the consumption never falls below αM_{t-}. When the wealth is sufficiently high ($W^*_t - m(\tau)/r \geq r^*_{D3} M_{t-}$), it is optimal to adopt the Merton strategy in the unconstrained case of investing a fraction $\frac{\mu - r}{\gamma\sigma^2}$ of $W^*_t - m(\tau)/r$ in the stock.

B. Before LTD but after retirement

In this subsection, we solve the investor's problem after retirement but before the LTD shock. In this period, the investor's problem becomes

$$\max_{\{c_t,\theta_t\}} \mathrm{E}\left[\int_0^\tau e^{-\rho t}\frac{c_t^{1-\gamma}}{1-\gamma}dt + e^{-\rho\tau}V^D(W_\tau + I_\tau, M_\tau, T)\right] \tag{30}$$

subject to

$$dW_t = rW_t dt + \theta_t(\mu - r)dt + \theta_t \sigma d\mathcal{B}_t - c_t dt + w_R\, dt - pI_t dt, \tag{31}$$

$$c_t \geq \alpha M_t, \tag{32}$$

and

$$W_t \geq -\left(\frac{w_R}{r+p} - \frac{p\, m(T)}{r(r+p)}\right), \tag{33}$$

where $\frac{w_R}{r+p} - \frac{p\, m(T)}{r(r+p)} > 0$ represents the present value of the labor income net of the medical expense,

$$M_t = \max\left(M_{0-}, \sup_{0\leq s\leq t} c_s\right).$$

Let

$$\eta \equiv \frac{\gamma^2}{\rho + \lambda - (1-\gamma)(r + p + \frac{\kappa^2}{2\gamma})}. \tag{34}$$

For the existence of a solution, the following assumption is necessary and sufficient:

Assumption 2 *$\eta > 0$ and $\eta_D > 0$.*

$\tilde{V}^R(\hat{z}, M)$ be the dual value function. Then $\tilde{V}^R(\hat{z}, M)$ satisfies the following HJB equation:

$$\frac{1}{2}\kappa^2 \hat{z}^2 \tilde{V}^R_{\hat{z}\hat{z}} - (r+p-\lambda-\rho)\hat{z}\tilde{V}^R_{\hat{z}} - (\rho+\lambda)\tilde{V}^R + \max_{\alpha M \le c \le M}[u(c) - c\hat{z}] + w_R \hat{z} + \lambda \tilde{V}^D\left(\frac{p}{\lambda}\hat{z}, M, T\right) = 0. \tag{35}$$

Due to homogeneity, we can write $\tilde{V}(\hat{z}, M)$ in the following form:

$$\tilde{V}^R(\hat{z}, M) = M^{1-\gamma}\varphi_R(y) - \left(\frac{p\, m(T)}{r(r+p)} - \frac{w_R}{r+p}\right)\hat{z}, \quad y = \frac{z}{M}, z = \hat{z}^{-1/\gamma}, \tag{36}$$

for some function φ_R. The optimal consumption can be shown to be

$$c_t^* = \begin{cases} \frac{z_t}{\bar{y}_R} & \text{if } \frac{z_t}{M_{t-}} \ge \bar{y}_R \\ M_{t-} & \text{if } \bar{y}_R > \frac{z_t}{M_{t-}} \ge 1, \\ z_t & \text{if } 1 > \frac{z_t}{M_{t-}} \ge \alpha, \\ \alpha M_{t-} & \text{if } \frac{z_t}{M_{t-}} < \alpha. \end{cases} \tag{37}$$

Then the HJB equation can be simplified to

$$\frac{1}{2}\kappa^2 y^2 \varphi_R''(y) + \left((r+p-\lambda-\rho)\gamma + \frac{1}{2}(1+\gamma)\kappa^2\right) y\varphi_R'(y) - \gamma^2(\rho+\lambda)\varphi_R(y)$$
$$+\gamma^2\lambda\varphi_D(\xi y; 0) + \frac{\gamma^2}{1-\gamma}(f_R(y)^{1-\gamma} - f_R(y)y^{-\gamma}(1-\gamma)) = 0, \tag{38}$$

where

$$f_R(y) = \begin{cases} \frac{y}{\bar{y}_R} & \text{if } y \ge \bar{y}_R \\ 1 & \text{if } \bar{y}_R > y \ge 1, \\ y & \text{if } 1 > y \ge \alpha, \\ \alpha & \text{if } y < \alpha, \end{cases} \tag{39}$$

and

$$\xi = \left(\frac{p}{\lambda}\right)^{-1/\gamma}.$$

When borrowing against income is allowed or when $\frac{w_R}{r+p} - \frac{p\,m(T)}{r(r+p)} \leq 0$, we have the following boundary conditions:

$$(1-\gamma)\varphi_R(\bar{y}_R,\tau) = \bar{y}_R\varphi'_R(\bar{y}_R,\tau) \tag{40}$$

$$-\gamma\varphi'_R(\bar{y}_R,\tau) = \bar{y}_R\varphi''_R(\bar{y}_R,\tau) \tag{41}$$

$$\lim_{y\downarrow 0}\left(\varphi_R(y,\tau) + \frac{1}{\gamma}y\varphi'_R(y,\tau)\right) = \frac{(k\alpha)^{1-\gamma}}{\rho(1-\gamma)}, \tag{42}$$

where (40) and (41) follow from the smooth pasting conditions at $\bar{y}_R$, and (42) follows because at $y=0$, the investor can only consume αM forever and thus $V^D(W,M,\tau) = M^{1-\gamma}\left(\varphi_R(y,\tau) + \frac{1}{\gamma}y\varphi'_R(y,\tau)\right) = \frac{(k\alpha M)^{1-\gamma}}{\rho(1-\gamma)}$.

If $\frac{w_R}{r+p} - \frac{p\,m(T)}{r(r+p)} > 0$, i.e., income is greater than the medical expense, and the investor cannot borrow against future net income, then we have the same boundary conditions at $\bar{y}_R$, but the lower boundary conditions are different. In this case, there exist a lower boundary $\underline{y}_R$ at which the financial wealth W_t is 0 and at $y = \underline{y}_R$, the investor cannot invest any amount in stock, otherwise any loss would make the investor violate the consumption constraint. Therefore, we have the following boundary conditions:

$$\frac{1}{\gamma}\underline{y}_R^{1+\gamma}\varphi'_R(\underline{y}_R,\tau)M_{t-} + \frac{m(\tau)}{r} = 0 \tag{43}$$

$$(1+\gamma)\varphi'_R(\underline{y}_R,\tau) + \underline{y}_R\varphi''_R(\underline{y}_R,\tau) = 0. \tag{44}$$

We can then solve the HJB equation (38) subject to the above boundary conditions. We next show that there exists an explicit solution up to one constant ($\bar{y}_R$) for the case without borrowing constraint and up to two constants ($\bar{y}_R$ and $\underline{y}_R$) for the case with borrowing constraint.

