Measuring Ambiguity Aversion

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Abstract

We confront the generalized dynamic intertemporal "smooth ambiguity aversion" preferences of Klibanoff, Marinacci, and Mukerji (2005, 2009) with data using Bayesian methods introduced by Gallant and McCulloch (2009) to close two existing gaps in the literature. First, we estimate the size of ambiguity aversion implied by financial data for the representative agent in a consumption-based equilibrium asset pricing model. Second, we investigate the contribution of ambiguity aversion in explaining variations in equity premium and consumption growth. Our estimates are comparable with those from existing empirical research and suggest ample scope for ambiguity aversion.

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1 Introduction

In this paper, we confront the "smooth ambiguity aversion" model of Klibanoff, Marinacci, and Mukerji (2005, 2009), (henceforth, KMM), in its generalized form advanced by Hayashi and Miao (2011) and Ju and Miao (2012), with data to close two existing gaps in the literature. First, we empirically estimate the size of ambiguity aversion implied by financial data for a representative agent endowed with smooth ambiguity aversion preferences in a consumption-based equilibrium asset pricing model. Second, we empirically investigate the contribution of smooth ambiguity aversion in explaining variations in equity premium and consumption growth. Given the rising popularity of smooth ambiguity preferences in economics and finance, it is important to characterize this model's empirical strengths and contributions, as well as its shortcomings. One salient feature of smooth ambiguity aversion is the separation of ambiguity and ambiguity aversion, where the former is a characteristic of the representative agent's subjective beliefs, while the latter derives from the agent's tastes. This study provides the first data-based estimation of this ambiguity aversion parameter in a dynamic asset pricing model. Our estimated ambiguity aversion parameter is higher than its calibrated counterparts in existing endowment or production-based asset pricing studies. Other estimated structural parameters are comparable with estimated values reported in the literature. To our knowledge, this is the first paper to conduct an empirical investigation of this class of preferences based on market data.

Ambiguity aversion matters. Jeong, Kim, and Park (2014) show that in an equilibrium asset pricing model where agents are endowed with a different but related class of ambiguity aversion preferences, ambiguity aversion accounts for 45% of average equity premium. They also find that it is economically and statistically significant. Our findings confirm theirs, and extend the literature in new directions. In our benchmark model, ambiguity arises due to a mixture of distributions for dividend growth. The state determining the distribution of dividend growth is unobservable. The agent can learn about the hidden state, but this ability does not eliminate the difficulties in forming forecasts. In turn, these difficulties generate the scope for ambiguity aversion. Ambiguity aversion gives rise to intertemporal choices that differ dramatically from those made by an ambiguity-neutral agent.

KMM preferences and "multiple-priors utility" of Chen and Epstein (2002) (henceforth, MPU) have drawn considerable attention in the literature. In practice, smooth ambiguity aversion of

KMM has two important advantages over MPU. Critically, MPU does not admit a sharp separation between ambiguity and ambiguity aversion. In the MPU framework, the set of priors, which characterizes ambiguity, also determines the degree of ambiguity aversion. Thus, in empirical studies based on MPU such as Jeong et al. (2014), one only obtains the estimate of the size of ambiguity instead of the magnitude of ambiguity aversion. In the MPU framework, it is therefore infeasible to do comparative statics analysis by holding the family of alternative distributions constant while varying the degree of ambiguity aversion. The most important feature of the generalized recursive smooth ambiguity preferences is precisely a separation between ambiguity and ambiguity aversion, and moreover a three-way separation among risk aversion, ambiguity aversion, and the elasticity of intertemporal substitution. The second advantage of KMM over MPU is tractability. Asset pricing models with MPU are generally difficult to solve with refined processes of fundamentals because MPU features kinked preferences.[1]

KMM preferences are relatively new. Klibanoff et al. (2005, 2009) introduced this class of preferences. Hayashi and Miao (2011) generalized these utility functions, and Hanany and Klibanoff (2009), Klibanoff, Marinacci, and Mukerji (2011), Cerreia-Vioglio, Ghirardato, Maccheroni, Marinacci, and Siniscalchi (2011) further refined them. This class of preferences is becoming increasingly popular in finance and macroeconomics. Applications include endowment economy asset pricing (Ju and Miao (2012), Collard, Mukerji, Sheppard, and Tallon (2012), and Ruffino (2013)), production-based asset pricing (Jahan-Parvar and Liu (2014) and Backus, Ferriere, and Zin (2014)), and portfolio choice (Gollier (2011), Maccheroni, Marinacci, and Ruffino (2013), Chen, Ju, and Miao (2014), and Guidolin and Liu (2014)), among others.

Our estimation of the level of ambiguity aversion facilitates using this class of preferences by linking it directly to the data. Most of existing applications of smooth ambiguity preferences rely on the methodology of calibration. The popular methods of calibrating the degree of ambiguity aversion include the "detection-error probability" method of Anderson, Hansen, and Sargent (2003) and Hansen (2007) (see Jahan-Parvar and Liu (2014) for an application) and "thought experiments" similar to Halevy (2007) (see Ju and Miao (2012) and Chen et al. (2014) for applications). Clearly, the contribution of our study is methodological in that we use both financial and macroeconomic data to estimate the degree of ambiguity aversion together with other structural parameters in a dynamic asset pricing model with learning.

[1] Strzalecki (2013) provides a rigorous and comprehensive discussion of ambiguity-based preferences.

Similar to other macro-finance applications, we face sparsity of data. As has become standard in the macro-finance empirical literature, we use prior information and a Bayesian approach to overcome data sparsity. Specifically, we use the "General Scientific Models" (henceforth, GSM) Bayesian estimation methodology developed by Gallant and McCulloch (2009). GSM is the Bayesian counterpart to the classical "indirect inference" and "efficient method of moments" (hereafter, EMM) methods introduced by Gouriéroux, Monfort, and Renault (1993) and Gallant and Tauchen (1996, 1998, 2010). These are simulation-based inference methods that rely on an auxiliary model for implementation. GSM follows the logic of the EMM variant of indirect inference and relies on the theoretical results of Gallant and Long (1997) in its construction of a likelihood. A comparison of Aldrich and Gallant (2011) with Bansal, Gallant, and Tauchen (2007) displays the advantages of a Bayesian EMM approach relative to a frequentist EMM approach, particularly for the purpose of model comparison. An indirect inference approach is an appropriate estimation methodology in the context of this study since the estimated equilibrium model is highly nonlinear and does not admit of analytically tractable solutions thereby severely inhibiting accurate, numerical construction of a likelihood by means other than GSM. GSM uses a sieve (Section 3.3) specially tailored to macro and finance time series applications as the auxiliary model. When a suitable sieve is used as the auxiliary model, as here, the GSM method synthesizes the exact likelihood implied by the model.[2] In this instance, the synthesized likelihood model departs significantly from a normal-errors likelihood, which suggests that alternative econometric methods based on normal approximations will give biased results. In particular, in addition to GARCH and leverage effects, the three-dimensional error distribution implied by the smooth ambiguity aversion model is skewed in all three components and has fat-tails for consumption growth and stock returns and thin tails for bond returns.

Ahn, Choi, Gale, and Kariv (2014) estimate the level of ambiguity aversion for several static specifications of ambiguity aversion preferences based on experimental data. Their findings differ from ours since a) they use a static specification and ignore the intertemporal choice, and b) their empirical findings imply that ambiguity aversion parameter estimates for the smooth ambiguity specification are not significantly different from zero.[3] In addition, the mapping between their static estimation results and our dynamic model estimates is not clear. Similar to our findings and

[2] Gallant and McCulloch (2009) use the terms "scientific model" and "statistical model" instead of the terms "structural model" and "auxiliary model" used in the econometric indirect inference literature. We will follow the conventions of the econometric literature. The structural models here are the "smooth ambiguity aversion" model and a restricted version of that model.

[3] They get much tighter estimates and hence statistically significant results for their kinked ambiguity specification.

as expected according to the theory, they report the estimates of ambiguity aversion that are larger than the estimates of risk aversion. However, the magnitudes of their estimates are very different from ours and those reported in calibration studies. Such discrepancies between market data-based and experimental estimates are common.

Two recent papers, Jeong, Kim, and Park (2014) and Viale, Garcia-Feijoo, and Giannetti (2014), provide time series and cross-sectional estimates for the MPU model. As mentioned above, these studies do not provide a direct measurement of ambiguity aversion because MPU does not admit a functional separation between ambiguity and ambiguity aversion. In a production based general equilibrium setting, Ilut and Schneider (2014) assume that ambiguity is an exogenously determined autoregressive process, while Bianchi, Ilut, and Schneider (2014) define ambiguity as parameter uncertainty or measurement error in volatility of marginal product of capital and in operating costs. Anderson, Ghysels, and Juergens (2009) estimate the magnitude of ambiguity aversion using forecasts of professional forecasters.

The rest of the paper proceeds as follows. Section 2 introduces the data used in our GSM Bayesian estimation. Section 3 presents the consumption-based asset pricing model with generalized recursive smooth ambiguity preferences developed by Ju and Miao (2012), and the numerical method used to solve the model. Section 4 discusses the estimation methodology and presents our empirical findings. Section 5 presents model comparison results, forecasts, and asset pricing implications. Section 6 concludes.

2 Data

Throughout this paper, lower case denotes the logarithm of an upper case quantity; e.g., $c_t = \ln(C_t)$, where C_t is consumption in period t, and $d_t = \ln(D_t)$, where D_t is dividends paid in period t. Similarly, we use logarithmic risk-free interest rate $(r_{f,t})$ and aggregate equity market return inclusive of dividends $(r_{e,t} = \ln(P_{e,t} + D_t) - \ln P_{e,t-1})$ in the analysis, where $P_{e,t}$ is the stock price in period t.

