Liquidity Backstops and Dynamic Debt Runs*

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Abstract

Liquidity backstops have important implications for financial stability. The different experiences of the municipal bond markets for variable rate demand obligations (VRDOs) and auction rate securities (ARS) during the recent financial crisis of 2007-09 provide a natural experiment to study the role of a liquidity backstop in mitigating runs: the liquidity-backstop-lacking ARS market collapsed subsequently, while the liquidity-backstop-possessing VRDO market survived. In this paper we develop a tractable dynamic model of debt runs in these markets and show that the lack of a liquidity backstop makes the ARS market more susceptible to runs than the VRDO market. Intuitively, absent a liquidity backstop, ARS creditors face a liquidity risk of being unable to liquidate their holdings if auctions fail en masse in the future. We conceptualize and evaluate the value of a liquidity backstop. The calibration results shed light on one central difference between shadow banks and traditional banks that have differential access to public liquidity backstops.

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1 Introduction

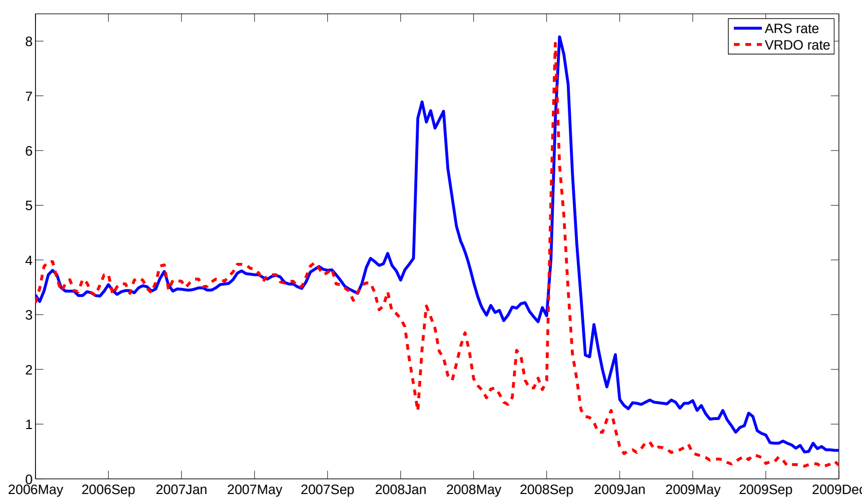
Liquidity backstops have important implications for financial stability, as demonstrated by the recent financial crisis of 2007-09. On the one hand, concerns about the provision of private liquidity backstops by banks (e.g., committed lines of credit) contributed to runs on asset-backed commercial paper programs in the summer 2007 (Covitz, Liang, and Suarez (2013)). These runs together with those in several other markets (e.g., markets for repo, money market funds) tightened credit going to firms and households and inflicted widespread damage to the US and global economy. On the other hand, during the financial crisis banks had relied on public liquidity backstops by the government and agencies (e.g., liquidity facilities) to be able to continue to honor their liquidity commitments: "the role of banks as liquidity providers was itself in crisis" (Acharya and Mora (2014)). In this paper we study the important role of liquidity backstops in mitigating runs (or, conversely, the role of the lack of liquidity backstops in exacerbating runs): How does a liquidity backstop work to mitigate runs? How to define and evaluate the value of a liquidity backstop in mitigating runs?

Addressing these questions helps us better understand the fragility in the shadow banking system. Runs on the traditional banking system were ended due to the existence of public liquidity backstops (e.g., federal deposit insurance and the central bank's lender-of-last-resort capacity). However, the shadow banking system lacks such liquidity backstops; hence, the money-like securities it created are runnable (Moreira and Savov (2013)). In fact, the recent financial crisis can be considered as modern bank runs on the shadow banking system (Gorton and Metrick (2010, 2012)). In this paper we provide the micro-foundation of runs triggered by the lack of a liquidity backstop that emphasizes the dynamic feature of runs on shadow banks. The value of a liquidity backstop we conceptualize and estimate in this paper points directly to how vulnerable the shadow banking system is compared to the traditional banking system.