Define

$$\beta_- = \frac{-\left(r+p-\lambda-\rho+\kappa^2/2\right)-\sqrt{\left(r+p-\lambda-\rho+\kappa^2/2\right)^2+2\,\kappa^2(\rho+\lambda)}}{\kappa^2}\gamma;$$

$$\beta_+ = \frac{-\left(r+p-\lambda-\rho+\kappa^2/2\right)+\sqrt{\left(r+p-\lambda-\rho+\kappa^2/2\right)^2+2\,\kappa^2(\rho+\lambda)}}{\kappa^2}\gamma;$$

The investor's problem can be divided into several cases by certain parameter values such that for each case an explicit solution (up to some numerically determined constants) can be obtained. Next we state the explicit form for one of the main cases to illustrate the form of the solutions. Suppose $\bar{y}_R > \bar{y}_D > k^b > 1 > \alpha k^b > \alpha$. Define

$$\varphi_R(y) = \begin{cases} C_0 \frac{y^{1-\gamma}}{1-\gamma} & \text{if } y \geq \bar{y}_R \\ D_1 y^{\beta_-} + D_2 y^{\beta_+} + h_1(y) + g_0(y) & \text{if } \bar{y}_R > y \geq \bar{y}_D \\ E_1 y^{\beta_-} + E_2 y^{\beta_+} + h_1(y) + g_1(y) & \text{if } \bar{y}_D > y \geq k^b \\ F_1 y^{\beta_-} + F_2 y^{\beta_+} + h_1(y) + g_2(y) & \text{if } k^b > y \geq 1 \\ G_1 y^{\beta_-} + G_2 y^{\beta_+} + h_2(y) + g_2(y) & \text{if } 1 > y \geq \alpha k^b \\ H_1 y^{\beta_-} + H_2 y^{\beta_+} + h_2(y) + g_3(y) & \text{if } \alpha k^b > y \geq \alpha \\ I_1 y^{\beta_-} + I_2 y^{\beta_+} + h_3(y) + g_3(y) & \text{if } y < \alpha \end{cases} \tag{45}$$

where functions $h_1(\cdot) - h_3(\cdot)$ and $g_0(\cdot) - g_3(\cdot)$ are specified in the Appendix, C_0, D_1, D_2, E_1, E_2, F_1, F_2, G_1, G_2, H_1, H_2, I_1, I_2, and $\bar{y}_R$ are constants to be determined.

We can then solve for C_0, D_1, D_2, E_1, E_2, F_1, F_2, G_1, G_2, H_1, H_2, I_1, I_2, $\bar{y}_R$, and $\underline{y}_R$ (with borrowing constraints) subject to the above boundary conditions and C^1 conditions across α, αk^b, 1, k^b, and $\bar{y}_D$.[6] Since these smooth pasting conditions are linear in C_0, D_1, D_2, E_1, E_2, F_1, F_2, G_1, G_2, H_1, H_2, I_1, and I_2, one can solve explicitly these constants in terms of $\bar{y}_R$ and then numerically solve one algebraic equation for $\bar{y}_R$ in the absence of borrowing constraints. Note that in the absence of borrowing constraint, the condition (42) reduces to $I_1 = 0$. With borrowing constraints, the

[6] As in the after LTD case, the matching of the values and the first derivatives across the break points ensures that $\varphi_R(y)$ is also C^2 across these points.

system of equations can be reduced to two nonlinear equations in $\underline{y}_R$ and $\bar{y}_R$ to be numerically solved.

Then the solution to the investor's problem (30) can be written in terms of the dual variable z_t and M_t.

Theorem 2 *Suppose*

$$W_0 \geq \frac{\alpha M_{0-}}{r} + \frac{p\left(m(T)B + m(T)^{+}(1-B)\right)}{r(r+p)} - \frac{w_R}{r+p}. \tag{46}$$

Suppose there exists a unique solution to C_0, D_1, D_2, E_1, E_2, F_1, F_2, G_1, G_2, H_1, H_2, I_1, I_2, $\bar{y}_R$, *and* $\underline{y}_R$ *such that* $\bar{y}_R > \bar{y}_D > k^b > 1 > \alpha k^b > \alpha$ *and* $\varphi_R(y)$ *as defined in (45) is* C^2 *across* α, αk^b, 1, k^b, *and* $\bar{y}_D$ *and satisfies the boundary conditions* (40) *to* (42) *for the case without borrowing constraint, and the boundary conditions* (40), (41), (43), (44) *for the case with borrowing constraint. Then the optimal consumption policy is as in (37), which implies that*

$$M_t^* = \max\left(M_{0-}, \frac{1}{\bar{y}_R} \sup_{0 \leq s \leq t} z_t\right), \tag{47}$$

the optimal wealth is

$$W_t^* = \frac{1}{\gamma} y_t^{1+\gamma} \varphi_R'(y_t)\, M_{t-} + \frac{p\, m(T)}{r(r+p)} - \frac{w_R}{r+p}, \tag{48}$$

the optimal trading strategy is

$$\frac{\theta_t^*}{W_t^* - \left(\frac{p\, m(T)}{r(r+p)} - \frac{w_R}{r+\lambda}\right)} = \frac{\mu - r}{\gamma \sigma^2} - \frac{\mu - r}{\gamma \sigma^2}\left(\psi(y_t) - \gamma\right), \tag{49}$$

and the optimal LTD insurance is

$$I_t^* = \frac{1}{\gamma}(\xi y_t)^{1+\gamma} \varphi_D'(\xi y_t; 0)\, M_{t-} - \frac{1}{\gamma} y_t^{1+\gamma} \varphi_R'(y_t)\, M_{t-} + \frac{w_R + m(T)}{r+p}, \tag{50}$$

where

$$\psi_R(y) \equiv -\frac{y \varphi_R''(y)}{\varphi_R'(y)}$$

represents the risk aversion coefficient of φ_D, and furthermore, the value function can be written in the following explicit parametric form

$$V^R(W, M) = M^{1-\gamma}\left(\varphi_R(y) + \frac{1}{\gamma}y\varphi_R'(y)\right), \tag{51}$$

with

$$W = \frac{1}{\gamma}y^{1+\gamma}\varphi_R'(y)\,M + \frac{p\,m(T)}{r(r+p)} - \frac{w_R}{r+p}. \tag{52}$$

Theorem 2 implies that there are four regions for z_t/M_{t-}: $(0, \alpha)$, $[\alpha, 1)$, $[1, \bar{y}_R)$, and $[\bar{y}_R, \infty)$, across which the optimal consumption and optimal investment strategy may differ. It can be shown that these four regions correspond to four regions for the ratio π_t of the adjusted wealth $W_t - (\frac{p\,m(T)}{r(r+p)} - \frac{w}{r+p})$ to the historical high M_t, where

$$\pi_t^R = \frac{W_t - \left(\frac{p\,m(T)}{r(r+p)} - \frac{w_R}{r+p}\right)}{M_{t-}}.$$

specifically, let

$$r_{R1}^* = \frac{1}{\gamma}\alpha^{1+\gamma}\varphi_R'(\alpha)\,,\; r_{R2}^* = \frac{1}{\gamma}\varphi_R'(1)\,,\; r_{R3}^* = \frac{1}{\gamma}\bar{y}_R^{1+\gamma}\varphi_R'(\bar{y}_R)\,.$$