We use real annual data from 1929 to 2011 for the purpose of inference, indexed by 2005 price levels. We use the data from 1929 to 1949 to provide initial lags for the recursive parts of the model and 1950-2011 data for estimation of parameters and for diagnostics. Our measure for the risk-free rate is one-year U.S. Treasury Bill rate. Our proxy for risky asset returns is the value weighted returns on CRSP-Compustat stock universe. We use the sum of nondurable and services consumption from Bureau of Economic Analysis (BEA) and deflated the series using the appropriate price deflator (also provided by the BEA). We use mid-year population data to obtain per capita consumption values. As noted in Garner, Janini, Passero, Paszkiewicz, and Vendemia (2006), there are notable discrepancies between measures of consumption released by different agencies. Thus, throughout the paper, we assume a 5% measurement error in consumption.[4] We construct price-dividend ratios following Bansal and Yaron (2004).

Table 1 presents the summary statistics of the data used in this study. Reported mean and standard deviations of risk-free rates ($r_{f,t}$), aggregate market returns($r_{e,t}$), excess returns ($r_{e,t}-r_{f,t}$), and real, per capita, log consumption growth (Δc_t) are in percentages. The reported p-values of Jarque and Bera (1980) test of normality imply that the assumption of normality is rejected for risk-free rate and log consumption growth series, but it cannot be rejected for aggregate market returns and excess returns. The plots of the data are shown in Figure 1.

3 Model

The intuitive notions behind any consumption-based asset pricing model are that agents receive income (wage, interest, and dividends) which they use to purchase consumption goods. Agents reallocate their consumption over time by trading shares of stock that pay random dividends and bonds that pay interest with certainty. This is done for consumption smoothing over time (for example, insurance against unemployment, saving for retirement, $\cdots$). Trading activity enters the model via the agent's budget constraint which implies that an agent's purchase of consumption, bonds, and stock cannot exceed income (in the form of aggregated wage, interest, and dividends) in any period. When applied to a national closed economy, consumption and dividends can be used as the driving processes instead of wages and dividends. Agents are endowed with a utility function that depends on the entire consumption process. The first order conditions of their utility maximization problem determine a map from the current state and history of the driving processes to the current price of a stock and a bond. These models are simulated by first simulating the driving processes and then evaluating the map that determines stock and bond prices.

[4] We also experimented with 1% and 10% error levels. Empirical results are robust to the level of measurement errors. Findings based on 1% and 10% measurement error are available upon request. We assume a linear error structure. That is, $C_t^* = C_t + u_t$ where C_t^* is the observed value, C_t is the true value, and u_t is the measurement error term.

3.1 The Benchmark Structural Model

We consider the representative-agent pure exchange economy model of Ju and Miao (2012). Aggregate consumption follows the process

$$\Delta c_{t+1} \equiv \ln\left(\frac{C_{t+1}}{C_t}\right) = \kappa_{z_{t+1}} + \sigma_{\Delta c}\epsilon_{t+1}, \tag{1}$$

where ϵ_t is an *i.i.d.* standard normal random variable, and z_{t+1} follows a two-state Markov chain with state 1 being the good state and state 2 being the bad state ($\kappa_1 > \kappa_2$). The transition matrix is

$$\mathbb{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix},$$

where p_{ij} denotes the probability of switching from state i to state j, and $p_{12} = 1 - p_{11}$ and $p_{21} = 1 - p_{22}$.

Dividend growth is modeled as containing a component proportional to consumption growth and an idiosyncratic component,

$$\Delta d_{t+1} \equiv \ln\left(\frac{D_{t+1}}{D_t}\right) = \lambda \Delta c_{t+1} + g_d + \sigma_{\Delta d}\varepsilon_{d,t+1}, \tag{2}$$

where $\varepsilon_{d,t+1}$ is an *i.i.d.* standard normal random variable and is independent of all other shocks in the model. The parameter λ can be interpreted as the leverage ratio on expected consumption growth as in Abel (1999) and Bansal and Yaron (2004). The parameters g_d and σ_d can be pinned down by calibrating the model to the first and second moments of dividend growth.

Ju and Miao (2012) assume that economic regimes are not observable, but the agent can learn about the state (z_t) through observing the history of consumption and dividends. The agent also knows the parameters of the model, namely, $\{\kappa_1, \kappa_2, \sigma, \lambda, g_d, \sigma_d\}$. The agent updates beliefs $\mu_t = \Pr(z_{t+1}|\Omega_t)$ according to Bayes' rule:

$$\mu_{t+1} = \frac{p_{11} f(\Delta c_{t+1}, 1)\mu_t + p_{21} f(\Delta c_{t+1}, 2)(1-\mu_t)}{f(\Delta c_{t+1}, 1)\mu_t + f(\Delta c_{t+1}, 2)(1-\mu_t)}, \tag{3}$$

where $f(\Delta c_{t+1}, i)$, $i = 1, 2$ is the normal density function of consumption growth conditional on state i.

The agent's preferences are represented by the generalized recursive smooth ambiguity utility function,

$$V_t(C) = \left[(1-\beta)C_t^{1-1/\psi} + \beta\left\{\mathcal{R}_t\left(V_{t+1}(C)\right)\right\}^{1-1/\psi}\right]^{\frac{1}{1-1/\psi}}, \tag{4}$$

$$\mathcal{R}_t\left(V_{t+1}(C)\right) = \left(\mathbb{E}_{\mu_t}\left[\left(\mathbb{E}_{z_{t+1},t}\left[V_{t+1}^{1-\gamma}(C)\right]\right)^{\frac{1-\eta}{1-\gamma}}\right]\right)^{\frac{1}{1-\eta}}, \tag{5}$$

where $\beta \in (0,1)$ is the subjective discount factor, ψ is the IES parameter, γ is the coefficient of relative risk aversion, and η is the uncertainty aversion parameter and must satisfy $\eta \geq \gamma$. Equation (5) characterizes the certainty equivalent of future continuation value, which is the key ingredient that distinguishes this utility function from Epstein-Zin's recursive utility. In Equation (5), the expectation operator $\mathbb{E}_{z_{t+1},t}[\cdot]$ is with respect to the distribution of consumption conditioning on the next period's state z_{t+1}, and the expectation operator $\mathbb{E}_{\mu_t}$ is with respect to the filtered probabilities about the unobservable state.

Under this utility function, the stochastic discount factor (SDF) is given by

$$M_{z_{t+1},t+1} = \beta\left(\frac{C_{t+1}}{C_t}\right)^{-1/\psi}\left(\frac{V_{t+1}}{\mathcal{R}_t\left(V_{t+1}\right)}\right)^{1/\psi-\gamma}\left(\frac{\left(\mathbb{E}_{z_{t+1},t}\left[V_{t+1}^{1-\gamma}\right]\right)^{\frac{1}{1-\gamma}}}{\mathcal{R}_t\left(V_{t+1}\right)}\right)^{-(\eta-\gamma)}. \tag{6}$$

Stock returns, $R_{e,t+1}$, are defined by

$$R_{e,t+1} = \frac{P_{t+1}^e + D_{t+1}}{P_t^e} = \frac{1+\varphi\left(\mu_{t+1}\right)}{\varphi\left(\mu_t\right)}\frac{D_{t+1}}{D_t},$$

where $\varphi\left(\mu_t\right)$ denotes the price-dividend ratio. Stock returns satisfy the Euler equation

$$\mathbb{E}_{\mu_t,t}\left[M_{z_{t+1},t+1}R_{e,t+1}\right] = 1.$$

The risk-free rate, $R_{f,t}$, is the reciprocal of the expectation of the SDF:

$$R_{f,t} = \frac{1}{\mathbb{E}_{\mu_t,t}\left[M_{z_{t+1},t+1}\right]}.$$

We can rewrite the Euler equation as

$$0 = \tilde{\mu}_t\mathbb{E}_{1,t}\left[M_{z_{t+1},t+1}^{EZ}\left(R_{e,t+1}-R_{f,t}\right)\right] + \left(1-\tilde{\mu}_t\right)\mathbb{E}_{2,t}\left[M_{z_{t+1},t+1}^{EZ}\left(R_{e,t+1}-R_{f,t}\right)\right],$$

where $M^{EZ}_{z_{t+1},t+1}$ can be interpreted as the SDF under Epstein-Zin recursive utility:

$$M^{EZ}_{z_{t+1},t+1} = \beta \left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1}{\psi}} \left(\frac{V_{t+1}}{\mathcal{R}_t\left(V_{t+1}\right)}\right)^{\frac{1}{\psi}-\gamma},$$

and $\tilde{\mu}_t$ can be interpreted as ambiguity distorted beliefs and represented by:

$$\tilde{\mu}_t = \frac{\mu_t \left(\mathbb{E}_{1,t}\left[V_{t+1}^{1-\gamma}\right]\right)^{-\frac{\eta-\gamma}{1-\gamma}}}{\mu_t \left(\mathbb{E}_{1,t}\left[V_{t+1}^{1-\gamma}\right]\right)^{-\frac{\eta-\gamma}{1-\gamma}} + \left(1-\mu_t\right)\left(\mathbb{E}_{2,t}\left[V_{t+1}^{1-\gamma}\right]\right)^{-\frac{\eta-\gamma}{1-\gamma}}}. \tag{7}$$

As long as $\eta > \gamma$, distorted beliefs are not equivalent to Bayesian beliefs. This distortion is an equilibrium outcome and is entirely driven by ambiguity aversion. Figure 2 shows the Bayesian belief and the ambiguity-distorted belief filtered using the historical consumption growth data for the period 1929–2011. The parameter values in the model are set to the estimates using the GSM Bayesian estimation method, which are shown below. It is obvious from Figure 2 that ambiguity aversion distorts the Bayesian belief in a pessimistic way, and thus an ambiguity averse agent slants his beliefs towards the bad regime.