To address these questions, in this paper we develop a dynamic model of debt runs in the municipal bond markets for variable rate demand obligations (VRDOs) and auction rate securities (ARS). As we will describe shortly, both markets experienced turmoil when the liquidity backup mechanism of the banking system broke down in the crisis of 2007-08. However, their experiences during the crisis are different due to the different extent of liquidity support they received from the banking system. Therefore, these markets provide an ideal laboratory to study and identify the value of a liquidity backstop in mitigating runs.

Figure 1: Historical Average Interest Rates

Figure 1 plots the average interest rates on the indexes of weekly resettable high-grade ARS (solid line) and VRDO (dashed line) between May 2006 and December 2009, maintained by the Securities Industry and Financial Markets Association (SIFMA).



VRDOs and ARS are money-like municipal bonds with floating interest rates that are reset on a periodic basis (typically weekly). VRDOs are structured with a liquidity backstop facility committed by a liquidity provider (usually a large bank) who acts as a "buyer of last resort" by buying the securities. By contrast, there are no explicit liquidity backstops in the ARS market: in the event when there are insufficient bids at an auction, the auction will fail and selling creditors will be stuck with their holdings until the next successful auction.[1]

The distinction between ARS and VRDOs had been blurred in the period prior to 2007 when auction failures were very rare and ARS were often marketed by broker-dealers as "cash equivalents.": investors had a wrong perception that auction agents would always step in to help prevent ARS auctions from failing. It is apparent from Figure 1 that before the end of 2007, the average ARS and VRDO interest rates were very close, suggesting that market participants considered the possibility of auction failures remote, and viewed both ARS and VRDO as almost identical securities. However, such a seemingly "explicit" liquidity provision viewed by ARS creditors is illusory. At the onset of the crisis, banks exposed to commitment drawdowns were particularly hit and were forced to cut back on new lending including uncommitted lending in the ARS market. Consequently, several major banks who are also auction agents in the ARS market decided not to intervene and instead let the auctions fail in early 2008 — the ARS market experienced a wave of auction failures in mid-February 2008 when "about two-thirds of auctions have failed per day" at the peak.[2] The sheer volume of failed auctions and fear of future auction failures propelled more investors to run on ARS. As shown in Figure 1, the ARS rate spiked as high as 6.6% in the second half of February and March in 2008, while, at the same time, the explicit liquidity provision in the VRDO market helped to stabilize its interest rate around 2%.

Later, in September 2008, following the bankruptcy of Lehman Brothers, the

[1] See Section 2 for more detail about the markets for VRDOs and ARS.

[2] See "Florida Schools, California Convert Auction-Rate Debt," Bloomberg L.P., February 22, 2008.

strength of explicit liquidity backstops that helped stabilize the VRDO market in February was in doubt as well. Investors worried whether banking institutions with explicit liquidity facility commitment would be able to meet their obligations — as a result, the average VRDO and ARS rates spiked around 8% on September 24, 2008, as shown in Figure 1. The runs on ARS and VRDO in 2008 allow us to distinguish the different effects of explicit and implicit liquidity provisions on the running decision of creditors, and make it possible to evaluate the commitment value of providing an explicit liquidity backstop.

In this paper, we develop a continuous time model of dynamic debt runs in the markets for ARS and VRDO based on He and Xiong (2012, HX hereafter). Our model captures several key characteristics of ARS and VRDO: the floating interest rate, the pre-specified interest rate cap or maximum interest rate, and more importantly, the underlying (explicit or implicit) liquidity provision. The model is very tractable. The decision of investors to run is solved in closed form in the unique threshold equilibrium where creditors decide to run if the fundamental falls below a certain endogenous threshold. The equilibrium threshold is determined by taking into account the difference between explicit and implicit liquidity provision and the time-varying interest rate.

We obtain three main results. First, we show that the lack of a liquidity backstop can exacerbate runs. The intuition can be understood as follows in the context of the ARS market. Absent a liquidity backstop, an ARS creditor faces not only credit risk that the project would fail, but also liquidity risk that auctions would fail, if future creditors choose not to roll over their debt. The former credit risk leads to possible credit loss in the event of default, while the latter liquidity risk results in losses due to illiquidity in the event of auction failures where creditor are stuck with their holdings. Because of these risks, the running decision of future creditors imposes a negative externality on the current creditor, making him more likely to run, ex ante. The existence of liquidity backstops, for instance in the VRDO market, largely mitigates the liquidity risk. As a result, the VRDO market is less susceptible to runs than the ARS market, which is consistent with the experiences of these markets during the crisis.