Then the four regions for z_t/M_{t-} correspond to the following four regions for π_t: $(\alpha/r, r_{R1}^*)$, $[r_{R1}^*, r_{R2}^*)$, $[r_{R2}^*, r_{R3}^*)$, and $[r_{R3}^*, \infty)$ respectively. Theorem 2 implies that the optimal consumption policy can be equivalently stated as follows, using the adjusted wealth to the historical high ratio to define the regions.

$$c_t^* = \begin{cases} \frac{W_t^* - \left(\frac{p\,m(T)}{r(r+p)} - \frac{w_R}{r+p}\right)}{r_{R3}^*} & \text{if} \quad \pi_t \geq r_{R3}^* \\ M_{t-} & \text{if} \quad r_{R3}^* > \pi_t \geq r_{R2}^*, \\ z_t & \text{if} \quad r_{R2}^* > \pi_t \geq r_{R1}^*, \\ \alpha M_{t-} & \text{if} \quad r_{R1}^* > \pi_t \geq \alpha/r. \end{cases} \tag{53}$$

Similar to the case after LTD, the above description of the optimal consumption policy suggests that when the wealth is low the investor consumes the minimum αM_{t-}.

When wealth increases beyond a certain level, the investor increases his consumption until it reaches the historical high M_{t-}. When wealth increases further, the investor keeps consuming at the level of the historical high without increasing it. If wealth is above a high threshold, the investor increases his consumption above the historical high such that the wealth to the new high ratio is equal to r^*_{R3}.

For the optimal trading strategy without borrowing constraint, it can be shown that $\lim_{y\downarrow 0}\psi(y) = 1+\gamma$ and $\forall y \geq \bar{y}_R$, $\psi(y) = \gamma$. Since as $y \downarrow 0$, wealth W^*_t goes to $\frac{\alpha M_{t-}}{r} + \frac{p\,m(T)}{r(r+p)} - \frac{w_R}{r+p}$, Theorem 2 implies that as W^*_t goes to $\frac{\alpha M_{t-}}{r} + \frac{p\,m(T)}{r(r+p)} - \frac{w_R}{r+p}$, the optimal amount invested in the stock goes to zero. Intuitively, this is the only feasible strategy to ensure the consumption never falls below αM_{t-}. When the wealth is sufficiently high ($W^*_t - \left(\frac{p\,m(T)}{r(r+p)} - \frac{w_R}{r+p}\right) \geq r^*_{R3}M_{t-}$), it is optimal to adopt the Merton strategy in the unconstrained case of investing a fraction $\frac{\mu-r}{\gamma\sigma^2}$ of $W^*_t - \left(\frac{p\,m(T)}{r(r+p)} - \frac{w_R}{r+p}\right)$ in the stock.

C. Before LTD and before retirement

In this section, we solve the investor's problem before the LTD shock and before retirement. In this period, the investor's problem becomes

$$\max_{\{c_t,\theta_t\}} \mathrm{E}\left[\int_0^\tau e^{-\rho t}\frac{c_t^{1-\gamma}}{1-\gamma}dt + e^{-\rho\tau}V^D(W_\tau + I_\tau, M_\tau, \tau)\right] \tag{54}$$

subject to

$$dW_t = rW_t dt + \theta_t(\mu - r)dt + \theta_t\sigma d\mathcal{B}_t - c_t dt + w(t)\,dt - pI_t dt, \tag{55}$$

$$c_t \geq \alpha M_t, \tag{56}$$

and

$$W_t \geq -\left(g(t) - \frac{p\,m(t)}{r(r+p)}\right), \tag{57}$$

where

$$M_t = \max\left(M_{0-}, \sup_{0\le s\le t} c_s\right),$$

and

$$g(t) = \int_t^T e^{-(r+p)(s-t)} w(s)ds + \frac{w_R}{r+p} e^{-(r+p)(T-t)}. \tag{58}$$

is the present value of the labor income.

$\tilde{V}(\hat{z}, M, t)$ be the dual value function. Then $\tilde{V}(\hat{z}, M, t)$ satisfies the following HJB equation:

$$\tilde{V}_t + \frac{1}{2}\kappa^2 \hat{z}^2 \tilde{V}_{\hat{z}\hat{z}} - (r+p-\rho-\lambda)\hat{z}\tilde{V}_{\hat{z}} - (\rho+\lambda)\tilde{V} + \max_{\alpha M\le c\le M}[u(c) - c\hat{z}] + w(t)\hat{z} + \lambda \tilde{V}^D\left(\frac{p}{\lambda}\hat{z}, M, t\right) = 0, \tag{59}$$

with terminal condition

$$\tilde{V}(\hat{z}, M, T) = \tilde{V}^R(\hat{z}, M).$$

Due to homogeneity, we can write $\tilde{V}(\hat{z}, M, t)$ in the following form:

$$\tilde{V}(\hat{z}, M, t) = M^{1-\gamma}\varphi(y,t) - \left(\frac{p\,m(t)}{r(r+p)} - g(t)\right)\hat{z}, \quad y = \frac{z}{M}, z = \hat{z}^{-1/\gamma}, \tag{60}$$

for some function φ. The optimal consumption can be shown to be

$$c_t^* = \begin{cases} \frac{z_t}{\bar{y}} & \text{if } \frac{z_t}{M_{t-}} \ge \bar{y}(t) \\ M_{t-} & \text{if } \bar{y}(t) > \frac{z_t}{M_{t-}} \ge 1, \\ z_t & \text{if } 1 > \frac{z_t}{M_{t-}} \ge \alpha, \\ \alpha M_{t-} & \text{if } \frac{z_t}{M_{t-}} < \alpha. \end{cases} \tag{61}$$

Then the HJB equation can be simplified to

$$\varphi_t + \frac{1}{2}\kappa^2 y^2 \varphi_{yy} + \left((r+p-\rho-\lambda)\gamma + \frac{1}{2}(1+\gamma)\kappa^2\right) y\varphi_y - \gamma^2(\rho+\lambda)\varphi$$
$$+\gamma^2\lambda\varphi_D(\xi y, t) + \frac{\gamma^2}{1-\gamma}\left(f(y,t)^{1-\gamma} - f(y,t)y^{-\gamma}(1-\gamma)\right) = 0, \tag{62}$$

with terminal condition

$$\varphi(y, T) = \varphi_R(y), \tag{63}$$

where

$$f(y,t)=\begin{cases}\frac{y}{\bar{y}} & \text{if } \ y\geq\bar{y}(t)\\ 1 & \text{if } \ \bar{y}(t)>y\geq 1,\\ y & \text{if } \ 1>y\geq\alpha,\\ \alpha & \text{if } \ y<\alpha.\end{cases} \tag{64}$$

In the absence of borrowing constraint, the investor can capitalize the entire labor income at time t_0 and thus the investor's problem reduces to a problem without labor income but a higher initial wealth. Therefore, in this case, $\varphi_D(y,\tau)$ is independent of τ, $\varphi(y,t)$ and $\bar{y}(t)$ are both independent of time. This implies that in the absence of borrowing constraint, we have $\varphi(y,t)=\varphi_R(y)$ (by (63)) and thus $\bar{y}(t)=\bar{y}_R$. The only difference from the After Retirement case is the present value of labor income.