We follow Ju and Miao (2012) and use the projection method with Chebyshev polynomials to solve the model. Specifically, homogeneity in utility preferences implies $V_t\left(C\right) = G\left(\mu_t\right) C_t$, and $G\left(\mu_t\right)$ satisfies the following functional equation

$$G\left(\mu_t\right) = \left[\left(1-\beta\right) + \beta\left(\mathbb{E}_{\mu_t}\left[\left(\mathbb{E}_{z_{t+1},t}\left[G\left(\mu_t\right)^{1-\gamma}\left(\frac{C_{t+1}}{C_t}\right)^{1-\gamma}\right]\right)^{\frac{1-\eta}{1-\gamma}}\right]\right)^{\frac{1-1/\psi}{1-\eta}}\right]^{\frac{1}{1-1/\psi}}.$$

To solve for the value function, we approximate $G\left(\mu_t\right)$ using Chebyshev polynomials in the state variable μ_t. The approximation takes the form

$$G\left(\mu\right) \simeq \sum_{j=0}^{p} \phi_j T_j\left(y\left(\mu\right)\right),$$

where p is the order of Chebyshev polynomials, T_j $(j = 0, ..., p)$ are Chebyshev polynomials, and $y\left(\mu\right)$ maps the sate variable μ onto the interval $[-1, 1]$. We then choose a set of collocation points for μ and solve for the coefficients $\left\{\phi_j\right\}_{j=0,\ldots,p}$ using a nonlinear equations solver. The expectation $\mathbb{E}_{z_{t+1},t}\left[\cdot\right]$ is approximated using Gauss-Hermite quadrature.

To solve for the equilibrium price-dividend ratio, we rewrite the Euler equation as

$$\frac{P_t^e}{D_t} = \mathbb{E}_t \left[M_{z_{t+1},t+1} \left(1 + \frac{P_{t+1}^e}{D_{t+1}} \right) \frac{D_{t+1}}{D_t} \right].$$

The price-dividend ratio can also be approximated using Chebyshev polynomials in the state variable μ_t. Since the pricing kernel $M_{z_{t+1},t+1}$ can be easily written as a functional of $G(\mu_{t+1})$ and consumption growth C_{t+1}/C_t, we can solve for the equilibrium price-dividend ratio in a similar way as we solve for the value function.

After solutions are found, we simulate logarithmic values of consumption growth, stock returns and risk-free rates:

$$\left\{ \ln \left(\frac{C_{t+1}}{C_t} \right), \ln \left(R_{e,t+1} \right), \ln \left(R_{f,t+1} \right) \right\}_{t=1}^{T}.$$

3.2 The Alternative Model with Ambiguity Neutrality

Our discussion of equations (4) and (5) conveys one important message: ambiguity aversion has an impact on the intertemporal decisions of the representative agent if and only if $\eta > \gamma$. If $\eta = \gamma$, then the agent is ambiguity neutral. As a result, the agent's preferences collapse to the familiar Kreps and Porteus (1978) and Epstein and Zin (1989) preferences:

$$\begin{aligned} V_t(C) &= \left[(1-\beta) C_t^{1-1/\psi} + \beta \left\{ \mathcal{R}_t \left(V_{t+1}(C) \right) \right\}^{1-1/\psi} \right]^{\frac{1}{1-1/\psi}}, \\ \mathcal{R}_t \left(V_{t+1}(C) \right) &= \mathbb{E}_t \left[V_{t+1}^{1-\gamma}(C) \right]^{\frac{1}{1-\gamma}}. \end{aligned}$$

It is immediately obvious that under ambiguity neutrality, the certainty equivalent in equation (5) collapses to the familiar temporal expectation. Thus, the agent is faced with a Markov switching structure in the aggregate consumption and dividend growth processes, following equations (1) and (2), and makes decisions based on Epstein-Zin's recursive preferences.

Given these preferences, the SDF is

$$M_{t+1} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-1/\psi} \left(\frac{V_{t+1}}{\mathbb{E}_t \left(V_{t+1} \right)} \right)^{1/\psi - \gamma}.$$

We use the same solution method that we use for the benchmark model to solve the problem for the ambiguity-neutral agent. After solving the model, we simulate logarithmic values of consumption growth, stock returns and risk-free rates.

3.3 Estimation of Structural Model Parameters

The ideas behind GSM are straightforward. If one can simulate the data implied by a model at a given value of its parameter θ, then one can determine the density implied by the model for the data at that value by statistical nonparametric density estimation methods. Particularly convenient in this context is the SNP nonparametric density estimator proposed by Gallant and Nychka (1987) and modified for time series applications by Gallant and Tauchen (1989). The SNP density is a sieve. A sieve is a parametric model with parameters arranged in an ordered sequence similar to the way the coefficients of a polynomial are ordered by degree. Rather than make a tuning parameter smaller to improve the fit to data as with kernel density estimators one increases the number of parameters with a sieve. One uses data a data driven rule to make the choice. Denote the SNP sieve by $f(\hat{y}_t \,|\, \hat{x}_{t-1}, \eta)$ where $\hat{y}_t$ denotes the simulated data at time t — consumption growth, stock returns, and bond returns in this instance — and $\hat{x}_{t-1}$ represents lagged values of $\hat{y}_t$. The data simulation and nonparametric estimation algorithms determine a map $\eta = g(\theta)$ from the parameters θ of the structural model to the parameters η of the SNP sieve. Denote the observed data by y_t, $t = 1, 2, \ldots, n$. Once the map $\eta = g(\theta)$ is determined, $\mathcal{L}(\theta) = \prod_{t=1}^{n} f[y_t \,|\, x_{t-1}, g(\theta)]$ is used as the likelihood for Bayesian inference. With a likelihood $\mathcal{L}(\theta)$ in hand, MCMC (Markov Chain Monte Carlo) is a convenient Bayesian estimation strategy.[5] Especially in this instance, because if one puts the parameters θ on a grid one only needs to determine the map $\eta = g(\theta)$ at a finite number of θ, many of which recur in the MCMC chain, in which case computation of $g(\theta)$ need not be repeated. The numerical implementation of these ideas is discussed in Aldrich and Gallant (2011) and the references therein. Code and a User's Guide implementing the GSM method, including generation of the MCMC chain, are at http://www.aronaldg.org/webfiles/gsm.

3.4 The Auxiliary Model

The trivariate SNP auxiliary model that we use is determined statistically and can be viewed as a one-lag VAR model with a BEKK (Engle and Kroner (1995)) variance structure that has one lag in both the ARCH and GARCH components of the BEKK and a leverage effect. The leverage effect enters through the ARCH component, where the ARCH coefficient takes one of two values

[5] MCMC generates a correlated sequence of draws from the posterior density from which estimates of location and scale from the posterior density can be computed. The sequence of draws is termed an MCMC chain. See Gamerman and Lopes (2006) for details. We used chains of length 100,000 past the point where transients died off using a move-one-at-at-time Gaussian proposal density.

depending on the sign of the innovation. The auxiliary model is determined from simulations of the structural model so issues of data sparsity do not arise; one can make the simulation length N as large as necessary to determine the parameters of the auxiliary model accurately.[6] Using the Bayesian information criteria (BIC) protocol for selecting an SNP density, we chose a model with the aforementioned (Section 1) location and scale specifications and non-Gaussian errors for innovations, which are of the degree four SNP type (Gallant and Nychka (1987)). The auxiliary model has 37 parameters.

4 Inference for Generalized Scientific Models

Here we describe the particulars of our implementation of the GSM Bayesian method described in Subsections 3.3. and 3.4.

Both the benchmark and alternative models are richly parameterized. For the benchmark model, parameter is $\theta = \{\beta, \gamma, \psi, \eta, p_{11}, p_{22}, \kappa_1, \kappa_2, \lambda, \sigma_{\Delta c}, \sigma_{\Delta d}\}$. Measures of location and scale of the prior for the benchmark model parameters are reported in the first numerical block of Table 2. The prior for the alternative model is the nearly the same except as noted in the legend of Table 2 due to the fact that γ and η are constrained by the alternative model prior to be nearly equal. Initially the priors for the benchmark and alternative models are Gaussian independence priors; i.e., the prior for the joint is the product of the marginals. However, due to support conditions described immediately below and exclusion of parameter values for which an equilibrium does not exist, the effective prior is not an independence prior. In general, this prior is in line with other Bayesian macro studies. Measures of location and scale for the posterior for the benchmark and alternative model are reported in the second and third numerical blocks of Table 2, respectively. The mean of the posterior is what is typically reported but the model is never actually simulated at the mean. The mode has the advantage that it must satisfy support conditions and the model has been simulated at that value.

In addition to the prior, we impose the following support conditions. The subjective discount factor β is between 0.00 and 1.00. We bound the coefficient of risk aversion γ to be above 0.00.[7] We impose the following on the coefficient of ambiguity aversion η: in the benchmark model, $\eta > \gamma$. We need this restriction for ambiguity aversion to exist. Hayashi and Miao (2011) and Ju and Miao

[6] We used N = 1000. We found that using larger values of N did not change results other than increase run times.
[7] In line with recommendations of Mehra and Prescott (1985), the mean of the prior distribution for this coefficient is set to 2.00.