Second, our model shows that introducing a floating interest rate that is inversely related to the issuer's fundamental generally makes runs occur less frequently. Intuitively, when the issuer's fundamental deteriorates, the interest rate increases to compensate investors for the higher default risk, and thus makes them more willing to roll over. Consistent with this intuition, we analytically prove that, all else equal, the likelihood of runs decreases with the maximum interest rate, which is consistent with the empirical findings in McConnell and Saretto (2010).

Lastly, we conceptualize and evaluate the value of a liquidity backstop based on a calibration of the model. As shown in Figure 1, following the eruption of auction failures, the ARS rate started to diverge from the VRDO rate since November 2007 when ARS creditors took into account the possibility of auction failures they previously ignored. This structural shift helps us to identify the value of a liquidity backstop in a spirit similar to the "differences-in-differences" approach. In the pre-crisis period prior to November 2007, ARS were considered as almost identical to VRDOs and as a result creditors in both markets make the same running decision, characterized by a common rollover threshold. However, as the crisis broke out and ARS creditors started to recognize the lack of a liquidity backstop in the ARS market, they thus face a higher rollover threshold than that in the VRDO market. Put differently, the lack of a liquidity backstop prompts ARS creditors to run more often. This is consistent with our main model prediction that the lack of a liquidity backstops makes creditors more likely to run, i.e., the run threshold is higher.

The value of a liquidity backstop can now be conceptualized in the following thought experiment. To drive down the run threshold in the ARS market to that in the VRDO market, the ARS rate need to be increased by a certain amount at each point of time, or alternatively, the ARS issuer can pay a fee to acquire the same liquidity backstop facility as in the VRDO market. A risk neutral ARS issuer would be indifferent between these two methods, as long as the fee for acquiring the liquidity backstop is the same as the constant increment in the ARS rate. Therefore, the value of a liquidity backstop equals the increment in the ARS rate that equalizes the rollover thresholds in both markets. Based on the calibrated paramteers, we show that a liquidity backstop is valued at about 40-60 basis points per annum. Our study complements Veronesi and Zingales (2010) that empirically estimates the cost (and benefit) of government intervention during the financial crisis.[3]

Our paper contributes to the theoretical debt-run literature that examines the determinants of runs.[4] Our model is built upon He and Xiong (2012), which extends the literature on static bank-run models (Diamond and Dybvig (1983); Rochet and Vives (2004); Goldstein and Pauzner (2005), etc.). The He-Xiong model highlights the dynamic coordination problem and one main finding is that fear of future rollover risk could motivate each creditor to run ahead of others. The extension in our paper has two key departures: one is the introduction of floating interest rate, and more importantly, the other is the possible failure of an implicit (i.e., uncommitted) liquidity provision. We show that absent a liquidity backstop, a run in the future generates a liquidity risk for current creditors and thus, in anticipation of the future liquidity risk, current creditors would tend to run earlier. Schroth, Suarez, and Taylor (2014), a closely related paper, extends and applies the He-Xiong model to the ABCP market.

[3]Note that we only consider liquidity backstops provided by banks in the private sector and thus our estimate applies to the value of a *private* liquidity backstop and can be considered as a lower bound for the value of a *public* liquidity backstop if the latter is perceived to be more robust.

[4]For empirical studies, please see Carey, Correa, and Kotter (2009) and Covitz, Liang, and Suarez (2012) for the run on ABCP, Gorton and Metrick (2012) for the run on repo, McCabe (2010), Kacperczyk and Schnabl (2012), Wermers (2012) for the run on money market mutual funds, and Shin (2009) on the run on Northern Rock, and Han and Li (2009) and McConnell and Saretto (2010) for the run on ARS.

Complementary to theirs, our paper has a different focus on the effect of liquidity backstops on the likelihood of runs.