With borrowing constraint, both the value function and the free boundaries become time dependent. We have the following similar boundary conditions:

$$(1-\gamma)\varphi(\bar{y}(t),t)=\bar{y}(t)\varphi'(\bar{y}(t),t) \tag{65}$$

$$-\gamma\varphi'(\bar{y}(t),t)=\bar{y}(t)\varphi''(\bar{y}(t),t) \tag{66}$$

$$\frac{1}{\gamma}\underline{y}(t)^{1+\gamma}\varphi'(\underline{y}(t),t)M_{t-}+\frac{m(\tau)}{r}=0 \tag{67}$$

$$(1+\gamma)\varphi'(\underline{y}(t),t)+\underline{y}(t)\varphi''(\underline{y}(t),t)=0, \tag{68}$$

where (65) and (66) follow from the smooth pasting conditions at $\bar{y}(t)$ and (67) and (68) from the requirement that at $y=\underline{y}(t)$ the financial wealth W_t be 0 and the investor cannot invest any amount in stock.

We can then solve the HJB equation (62) subject to the above boundary conditions. With the additional time dimension, we solve backward from time T for each fixed M value, which essentially reduces the problem to be a series of solving an ODE with free boundaries.

IV. Numerical Analysis

In this section, we conduct analysis of the optimal consumption and investment policies.

Following Cocco, Gomes, and Maenhout (2005), we use the following default parameter values for numerical analysis: $r = 0.02$, $\mu = 0.06$, $\sigma = 0.157$, $\iota = 0.6$, $\gamma = 5$, $\rho = 0.04$, $T = 65$ and $t_0 = 20$; wage rate function $w(t) = e^{-1.9348+0.3194t-0.00577t^2+0.000033t^3}$. According to disability statistics, we choose $\lambda = 0.0064$ and $p = \lambda$. In addition, we set $e = 0$, $k = 0.5$, $\alpha = 0.75$, $M_0 = 20$, and $W_0 = 1500$. We set initial M_0 to be the initial consumption level when $\alpha = 0$.

A. After LTD

We first examine the optimal consumption and investment policies after the LTD shock. In Figure 1, we plot the free boundaries $\underline{y}_D$ and $\bar{y}_D$ against M with and without the borrowing constraint. Without the borrowing constraint, the investor capitalizes all the future income and increases the consumption above the historically highest level of M only when y exceeds 1.22, which is independent of M. With borrowing constraint, Figure 1 shows that the wealth level (which y is proportional to) at which the investor increases consumption beyond HHL is higher than without the constraint because the value of future income is effectively reduced. The boundary $\underline{y}_D$ at which the borrowing constraint is borrowing decreases with M. This is because when M is small (relative to the income), the investor can afford to using most of the financial wealth because future consumption can be supported by future income and thus the borrowing constraint binds at a higher wealth level. Note that when M is low, the investor can still consume more than the minimum living standard αM even when the borrowing constraint binds. This is because the future income is relatively large and is more than enough to support the minimum consumption level in the future.

In Figure 2, we plot r^*_{D1}, r^*_{D2}, and r^*_{D3} without borrowing constraint against α. Figure 2 shows that all three boundaries increase with α. When $\alpha = 0$, we recover the standard result in the absence of consumption ratcheting and the investor does not leave a buffer between the wealth level at which he increase beyond M and the wealth level above which he consumes M. When $\alpha = 1$, we recover the results in Dybvig (1993) where the critical wealth above which the investor consumes M and the critical wealth below which the investor consumes αM coincide. In Figure 3, we plot the fraction of financial wealth invested in stock against the LTD shock time from 20 to 80 for $\alpha = 0.75, 1$. Recall that the earlier the LTD shock, the lower the investor's income after LTD. This figure suggests that as the after LTD income increases, the investor invests more in stock. As α increases, the investment decreases.

In Figure 4, we plot the consumption to wealth ratio against the LTD shock time from 20 to 80 for $\alpha = 0.75, 1$. Figure 4 shows that when after LTD income is low, the investor consumes the minimum level of αM. As income increases, the consumption also increases. An increase in α decreases consumption.

In Figures 5 and 6, we plot the fraction of financial wealth invested in stock and consumed respectively against financial wealth W from $\$4,500$ to $\$45,000$ for $\alpha = 0, 0.75, 1$. Figures 5 and 6 show that the fractions decrease with financial wealth. This is because the investor keeps a constant fraction of the total wealth in stock and for consumption. With a constant present value of income, the fraction of financial wealth invested in stock and consumed decreases with the financial wealth.

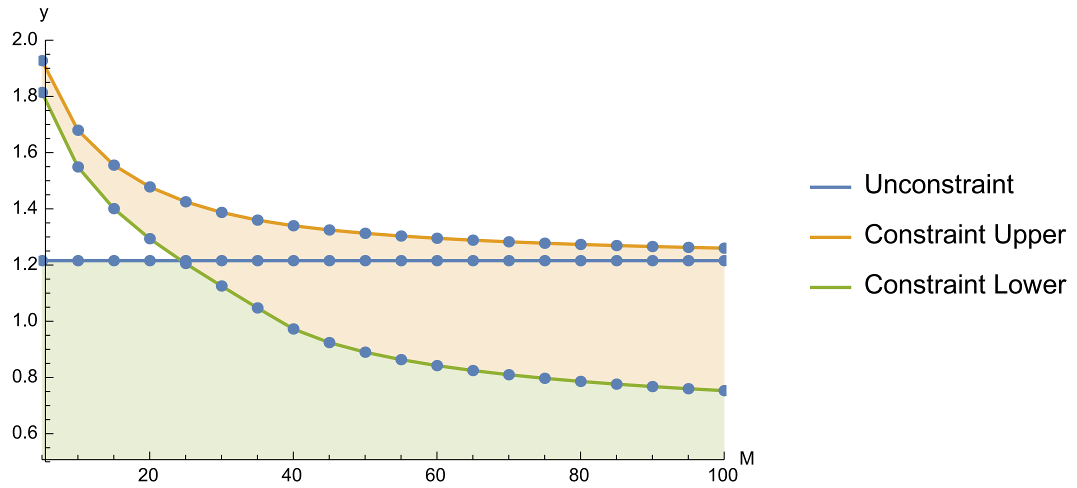

Figure 1: The free boundaries $\underline{y}_D$ and $\bar{y}_D$ against M with and without the borrowing constraint. The default parameter values are: $\rho = 0.04$, $r = 0.02$, $\mu = 0.06$, $\sigma = 0.157$, $\iota = 0.6$, $\gamma = 5$, $t_0 = 20$, $T = 65$, $w(t) = e^{-1.9348+0.3194t-0.00577t^2+0.000033t^3}$, $\lambda = 0.0064$, $p = \lambda$, $e = 0$, $k = 0.5$, $M_0 = 20$, $W_0 = 1500$ and $\alpha = 0.75$.