(2012) furnish detailed discussions of this requirement. Briefly, with $\eta \leq \gamma$, compound predictive probability distributions are reduced to an ordinary predictive probability, removing ambiguity from the model and leaving no room for ambiguity aversion to play a part in agent's allocations. To preserve concavity in attitudes toward ambiguity, we need $\eta > \gamma$. Naturally, in the alternative model we force these two parameters to be nearly equal to obtain ambiguity neutrality. Kreps and Porteus (1978) and Epstein and Zin (1989) preferences require a separation between risk aversion and intertemporal substitution. As a result, intertemporal elasticity of substitution parameter ψ cannot assume either 0 or 1 values. For ψ we impose $1.00 < \psi$. We constrain $0.9396 < p_{1,1} < 0.999620$, $0.2514 < p_{2,2} < 0.7806$, $0.01596 < \kappa_1 < 0.02906$, $-0.1055 < \kappa_2 < -0.0302$, $0 < \lambda$, $0.02646 < \sigma_{\Delta c} < 0.03608$, and $0.06542 < \sigma_{\Delta d} < 0.1746$ based on numerical experience acquired with these model in previous studies.

4.1 Empirical Results

We report estimation results for the benchmark model featuring ambiguity aversion and the alternative model with ambiguity neutral agent in Table 2. As mentioned above, we assume a 5% measurement error in the real, per capita, consumption growth process. We estimate a total of 11 parameters. Log dividend growth process, Δd_t, is the latent variable in our estimation. Estimation results for the benchmark structural model featuring ambiguity aversion and learning are reported in columns 5 to 7 in Table 2. We report estimated parameters for the alternative structural model with no ambiguity aversion in the last three columns of Table 2.[8]

Estimates of mode and mean measures of subjective discount factor β are stable across our benchmark and alternative models, and reasonably close to values reported by Aldrich and Gallant (2011) and Bansal et al. (2007). Thus, they do not cause any concern for us and imply precise measurements of the target parameter.

We observe the following regularities in estimated parameters: The magnitude of risk aversion parameter, γ, is sensitive to the presence of ambiguity aversion. The mode and the mean of the posterior of γ in the benchmark model are over 50% smaller than the prior values and only about 12% of the value of the corresponding estimate in the alternative model with no ambiguity. This result is related to estimation outcomes reported in Jeong et al. (2014) for their baseline models II

[8] In practice, we force η to be close to γ by imposing a tight prior. As a result, for the ambiguity neutral case (alternative model), we get estimated mode, mean, and standard deviations of η that are very close to the values reported for γ in the last three columns of Table 2.

(recursive utility) and III (MPU), where aggregate wealth consists of financial wealth alone. Using MPU, Jeong et al. (2014) report γ ranging between 0.2 to 2.9, while using only recursive utility this value is 4.9. Thus, mode and mean values for γ equal to 0.3203 and 0.8258 respectively are in agreement with Jeong et al. (2014) estimates. In comparison with Aldrich and Gallant (2011), our estimates for γ are substantially smaller than what they report for both habit formation model and long-run risk model, but similar to prospect theory-based results. The difference between estimates of mode and mean of posterior values of γ in our alternative model and the LRR model in Aldrich and Gallant (2011) are still non-negligible.

Under the benchmark model, we obtain estimates for the mode and the mean of IES, ψ, that are larger than unity as is advocated by the long run risk literature. These estimated values are much larger than estimates reported by Aldrich and Gallant (2011), which are in the neighborhood of 1.50. However, our mode and mean estimates under both benchmark and alternative models are significantly more stable than values reported by Jeong et al. (2014). Their estimates, across five models and two assumptions for volatility dynamics (time-varying volatility and nonlinear stochastic volatility) range between 0.00 to ∞. Their benchmark MPU model ψ estimates, when all parameters are estimated from the data and wealth is only a function of returns to financial investments, are equal to 0.68 with time-varying volatility and 11.161 with nonlinear stochastic volatility. When wealth is assumed as the sum of financial and labor income, and hence some parameters are calibrated rather than estimated, their EIS estimates range between 1.17 to 15.13. As such, our estimates of posterior means that range between 3.70 and 3.85 are both more stable and more plausible, especially in comparison with EIS estimates in financial wealth-only, fully estimated cases in Jeong et al. (2014) that are directly comparable to our estimated models.

The EIS parameter ψ, as discussed in Liu and Miao (2014), is a crucial parameter for matching macroeconomic and financial moments. The risk aversion parameter and the EIS both determine the representative agent's preference for the timing of resolution of uncertainty. If $\gamma > 1/\psi$, the agent prefers an early resolution of uncertainty (see Epstein and Zin (1989) and Bansal and Yaron (2004)). By this measure, both benchmark and alternative models point to a representative agent who desires an early resolution of uncertainty. Based on our estimation results, and as expected due to the impact of ignoring ambiguity aversion on the estimated value of risk aversion parameter, this effect is stronger in the alternative model. Thus, based on the GSM estimation of the structural models with and without ambiguity aversion, we find strong support for a large literature on long run risks.[9]

We report the first direct estimates of the ambiguity aversion parameter η. The posterior mode and mean values of this parameter are 28.91 and 23.30 in our benchmark model. In comparison with calibration exercises in the literature, our estimates are larger than calibrated values in an endowment economy ($\eta = 8.86$ in Ju and Miao (2012)) and in a production economy ($\eta = 19$ in Jahan-Parvar and Liu (2014)). Obviously, our estimated results meet the criteria that $\eta \gg \gamma$.

Jeong et al. (2014) report an indirect measure for ambiguity aversion based on MPU equal to 0.34. Their estimation result is not directly comparable to ours, since it is based on a very different functional form and underlying assumptions. In fact, it is best described as a lower bound on the beliefs about the probability of the state of the economy in the MPU model, and not a measure for ambiguity aversion. Thus, as a probability measure, it is bound between 0 and 1. Similarly, Viale et al. (2014) characterize cross-sectional ambiguity as the likelihood ratio between MPU-distorted and reference models. Ilut and Schneider (2014) and Bianchi et al. (2014) present estimates of MPU ambiguity measures in DSGE settings. Ilut and Schneider (2014) posits an autoregressive and exogenous process for ambiguity, while Bianchi et al. (2014) – similar to Jeong et al. (2014) – find bounds for beliefs about uncertainty. As such, these findings are not directly comparable with ours.

Ahn et al. (2014) study smooth ambiguity aversion as part of their experimental research on ambiguity. They report values for the ambiguity aversion parameter based on the static formulation of smooth ambiguity aversion ranging between 0.00 and 2.00, with mean value of 0.207 for all subjects in their experiment population. They choose their population such that both "ambiguity neutral" and "ambiguity loving" subjects are represented. Thus, estimates for ambiguity aversion parameter for 5 to 50th percentiles of their population are zero. In addition, the value of ambiguity aversion parameters – regardless of whether smooth or kinked specifications are estimated – are at least an order of magnitude smaller than dynamic model-based estimates such as ours.[10] We believe

[9] In the empirical literature, some papers (e.g. Hall (1988) and Ludvigson (1999)) find that the EIS estimate is close to zero using aggregate consumption data. Other papers find higher values using cohort- or household-level data (e.g., Attanasio and Weber (1993) and Vissing-Jorgensen (2002)). Attanasio and Vissing-Jorgensen (2003) find that the EIS estimate for stockholders is typically above 1. Bansal and Yaron (2004) argue that estimates of the EIS based on aggregate data will be biased down if the usual assumption that consumption growth and asset returns are homoscedastic is relaxed.

[10] We find that the difference in the magnitude of these estimates is similar to the difference between static estimates of Gul (1991) disappointment aversion parameter reported by Choi, Fisman, Gale, and Kariv (2007) and dynamic estimates reported by Feunou, Jahan-Parvar, and Tdongap (2013). Thus, these differences are more likely to be an outcome of ignoring the dynamics in the data, rather than a result of using alternative estimation methods. For example, while we use GSM Bayesian methodology, Feunou et al. (2013) implement a frequentist maximum likelihood estimator. Yet, similar discrepancies between statically and dynamically estimated preference parameters

that ignoring intertemporal dimensions of choice under ambiguity explains these differences in the magnitude of estimated parameters. As mentioned earlier, while estimated ambiguity aversion parameters based on kinked specification are statistically different from zero in Ahn et al. (2014), the estimated smooth ambiguity parameters are not. In comparison with these findings, η in our benchmark model is tightly estimated – standard deviations of the posterior are less than 15 percent of the value of the posterior mean.

Our estimates of the transition probabilities (p_{11} and p_{22}), high and low mean consumption growth (κ_1 and κ_2), and the volatility of consumption growth ($\sigma_{\Delta c}$) are close to empirical results reported in other studies such as Cecchetti, Lam, and Mark (2000) for both benchmark and alternative models. The main difference lies with the benchmark model which yields lower estimated values for p_{22}, κ_1 and $\sigma_{\Delta c}$. The difference between Cecchetti et al. (2000) estimates of these parameters and ours are not negligible. In particular, the difference between Cecchetti et al. (2000) estimate of p_{22} and ours points to about 50% difference. However, this is not surprising. Cecchetti et al. (2000) estimates (reported in Table 2, page 790 in their paper) are based on a reduced-form fitting of the data. Ours are based on a structural model. Besides, we use different data and methodology. Our data set is both more recent (1929-2011 vs. 1871-1993) and shorter.