B. Before LTD but after retirement

In Figure 7, we plot $r^*_{R1}, r^*_{R2}, and r^*_{R3}$ against M for $p = \lambda = 0.0064, 2x0.0064, 3x0.0064$ with and without borrowing constraints. As in the case after LTD, without borrowing constraints, the critical ratios are flat across different M levels as shown in the analytical section and with borrowing constraints, the ratios decrease with M. An increase in the LTD risk significantly increases these critical ratios. Intuitively, with greater LTD risk, the future income net of medical expense is lower and the investor needs greater consumption to achieve the same utility after LTD. Therefore, the investor is more conservative in increasing consumption and thus the critical ratios increase.

C. Before LTD and before retirement

In Figure 8, we plot the fraction of financial wealth invested in stock and the consumption to wealth ratio against age from 20 to 80 for $p = \lambda = 0.0064, 2x0.0064, 3x0.0064$.

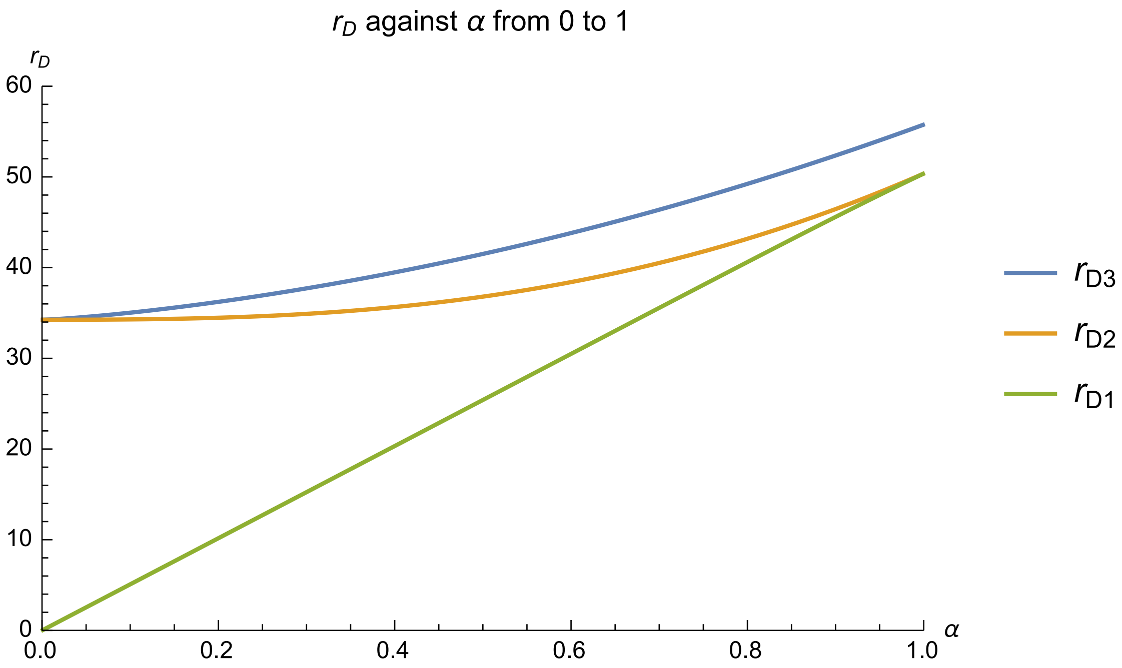

Figure 2: r^*_{D1}, r^*_{D2}, and r^*_{D3} without borrowing constraint against α. The default parameter values are: $\rho = 0.04$, $r = 0.02$, $\mu = 0.06$, $\sigma = 0.157$, $\iota = 0.6$, $\gamma = 5$, $t_0 = 20$, $T = 65$, $w(t) = e^{-1.9348+0.3194t-0.00577t^2+0.000033t^3}$, $\lambda = 0.0064$, $p = \lambda$, $e = 0$, $k = 0.5$, $M_0 = 20$, $W_0 = 1500$ and $\alpha = 0.75$.

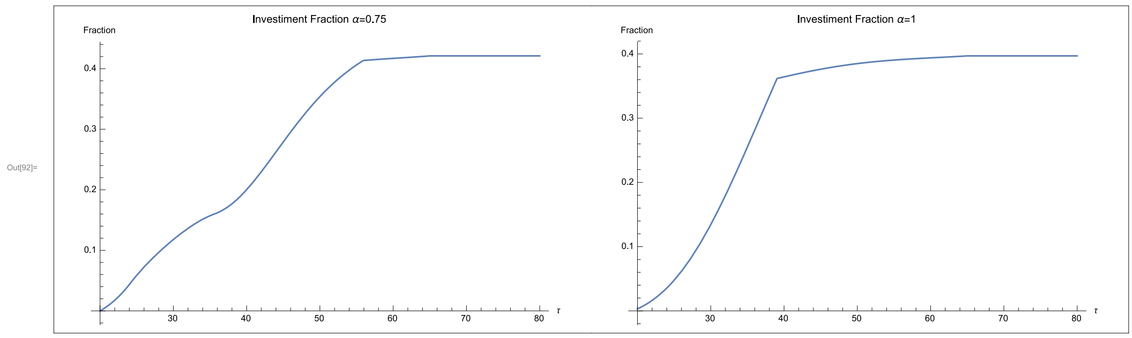

Figure 3: The fraction of financial wealth invested in stock against the LTD shock time from 20 to 80 for $\alpha = 0.75, 1$. The default parameter values are: $\rho = 0.04$, $r = 0.02$, $\mu = 0.06$, $\sigma = 0.157$, $\iota = 0.6$, $\gamma = 5$, $t_0 = 20$, $T = 65$, $w(t) = e^{-1.9348+0.3194t-0.00577t^2+0.000033t^3}$, $\lambda = 0.0064$, $p = \lambda$, $e = 0$, $k = 0.5$, $M_0 = 20$, and $W_0 = \alpha M_0 / r$.