Our estimates for leverage ratio (λ) and the volatility of dividend process ($\sigma_{\Delta d}$) are not directly comparable to estimates reported in Aldrich and Gallant (2011), Gallant and McCulloch (2009), or Bansal et al. (2007) due different specifications for modeling dividend growth. Specifically, the LRR model features time variation in the volatility of fundamentals, while we rely on Markov-switching and distortions in state beliefs to deliver the time-variation in the volatility of returns.[11] However, our estimated $\sigma_{\Delta d}$ is close in magnitude to the volatility of dividend process in the prospect theory model estimated in Aldrich and Gallant (2011). In their formulation of a prospect theory-based asset pricing model, Aldrich and Gallant posit constant volatility for this process. Thus, magnitudes of estimated parameters are comparable, since estimation methodology is essentially the same in both studies.

are observed.

[11] Jahan-Parvar and Liu (2014) discuss this feature of asset pricing models based on smooth ambiguity aversion preferences in detail both theoretically and based on simulation exercises.

5 Model Comparison and Robustness

5.1 Relative Model Comparison

Relative model comparison is standard Bayesian inference. The posterior probabilities of the models with and without ambiguity aversion are computed using the Newton and Raftery (1994) $\hat{p}^4$ method for computing the marginal likelihood from an MCMC chain when assigning equal prior probability to each model. The advantage of that method is that knowledge of the normalizing constants of the likelihood $\mathcal{L}(\theta)$ and the prior $\pi(\theta)$ are not required. We do not know these normalizing constants due to the imposition of support conditions. It is important, however, that the auxiliary model be the same for both models. Otherwise the normalizing constant of $\mathcal{L}(\theta)$ would be required. One divides the marginal density for each model by the sum for both models to get the probabilities for relative model assessment.

The computed odds ratio is $1/5.85e{-}184$, which strongly favors the benchmark model over the alternative model. This ratio implies that our benchmark model provides a better description of the available data in the framework of the equilibrium model discussed in Section 3. Given the values of the log likelihoods for the benchmark and alternative models reported in Table 2 one hardly needs to bother with odds ratios. The verdict is obvious.

5.2 Forecasts

We can view a forecast as a functional $\Upsilon : f(\cdot|\cdot,\eta) \mapsto \upsilon$ of the auxiliary model that can be computed from $f(\cdot|\cdot,\eta)$ either analytically or by simulation. Due to the map $\eta = g(\theta)$, we view such a forecast as both a forecast from the structural model and as a function of θ. Viewing it as a function of θ, we can compute υ at each draw in the posterior MCMC chain for θ which results in an MCMC chain for υ. From the latter chain and the mean, mode, and standard deviation of υ can be computed. The same quantities can be computed for draws from the prior. Two example are Figures 4 and 5, which plot the mean prior and posterior forecasts of the benchmark model (left hand side) and the alternative model (no ambiguity aversion, right hand side). These forecasts are generated for two periods, ending in 2006 and 2011 respectively, to illustrate the impact of going into and emerging from a period of significant economic uncertainty and the role of ambiguity aversion in such times.

Prior forecasts appear in Figure 4. As expected, they do not differ much between pre- and post-Great Recession periods. There are, however, differences between prior forecasts based on the benchmark model and the alternative model. The main difference is the disparity in the level of benchmark and alternative model-based forecasts of the short rate. The benchmark model forecasts a higher level for the short rate (and wider posterior standard deviations) than the alternative model. These forecasts are counterintuitive, since we expect the agent to have a higher demand for a safe asset that pays the short rate, and hence a lower short rate. Once we take the forecast standard deviations into account, they appear less counterintuitive. The second difference is the slight increase in consumption growth path forecast by the benchmark model, against the drop in consumption growth path forecasts by the alternative model.

Prior forecasts are not a measure of a model's success in capturing the data dynamics. For that purpose, we rely on posterior forecasts, which we report in Figure 5[12]. As the Figure shows, consumption growth dynamics differ both between the benchmark and the alternative model, and across pre- and post-Great Recession forecasts. Both observations are in line with the theory and our expectations. Our discussion is based on comparing mean posterior forecasts. The posterior forecast paths generated by both modes are on average similar, but the benchmark model implies more variation in consumption growth in the future.

The benchmark model yields little variation in consumption growth forecasts. Both benchmark and alternative models forecast a slight drop in consumption growth for pre-recession period. Similarly, both models forecast a flat trajectory for consumption growth based on available information by the end of 2011.

In pre-recession period, the benchmark model forecasts a steeper drop in equity returns in comparison with the alternative model, roughly 10% against 5%, respectively. For the post-Great Recession period, the benchmark model yields stock return posterior forecasts that are 50% lower than the alternative model. Simply put, ambiguity aversion implies modest equity premium (around 4.5%) for the foreseeable future, while ambiguity neutral alternative model predicts a bull market – equity premium close to 10%. The difference, reflects the role of the third term in SDF equation (6) for the benchmark model, and its absence in the SDF for the alternative model, both between models and across forecast periods. This gap in forecasts between the two models is in line with earlier findings. For example, Aldrich and Gallant (2011) report forecasts of roughly 6% for equity returns for 2009-2013 period for the long-run risk model of Bansal and Yaron (2004), based on data ending in the Great Recession period, which may be viewed as high given the recent past

[12] The dramatic change in the standard errors of the forecasts between prior and posterior gives one an intuitive feel for the information contained in the data.

experience.

It is clear from this figure that the benchmark model predicts both a drop in the risk-free rate and an overall lower risk-free rate in comparison with the predictions of the alternative model, across pre- and post-Great Recession periods.[13] This empirical regularity echoes the findings of earlier theoretical research. Ambiguity aversion implies a more pessimistic attitude, and as result, a higher precautionary saving demand than the precautionary demand that Epstein and Zin utility induces in the alternative model. As a result, the prices of the risk-free bonds are higher, leading to lower yields. The difference between forecasts are not negligible: in both pre- and post-recession periods, it amounts to roughly 1% in real interest. Given recent announcements by various practitioners, academicians and former policy makers about likelihood of interest rates reverting back to "old normal" levels, our benchmark model forecasts seem reasonable.[14]

Given that Bayesian model comparison prefers the benchmark model over ambiguity neutral alternative model, these forecasts merit attention. The two models lead to very different dynamics for consumption growth and asset returns. If we indeed live in a world populated by ambiguity averse agents – implied by our results – then policy and decision makers need to be aware of the inherently different implications generated by these two class of preferences.

5.3 Asset Pricing Implications

In this section, we discuss the asset pricing implications of our estimated models. As mentioned earlier, our benchmark model is the exchange economy of Ju and Miao (2012). We are interested in comparing how closely asset pricing quantities generated by calibrating this model using estimated parameters, reported in Table 2, match sample moments reported in Table 1. Besides, we compare the results generated from the alternative model with ambiguity neutrality with those from the benchmark model. In addition, we compare the performance of this calibration with results reported in the original study by Ju and Miao. These comparisons show that our estimated parameters imply better performance in the asset pricing dimension.

We report model generated unconditional means and standard deviations of risk-free rate, $r_{f,t}$ and equity premium, $r_{e,t} - r_{f,t}$. In addition, we report Sharpe ratios and the ratio of volatility of

[13] While this observation is in line with the zero-lower bound environment since the Great Recession, they should not be viewed as synonymous. We are forecasting real risk-free rates. They are not influenced by fiscal or monetary policy, and are endogenously determined.

[14] For example, according to Reuters in May 16, 2014, former Federal Reserve Chairman Ben Bernanke opined that low interest rate environment is likely to continue beyond many current forecasts.

the SDF to expected value of the SDF, $\sigma(M_t)/\mathbb{E}(M_t)$, which is interpreted as the market price of risk. The results are reported in Table 3.

In comparison with sample data presented in Table 1, and with Ju and Miao (2012), we observe the following.[15] First, the benchmark model generates risk-free rate moments that are much closer to sample moments than the alternative model. While the volatility of risk-free rate is largely controlled by the magnitude of the IES parameter, introduction of ambiguity aversion seems to improve the model generated excess volatility in risk-free rates in the alternative model.

Second, while both benchmark and alternative models yield model generated volatilities for equity premium that are very close to sample volatility for this quantity, they differ dramatically in terms of their performance in matching the mean equity premium. In our sample, the mean equity premium value is 7.88%. At 9.44%, the benchmark model generates a larger mean equity premium. However, this value is much closer to the sample mean than 3.83% mean equity premium generated by the alternative model. As documented in Bansal and Yaron (2004), it is a well-known fact that without high risk aversion parameter values, Epstein and Zin (1989) recursive preferences have difficulty in matching the mean equity premium. In their study, Bansal and Yaron have to set γ equal to 10 – at the end of admissible range suggested by Mehra and Prescott (1985) – to match the mean equity premium. Since the estimated γ for the alternative model is smaller ($\mathbb{E}(\gamma) = 7.52$ and $\sigma(\gamma) = 0.56$), the alternative model falls into this well-documented trap.

The market price of risk, defined as $\sigma(M)/\mathbb{E}(M)$, is closely related to the moments of asset returns via the Hansen-Jagannathan bound. Ju and Miao (2012) find that the market price of risk is about 0.60 as implied by the calibration with ambiguity aversion. In a production setting, Jahan-Parvar and Liu (2014) find that the market price of risk is about 0.94 with ambiguity aversion. In the data, the Sharpe ratio is about 0.40–0.50. The alternative model implies that the market price of risk is 0.37, which is lower than the empirical Sharpe ratio and violates the Hansen-Jagannathan bound. On the other hand, the benchmark model generates a market price of risk of 2.33, which satisfies the Hansen-Jagannathan bound and also enables the model to match the key financial moments.

Finally, an important question is: does our structural estimation imply a reasonable magnitude of ambiguity aversion? To address this question, we use detection-error probabilities to assess the room allowed for ambiguity aversion based on our structural estimation results. This exercise is

[15] Ju and Miao (2012) use data from 1871-1993 sample period. Thus, our findings are not directly comparable with theirs.

meaningful because our estimation is grounded in the data and thus is more informative about the behavior of economic agents and the dynamics of economic variables. For comparison, we also calculate detection-error probabilities using the calibrated parameter values in Ju and Miao (2012).

Detection-error probabilities are an approach developed by Anderson et al. (2003) and Hansen and Sargent (2010) to assess the likelihood of making errors in selecting statistically "close" (in terms of relative entropy) data generating processes (DGP). In this paper, a DGP refers to a Markov switching model for consumption growth as specified in Equation (1). Without ambiguity aversion, transition probabilities are defined as in the transition matrix $\mathbb{P}$ in Section 3.1, and in this case we obtain the reference DGP. However, ambiguity aversion implies distortion to the transition probabilities in a pessimistic way and thus gives rise to the distorted DGP. The Appendix shows that the reference DGP and the distorted DGP differ only in terms of transition probabilities. We adapt to the current endowment economy the approach of computing detection-error probabilities in Jahan-Parvar and Liu (2014). This approach enables us to simulate artificial data from the reference and distorted DGPs and to evaluate the likelihood explicitly. Details of the algorithm of computing detection-error probabilities are included in the Appendix.

A sizable detection-error probability associated with a certain value of the ambiguity aversion parameter, η, implies that there is a large chance of making mistakes in distinguishing the reference DGP from the distorted DGP, and thus ample room is allowed for ambiguity aversion. With the estimated structural parameters in Table 2, the detection-error probability is 22.41%. This finding implies that the ambiguity-averse agent in our estimation faces significant difficulty is distinguishing between the reference and distorted DGPs.

6 Conclusion

Smooth ambiguity preferences of Klibanoff et al. (2005, 2009) have gained considerable popularity in recent years. In part, this popularity is due to clear separation between ambiguity – a characteristic of a representative agent's subjective beliefs – and ambiguity aversion that derives from the agent's tastes. In this paper, we estimate the endowment equilibrium asset pricing model with smooth ambiguity preferences proposed by Ju and Miao (2012) using U.S. data and GSM Bayesian estimation methodology of Gallant and McCulloch (2009) to a) investigate the empirical properties of such an asset pricing model as an adequate characterization of the returns and consumption growth data and, b) provide an empirical estimation of the ambiguity aversion parameter and its relationship with other structural parameters in the model. Our study contributes to the existing literature by providing a formal empirical investigation for adequacy of this class of preferences for economic modeling, and presenting estimations for the structural parameters of this model. The estimated structural parameters are in line with theoretical expectations, and are comparable with estimated parameters in studies using similar or related estimation methods. With respect to measurement of ambiguity aversion, our results show a marked improvement over the existing literature. The existing empirical literature either provides measures of ambiguity (which is usually the size of the set of priors in the MPU framework) – but not ambiguity aversion of the agent – or statistically implausible estimates for smooth ambiguity aversion parameters. Our study addresses both shortcoming in the extant literature.

We find that Bayesian model comparison strongly favors the benchmark model over the alternative model featuring Epstein and Zin recursive preferences. In addition, we find that the estimated ambiguity aversion parameter is higher than the values in calibration studies. We explore forecasting and asset pricing implications using our estimated model. We find marked differences in forecasts generated from the benchmark model and the alternative model. In short, the benchmark model generates more conservative forecasts for equity premium and real interest rates in comparison with the alternative model. We find that based on the estimated parameters, the equilibrium asset pricing model can successfully reproduce main stylized facts about asset returns. In addition, detection-error probabilities computed using the estimated parameters imply ample scope for ambiguity aversion.

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7 Appendix: Detection Error Probabilities

- In constructing distorted transition probabilities, we consider a "full information model", where the agent is ambiguity averse but state z_t is observable. In this case, the Euler equation is

$$0 = p_{11}\mathbb{E}_{1,t}\left[M_{z_{t+1},t+1}\left(R_{e,t+1} - R_{f,t}\right)\right] + \left(1 - p_{11}\right)\mathbb{E}_{2,t}\left[M_{z_{t+1},t+1}\left(R_{e,t+1} - R_{f,t}\right)\right]$$

for $z_t = 1$ and

$$0 = \left(1 - p_{22}\right)\mathbb{E}_{1,t}\left[M_{z_{t+1},t+1}\left(R_{e,t+1} - R_{f,t}\right)\right] + p_{22}\mathbb{E}_{2,t}\left[M_{z_{t+1},t+1}\left(R_{e,t+1} - R_{f,t}\right)\right]$$

for $z_t = 2$. The Euler equation can be rewritten as

$$\begin{aligned} 0 &= \tilde{p}_{11}\mathbb{E}_{1,t}\left[M^{EZ}_{z_{t+1},t+1}\left(R_{e,t+1} - R_{f,t}\right)\right] + \left(1 - p_{11}\right)\mathbb{E}_{2,t}\left[M^{EZ}_{z_{t+1},t+1}\left(R_{e,t+1} - R_{f,t}\right)\right] \\ 0 &= \left(1 - \tilde{p}_{22}\right)\mathbb{E}_{1,t}\left[M^{EZ}_{z_{t+1},t+1}\left(R_{e,t+1} - R_{f,t}\right)\right] + \tilde{p}_{22}\mathbb{E}_{2,t}\left[M^{EZ}_{z_{t+1},t+1}\left(R_{e,t+1} - R_{f,t}\right)\right] \end{aligned}$$

where $M^{EZ}_{z_{t+1},t+1}$ is the SDF under recursive utility without ambiguity aversion, and $\tilde{p}_{11}$ and $\tilde{p}_{22}$ are distorted transition probabilities and are given by

$$\tilde{p}_{11} = \frac{p_{11}}{p_{11} + \left(1 - p_{11}\right)\left(\dfrac{\mathbb{E}_2\left[V^{1-\gamma}_{z_{t+1},t+1}\right]}{\mathbb{E}_1\left[V^{1-\gamma}_{z_{t+1},t+1}\right]}\right)^{-\frac{\eta-\gamma}{1-\gamma}}}, \tag{8}$$

$$\tilde{p}_{22} = \frac{p_{22}}{\left(1 - p_{22}\right)\left(\dfrac{\mathbb{E}_1\left[V^{1-\gamma}_{z_{t+1},t+1}\right]}{\mathbb{E}_2\left[V^{1-\gamma}_{z_{t+1},t+1}\right]}\right)^{-\frac{\eta-\gamma}{1-\gamma}} + p_{22}}, \tag{9}$$

where $V_{z_t,t}$, $(z_t = 1, 2)$ are solutions to the following value function under full information:

$$\begin{aligned} V_{z_t,t}(C) &= \left[(1-\beta)C_t^{1-\frac{1}{\psi}} + \beta\left\{\mathcal{R}_{z_t}\left(V_{z_{t+1},t+1}\left(C\right)\right)\right\}^{1-\frac{1}{\psi}}\right]^{\frac{1}{1-\frac{1}{\psi}}}, \\ \mathcal{R}_{z_t}\left(V_{z_{t+1},t+1}\left(C\right)\right) &= \left(\mathbb{E}_{z_t}\left[\left(\mathbb{E}_{z_{t+1},t}\left[V^{1-\gamma}_{z_{t+1},t+1}\left(C\right)\right]\right)^{\frac{1-\eta}{1-\gamma}}\right]\right)^{\frac{1}{1-\eta}}. \end{aligned}$$

- The numerical algorithm of calculating detection-error probabilities takes the following steps:

  1. Repeatedly draw $\{\Delta c_t\}_{t=1}^{T}$ under the reference data generating process (DGP), which is the two-state Markov switching model with transition probabilities p_{11} and p_{22}.
  2. Evaluate the log likelihood function under the reference DGP by computing

$$\ln L^r_T = \sum_{t=1}^{T} \ln\left\{\sum_{z_t=1}^{2} f\left(\Delta c_t | z_t\right)\Pr\left(z_t | \Omega_{t-1}\right)\right\}$$

  where $\pi_{t-1} = \Pr\left(z_t = 1 | \Omega_{t-1}\right)$ are filtered probabilities implied by the Markov switching model.

3. Evaluate the log likelihood function under the distorted DGP by computing

$$\ln L_T^d = \sum_{t=1}^{T} \ln \left\{ \sum_{z_t=1}^{2} f\left(\Delta c_t | z_t\right) \Pr \widetilde{\left(z_t | \Omega_{t-1}\right)} \right\}$$

   where $\Pr \widetilde{\left(z_t|\Omega_{t-1}\right)}$ are the filtered probabilities that are obtained by applying the distorted transition probabilities $\tilde{p}_{11,t}$ and $\tilde{p}_{22,t}$ (in place of the constant transition probabilities p_{11} and p_{22}) to the Markov switching model's filter.

4. Compute the fraction of simulations for which $\ln\left(\frac{L_T^d}{L_T^r}\right) > 0$ and denote it as p_r. The fraction approximates the probability that the econometrician believes that the distorted DGP generated the data, while the data are actually generated by the reference DGP.