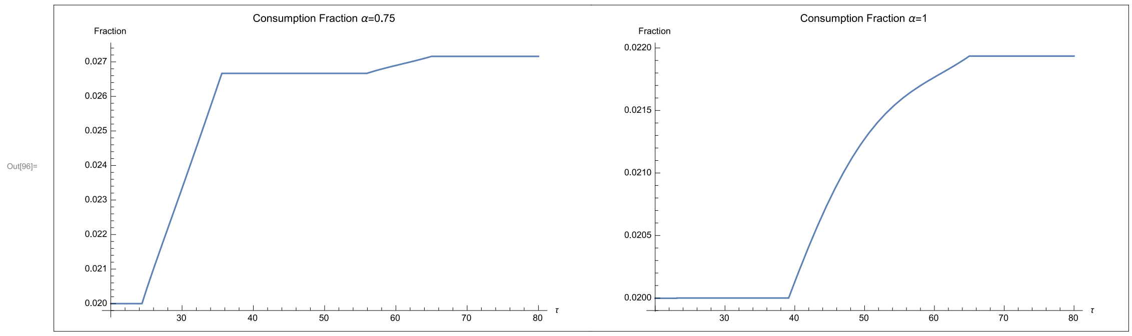

Figure 4: The consumption to financial wealth ratio against the LTD shock time from 20 to 80 for $\alpha = 0.75, 1$. The default parameter values are: $\rho = 0.04$, $r = 0.02$, $\mu = 0.06$, $\sigma = 0.157$, $\iota = 0.6$, $\gamma = 5$, $t_0 = 20$, $T = 65$, $w(t) = e^{-1.9348+0.3194t-0.00577t^2+0.000033t^3}$, $\lambda = 0.0064$, $p = \lambda$, $e = 0$, $k = 0.5$, $M_0 = 20$, and $W_0 = \alpha M_0/r$.

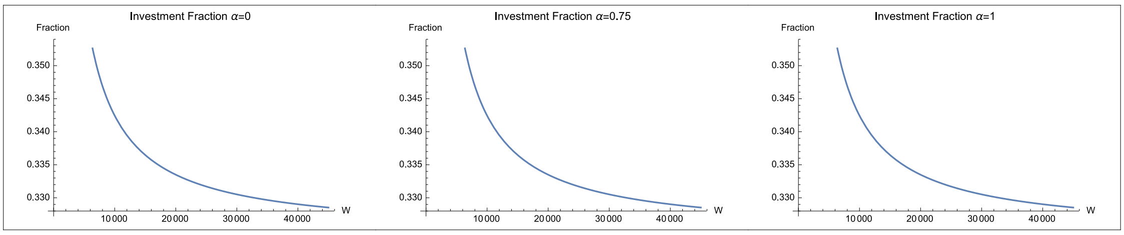

Figure 5: The fraction of financial wealth invested in stock against financial wealth W. The default parameter values are: $\rho = 0.04$, $r = 0.02$, $\mu = 0.06$, $\sigma = 0.157$, $\iota = 0.6$, $\gamma = 5$, $t_0 = 20$, $T = 65$, $w(t) = e^{-1.9348+0.3194t-0.00577t^2+0.000033t^3}$, $\lambda = 0.0064$, $p = \lambda$, $e = 0$, $k = 0.5$, $M_0 = 20$, and $W_0 = 1500$.

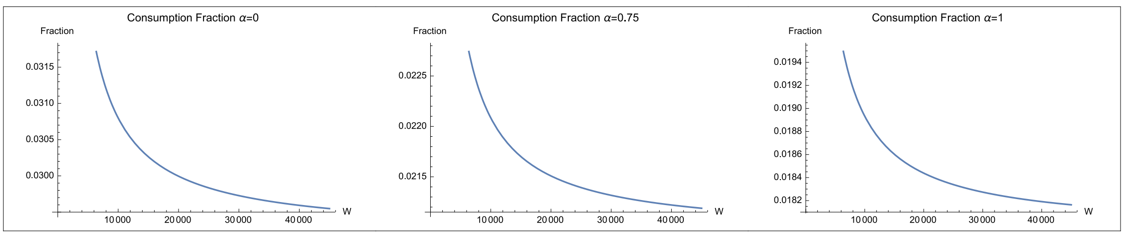

Figure 6: The fraction of financial wealth consumed against financial wealth W. The default parameter values are: $\rho = 0.04$, $r = 0.02$, $\mu = 0.06$, $\sigma = 0.157$, $\iota = 0.6$, $\gamma = 5$, $t_0 = 20$, $T = 65$, $w(t) = e^{-1.9348+0.3194t-0.00577t^2+0.000033t^3}$, $\lambda = 0.0064$, $p = \lambda$, $e = 0$, $k = 0.5$, $M_0 = 20$, and $W_0 = 1500$.

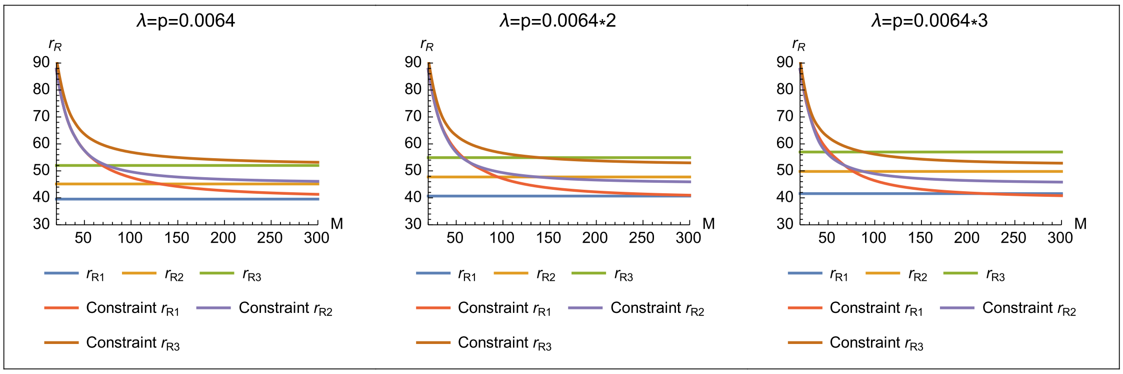

Figure 7: $r^*_{R1}, r^*_{R2}, and r^*_{R3}$ against M with and without borrowing constraints. The default parameter values are: $\rho = 0.04$, $r = 0.02$, $\mu = 0.06$, $\sigma = 0.157$, $\iota = 0.6$, $\gamma = 5$, $t_0 = 20$, $T = 65$, $w(t) = e^{-1.9348+0.3194t-0.00577t^2+0.000033t^3}$, $e = 0$, $k = 0.5$, $M_0 = 20$, $W_0 = 1500$ and $\alpha = 0.75$.

Figure 8 shows that our model implies a hump shaped consumption investment policy over life cycle, which is consistent with widely documented empirical evidence. The hump shape is driven by the hump shaped labor income before retirement. Thus the traditional financial advice that one should invest less as one ages is only partially correct. When an investor is at the beginning of his working life, the fraction of financial wealth invested in stock should increase with age, because of the labor income effect. As the LTD risk increases, both consumption and investment decrease.

In Figure 9, we plot the fraction of total wealth consumed, invested in stock and used to buy LTD insurance against financial wealth for $p = \lambda = 0.0064, 2x0.0064, 3x0.0064$. Figure 9 shows that the optimal consumption fraction is nonmonotonic in the financial wealth. The initial increasing part corresponds to the region between the minimum consumption of αM and M. As financial wealth increases, the consumption fraction increases. As financial wealth increases further, the consumption increases to M and stays there in a certain region, and thus the fraction starts to decrease. Finally, when the wealth is high enough, the investor further increases consumption such that the

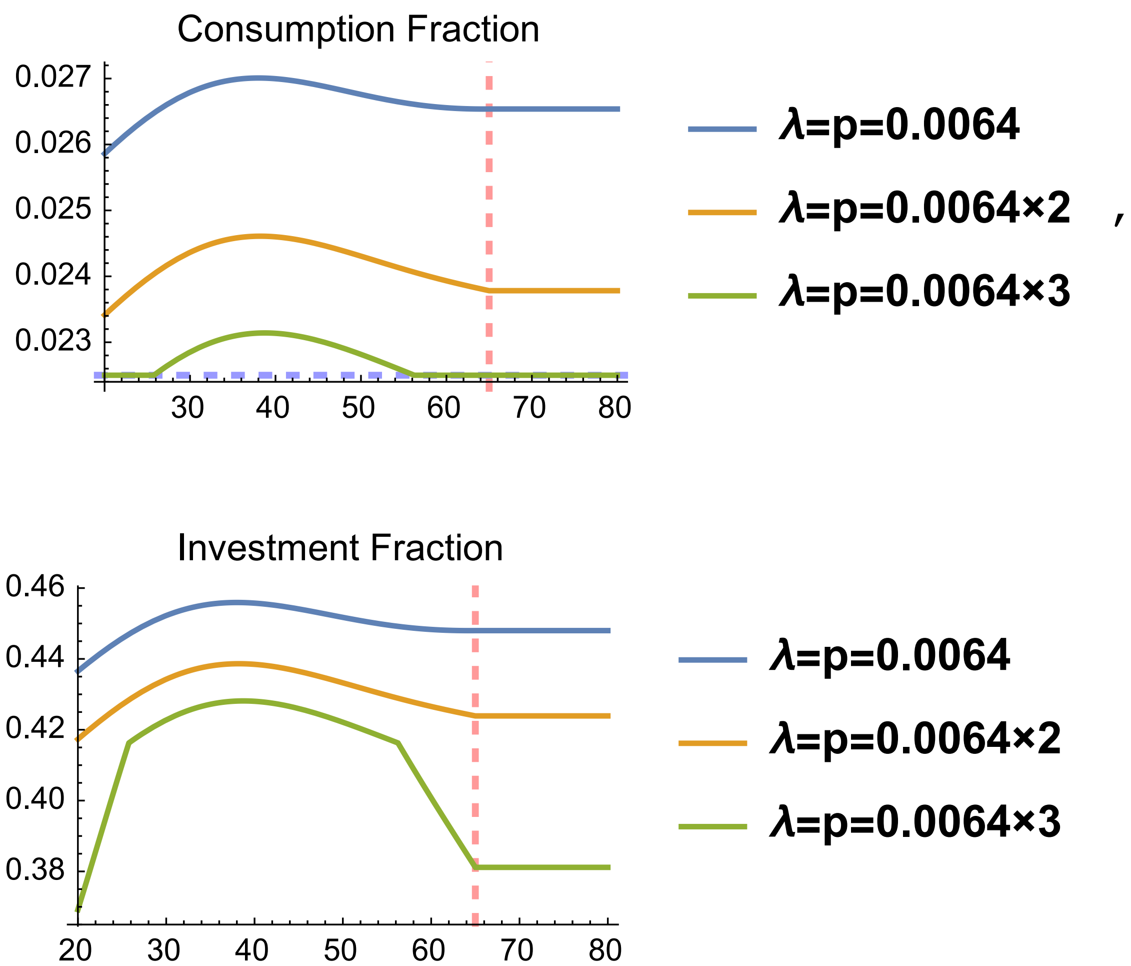

Figure 8: The fraction of financial wealth invested in stock and the consumption to wealth ratio against age. The default parameter values are: $\rho = 0.04$, $r = 0.02$, $\mu = 0.06$, $\sigma = 0.157$, $\iota = 0.6$, $\gamma = 5$, $t_0 = 20$, $T = 65$, $w(t) = e^{-1.9348+0.3194t-0.00577t^2+0.000033t^3}$, $\lambda = 0.0064$, $p = \lambda$, $e = 0$, $k = 0.5$, $M_0 = 20$, $W_0 = 1500$ and $\alpha = 0.75$.

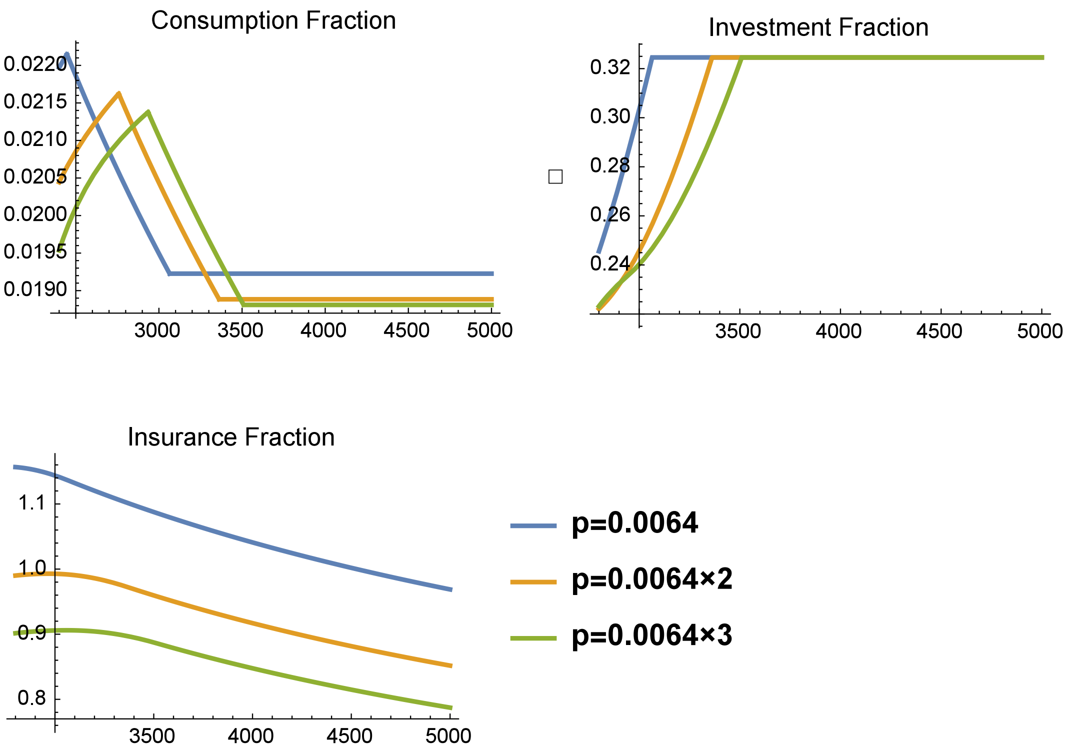

Figure 9: The default parameter values are: $\rho = 0.04$, $r = 0.02$, $\mu = 0.06$, $\sigma = 0.157$, $\iota = 0.6$, $\gamma = 5$, $t_0 = 20$, $T = 65$, $w(t) = e^{-1.9348+0.3194t-0.00577t^2+0.000033t^3}$, $\lambda = 0.0064$, $p = \lambda$, $e = 0$, $k = 0.5$, $M_0 = 20$, $W_0 = 1500$ and $\alpha = 0.75$.