5. Do a symmetrical computation and simulate $\{\Delta c_t\}_{t=1}^{T}$ under the distorted DGP. Compute the fraction of simulations for which $\ln\left(\frac{L_T^r}{L_T^d}\right) > 0$ and denote it as p_d. This fraction approximates the probability that the reference DGP generated the data when actually the distorted DGP generates the data.

   Assuming an equal prior on the reference and the distorted DGP, the detection error probability is defined by (see Anderson et al. (2003)):

$$p\left(\eta\right) = \frac{1}{2}\left(p_r + p_d\right). \tag{10}$$

   In the approximation, we set $T = 100$ years and simulate 20,000 samples of artificial data.

Table 1: Summary Statistics of the Data

1929-2011	$r_{f,t}$	$r_{e,t}$	$r_{e,t} - r_{f,t}$	Δc_t
Mean	1.01	7.86	6.85	1.85
St Dev	0.04	19.87	19.88	2.18
Skewness	-0.10	-0.23	-0.23	-1.31
Kurtosis	3.85	-0.08	-0.08	5.68
J-B test	0.0016	0.5000	0.5000	0.0001

This table reports summary statistics for annual U.S. data for 1929-2011 period. 1-year Treasury Bill rate ($r_{f,t}$), aggregate equity returns ($r_{e,t}$), excess returns ($r_{e,t} - r_{f,t}$), and real, per capita, log consumption growth (Δc_t) are expressed in percentages. The row titled "$J - B$ test" reports the p-values of Jarque and Bera (1980) test of normality.

Table 2: **GSM Estimation Results for the Benchmark Model and the Ambiguity Neutral Model**

	Prior			Benchmark Model			Ambiguity Neutrality		
Parameter	Mode	Mean	St Dev	Mode	Mean	St Dev	Mode	Mean	St Dev
β	0.9766	0.6785	0.2074	0.9554	0.9653	0.0049	0.9746	0.9781	0.0021
γ	1.8750	1.9540	0.8625	0.3203	0.8258	0.5115	6.4023	7.5229	0.5587
ψ	1.3750	1.7700	0.5079	4.1797	3.8509	0.4930	4.5391	3.7072	0.4320
η	10.0000	9.3950	3.7052	28.9063	23.2978	3.0176	7.0469	8.5586	0.8371
$p_{1,1}$	0.9805	0.9684	0.0179	0.9405	0.9406	0.0009	0.9671	0.9542	0.0113
$p_{2,2}$	0.4688	0.4994	0.1348	0.2732	0.2636	0.0133	0.4213	0.4389	0.0096
κ_1	0.0217	0.0228	0.0037	0.0213	0.0210	0.0028	0.0194	0.0171	0.0013
κ_2	-0.0615	-0.0672	0.0195	-0.0603	-0.0615	0.0016	-0.0638	-0.0507	0.0087
λ	2.8750	2.3579	0.9878	1.5313	1.5698	0.6541	2.7656	3.4094	0.8523
$\sigma_{\Delta c}$	0.0308	0.0310	0.0028	0.0273	0.0269	0.0004	0.0281	0.0330	0.0029
$\sigma_{\Delta d}$	0.1133	0.1259	0.0274	0.1726	0.1698	0.0042	0.1713	0.1682	0.0055
Log Lik Stat Mod		N/A			-417.2963			-560.6611	
Log Lik Sci Mod		N/A			-433.2399			-562.0021	

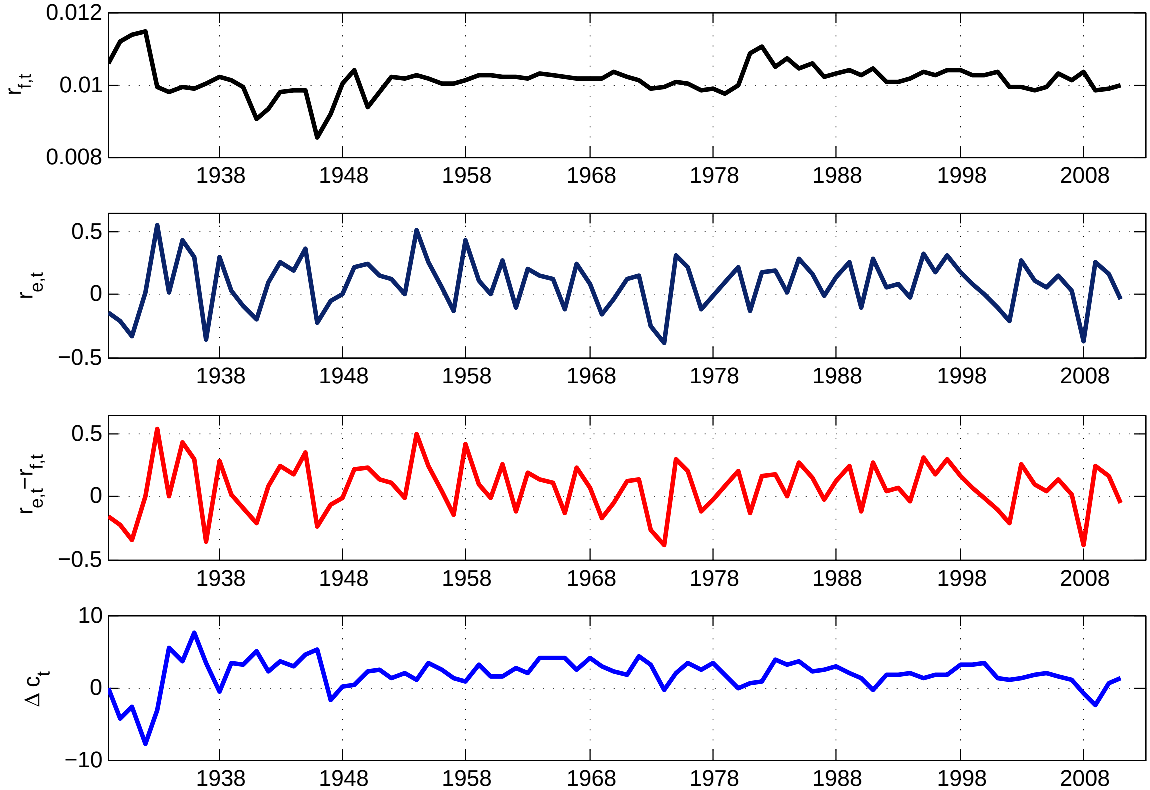
This table reports priors and posteriors on mode, mean, and standard deviation of preference, dividend growth, and consumption growth parameters for the benchmark model featuring ambiguity aversion and the alternative model with ambiguity neutral agents. The prior for the ambiguity neutral model is nearly identical to that for the benchmark model except that the lines for γ and η become nearly the same with mode, mean, and standard deviation 7.2, 5.4, and 3.6, respectively. Preference parameters (β, γ, ψ and η) represent subjective discount factor, coefficients of risk aversion, intertemporal elasticity of substitution, and ambiguity aversion respectively. $p_{1,1}$ and $p_{2,2}$ are transition probabilities from good-to-good and bad-to-bad states, respectively. κ_1 and κ_2 are good and bad state mean consumption growth parameters, respectively. λ is the leverage ratio in the model, and $\sigma_{\Delta c}$ and $\sigma_{\Delta d}$ are volatility parameters for consumption and dividend growth, respectively. We use 1929-2011 annual real-valued data. 1929-1949 data are used for priming the estimation process, and 1950-2011 data yield the estimated parameters. Log likelihood and log posterior values (when relevant) are also reported.

Table 3: Asset Pricing Implications

	Benchmark Model	Alternative Model	Ju and Miao
$\mathbb{E}(r_{f,t})$	0.98	2.14	2.66
$\sigma(r_{f,t})$	0.12	2.34	1.16
$\mathbb{E}(r_{e,t} - r_{f,t})$	9.44	3.83	5.75
$\sigma(r_{e,t} - r_{f,t})$	19.75	19.86	18.26
Sharpe Ratio	0.48	0.19	0.31
$\sigma(M_t)/\mathbb{E}(M_t)$	2.33	0.37	0.60

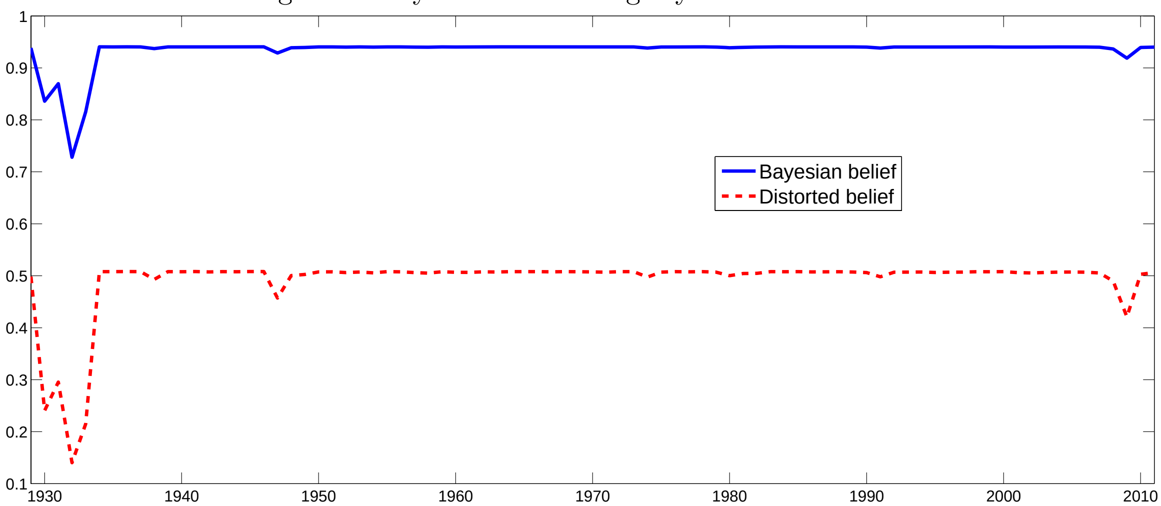
We present the asset pricing quantities implied by calibrating the model in Ju and Miao (2012) and the alternative model featuring ambiguity neutrality using the estimated parameters presented on Table 2. In addition, we present the original quantities reported by Ju and Miao (2012) for 1871–1993 sampling period. Unconditional means and standard deviations, $\mathbb{E}(x_t)$ and $\sigma(x_t)$, are in percents.