fraction stays constant. As financial wealth increases, the fraction of total wealth invested in stock increases because the investor has more to invest. When the financial wealth is high enough for the investor to increase the historically highest consumption level M, the fraction of total wealth invested in stock reaches the Merton level and stays flat afterwards. In contrast, the LTD insurance as a fraction of total wealth decreases as the financial wealth increases, because the marginal utility gain from additional insurance decreases due to the reduced relative risk aversion. Clearly, as the insurance premium p increases, both the insurance fraction and the stock fraction decrease because of the higher cost of insurance. While this is also true for the consumption for low and high financial wealth levels, it does not hold for the middle range of wealth levels. For example, at $W = 3,000$, the relationship is reversed. The reason is that at this financial wealth level, the consumption levels for all three p levels are the same at M, but the total wealth level is higher with a lower insurance premium.

V. Conclusion

We propose a lifecycle consumption and investment model in the presence of long term disability risk for an investor who needs to maintain a living standard that is at least a certain fraction of the historically highest level. We show that there exists an optimal wealth-to-historically-highest-consumption threshold ratio above which the investor increases consumption beyond the historically highest level. We find that the long-term disability risk significantly reduces consumption and investment. The inability to borrow against future income magnifies the impact of long term disability and further decreases consumption and investment. Our model generates hump shaped lifecycle consumption and investment patterns that are consistent with empirical evidence. In addition, the access to long term disability insurance is important for reducing the large negative impact of the LTD risk.

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Appendix A

PROOF OF THEOREMS 1 AND 2: We only provide a sketch of the proof that contains the main steps for Theorem 2. The proof for Theorem 1 only needs minor changes on income flow and is thus omitted to save space. It is tedious but straightforward to use the generalized Itô's lemma, equations (45), (48), (37), (49), (50) to verify that the claimed optimal strategy W_t^*, c_t^*, θ_t^*, and I_t^* in these two theorems satisfy the budget constraint (31). In addition, it can be shown that z_0 exists and is unique and W_t^* satisfies the borrowing constraint in each problem. Furthermore, there is a unique solution to the equations for $\bar{y}_R$ and $\underline{y}_R$ (with borrowing constraint).

After integrating out the LTD risk, the utility function can be written as

$$E \int_0^\infty e^{-(\rho+\lambda)s} \left(\frac{c_s^{1-\gamma}}{1-\gamma} + \lambda V^D(W_s + I_s, M_s, T) \right) ds. \tag{69}$$

Accordingly, define

$$\begin{aligned} N_t &= \int_0^t e^{-(\rho+\lambda)s} \left(\frac{c_s^{1-\gamma}}{1-\gamma} + \lambda V^D(W_s + I_s, M_s, T) \right) ds \\ &\quad + e^{-(\rho+\lambda)t} V(W_t, M_t, t). \end{aligned} \tag{70}$$

One can show that N_t is a supermartingale for any feasible policy (c, θ, I) and a martingale for the claimed optimal policy (c^*, θ^*, I^*), which implies that $N_0 \geq E[N_t]$, i.e.,

$$\begin{aligned} V(W_0, M_0, 0) &\geq E \int_0^t e^{-(\rho+\lambda)s} \left(\frac{c_s^{1-\gamma}}{1-\gamma} + \lambda V^D(W_s + I_s, M_s, T) \right) ds \\ &\quad + E[e^{-(\rho+\lambda)t} V(W_t, M_t, t)], \end{aligned} \tag{71}$$

and with equality for the claimed optimal policy. In addition, it can be shown that

$$\lim_{t \to \infty} E[e^{-(\rho+\lambda)t} V(W_t, M_t, t)] \geq 0,$$

with equality for the claimed optimal policy.

Therefore, taking the limit as $t \uparrow \infty$ in (71), we have

$$V(W_0, M_0, 0) \geq E\left[\int_0^{\infty} e^{-(\rho+\lambda)s}\left(\frac{c_s^{1-\gamma}}{1-\gamma} + \lambda V^D(W_s + I_s, M_s, T)\right) ds\right],$$

with equality for the claimed optimal policy (c^*, θ^*, I^*). This completes the proof.

□

A. Functions for Theorem 2

$$g_0(y) = \frac{A_0 \gamma \lambda}{\rho + \lambda - (1-\gamma)\left(\lambda + r + \frac{\kappa^2}{2\gamma}\right)} \frac{y^{1-\gamma}}{1-\gamma}; \tag{72}$$

$$g_1(y) = \frac{k^{1-\gamma}\lambda}{(1-\gamma)\rho(\lambda+\rho)} - \frac{\lambda}{r(\lambda+r)} y^{-\gamma} + \hat{A}_1 y^{\beta_m} + \hat{A}_2 y^{\beta_p}; \tag{73}$$

$$g_2(y) = \frac{\gamma \lambda \eta_D}{\rho + \lambda - (1-\gamma)\left(\lambda + r + \frac{\kappa^2}{2\gamma}\right)} \frac{y^{1-\gamma}}{1-\gamma} + \hat{B}_1 y^{\beta_m} + \hat{B}_2 y^{\beta_p} \tag{74}$$

$$g_3(y) = \frac{(\alpha k)^{1-\gamma}\lambda}{(1-\gamma)\rho(\lambda+\rho)} - \frac{\alpha\lambda}{r(\lambda+r)} y^{-\gamma} + \hat{C}_2 y^{\beta_p} \tag{75}$$

$$h_1 = \frac{1}{(1-\gamma)(\lambda+\rho)} - \frac{1}{\lambda+r} y^{-\gamma}; \tag{76}$$

$$h_2 = \frac{\gamma^2}{(1-\gamma)\left(\rho + \lambda - (1-\gamma)\left(\lambda + r + \frac{\kappa^2}{2\gamma}\right)\right)} y^{1-\gamma}; \tag{77}$$

$$h_3 = \frac{\alpha^{1-\gamma}}{(1-\gamma)(\lambda+\rho)} - \frac{\alpha}{\lambda+r} y^{-\gamma}, \tag{78}$$

where

$$\hat{A}_1 = A_1, \quad \hat{A}_2 = A_2$$

$$\hat{B}_1 = B_1, \quad \hat{B}_2 = B_2$$

$$\hat{C}_2 = C_2.$$