Figure 1: Risk Free Rates, Aggregate Equity Returns, Excess Returns, and Consumption Growth



The figure shows, from top to bottom, 1-year Treasury Bill rates, annual returns of CRSP-Compustat value weighted index returns, excess returns over 1-year T-Bill rate, and annual real per capita consumption growth for 1929-2011 period.

Figure 2: Bayesian and Ambiguity-Distorted Beliefs

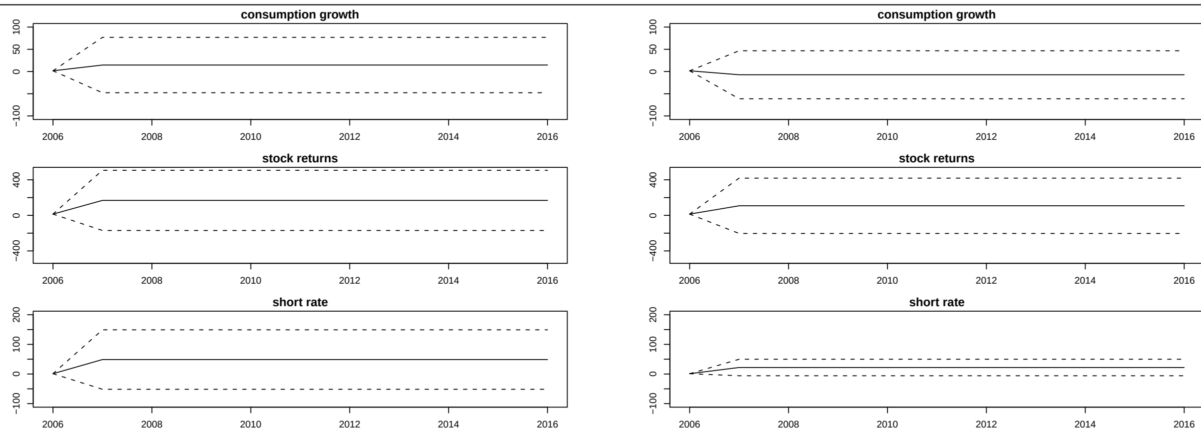


The figure plots Bayesian belief, defined in Equation (3), and ambiguity-distorted belief, defined in Equation (7), based on 1929-2011 historical consumption growth data.

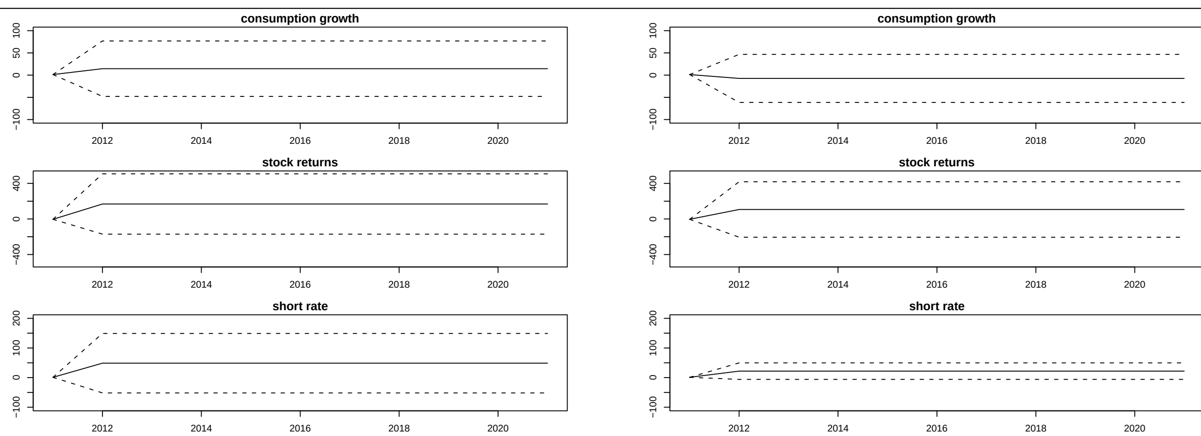
Table 4: Prior Forecasts

Benchmark Model | Alternative Model

Pre-Great Recession Forecasts



Post-Great Recession Forecasts

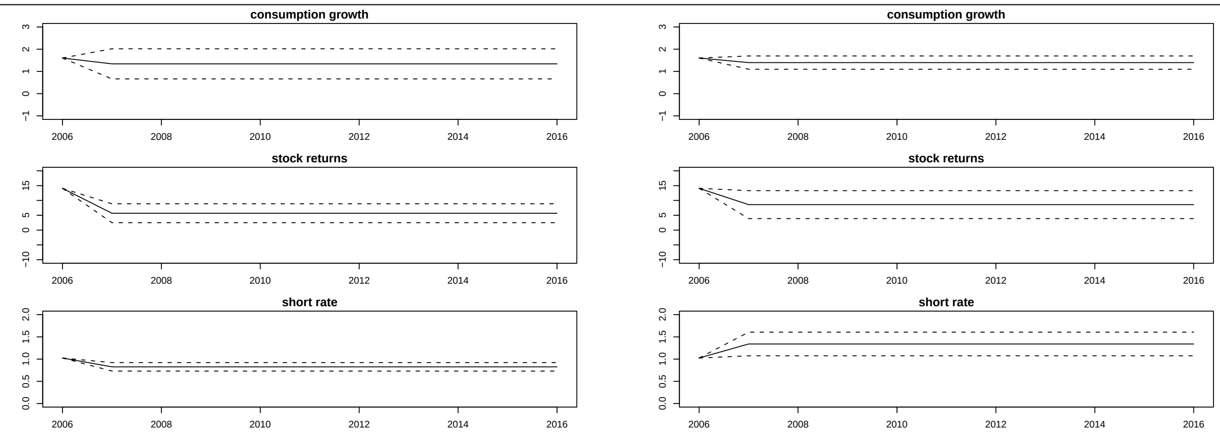


The figures show, from top to bottom, prior forecasts for consumption growth, equity returns and the short rate for both our benchmark model featuring ambiguity aversion and the alternative model with ambiguity neutral agents. The left hand panel contains forecasts for the benchmark model, and the right hand panel does the same for the alternative model. The dashed lines are the ± 1.96 posterior standard deviations.

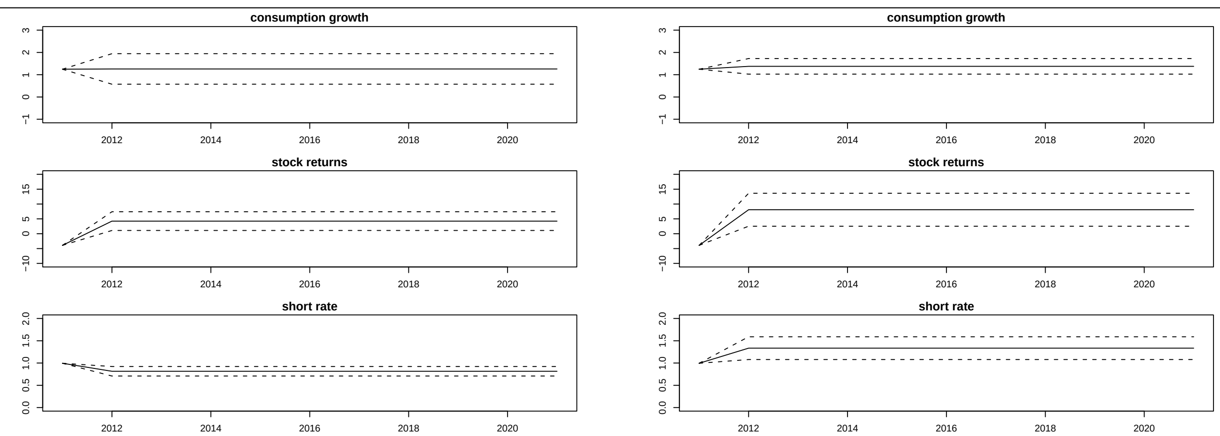
Table 5: Posterior Forecasts

Benchmark Model	Alternative Model

Pre-Great Recession Forecasts



Post-Great Recession Forecasts



The figures show, from top to bottom, prior forecasts generated for (rows 1 to 3) and posterior forecasts (rows 4 to 6) for consumption growth, equity returns and the short rate for both our benchmark model featuring ambiguity aversion and the alternative model with ambiguity neutral agents. The left hand panel contains forecasts for the benchmark model, and the right hand panel does the same for the alternative model. The dashed lines are the ± 1.96 posterior standard deviations.