This paper is also related to the literature on the role of banks as liquidity providers. Kashyap, Rajan, and Stein (2002) provides a convincing argument that banks have a natural advantage of acting as liquidity providers to provide liquidity on demand. The advantage stems from a synergy between deposit-taking and loan commitments to the extent that both types of activities require banks to hold large balances of liquid assets. The synergy exists as long as both activities are not too highly correlated, which holds up very well during normal times or several recent episodes of market stress.[5] However, as Acharya and Mora (2014) argues, during the banking crisis of 2007-08, the role of banks as liquidity providers was itself in crisis as both sides of their balance sheet were hit. In this paper, we argue that although they honored contractual obligations in the VRDO market, banks as liquidity providers cut back on uncommitted lending and failed to provide *implicit* liquidity support in the ARS market, resulting in the wave of auction failures and runs on ARS. By contrasting the run episodes in the VRDO and ARS markets, our paper is able to shed new light on how valuable the role of banks in providing backup liquidity is.

The remainder of this paper is structured as follows. In Section 2, we provide an overview of the VRDO and ARS markets and the turmoil in these markets during the financial crisis. Section 3 presents the model. Section 4 characterizes the equilibrium and discusses key model implications, including the externalities imposed on future creditors by the running decisions of current creditors. In Sections 5 and 6, we discuss calibration procedure and results. Section 7 concludes. Most proofs are in the appendix at the end of this paper. A companion internet appendix provides omitted proofs and additional derivations.

[5] See Gatev and Strahan (2006) and Gatev, Schuermann, and Strahan (2009) for evidence of a negative correlation between deposit withdrawals and commitment draw-downs in the commercial paper market.

2 Overview of the Markets for VRDOs and ARS

In this section, we first provide a description of VRDOs and ARS, and then an overview of these markets, and a narrative of the disruptions in these markets in 2008 during the recent financial crisis.

2.1 Background

In this subsection we provide some background information on VRDOs and ARS.

Auction Rate Securities. ARS are long-term municipal bonds with interest rates that are periodically reset through a Dutch auction process at short-term intervals, usually 7, 28 or 35 days. Following a successful auction, buyers purchase the bonds at par and receive the market clearing interest rate until the next interest reset date. ARS have nominally long-term maturities that usually range from 20 to 30 years. Nonetheless, the interest rate reset mechanism provides creditors with frequent opportunities to sell their holdings through auctions, and thus makes ARS priced and traded as short-term instruments.

At each auction, the auction agent accepts bids from market participants. Existing bond holders can submit one of three types of orders: a "hold at market" order if they wish to maintain their positions regardless of the market-clearing rate; a "sell at market" (market sell) order if they wish to sell regardless of the market-clearing rate; a "hold at rate" (limit sell) order if they commit to sell their positions under the condition that the market-clearing rate is equal to or lower than the specified rate. Potential buyers can submit a limit buyer order to buy the bond if the bid is less than or equal to the market-clearing rate. The auction agent then receives all the bids and can submit his/her own order.

The market-clearing interest rate is bounded from above by a pre-specified maximum interest rate, often shortened to "max rate" in Wall Street parlance. Throughout this paper we use the terms "maximum interest rate" and "max rate" interchangeably. The max rates are either fixed, or floating and usually tied to a reference rate (e.g., LIBOR). Fixed max rates are specified for all ARS, in a wide range of 9% to 25%. For ARS that also have floating max rates, the binding max rate is the minimum of the two.[6]

An auction fails when there are not sufficient bids to clear the market at a rate less than the max rate. In the case of auction failure, the max rate is imposed, however, importantly, creditors are stuck with the bonds until the next successful auction. Until the ARS market froze in mid-Februrary 2008, auction failures were extremely rare — there were only 13 failed auctions between 1984 and 2006.[7] However, as described shortly in the next subsection, after the financial crisis broke out, a tidal wave of auction failures hit the market.

Variable Rate Demand Obligations. VRDOs are very similar to ARS; they are also long-term floating-rate bonds with periodic interest rate resets. Unlike ARS, interest rates of VRDOs are reset periodically through "remarketing agents" so that the securities can be sold at par.

The key distinguishing characteristic of VRDOs is the existence of an explicit liquidity facility/backstop. VRDO creditors have a "tender" or "put" option which allows them to put the bonds at par value (plus any accrued interest) to the remarketing agent who then try to resell (remarket) the tendered bonds to new investors. To make the tender option feasible, VRDOs are usually structured with a liquidity facility provided by a third-party "liquidity provider." The liquidity provider, usually a large bank, acts as a buyer of last resort; it provides liquidity support by buying the bonds if the remarketing agent is unable to remarket them. In this case, the bonds become the so-called "bank bonds" showing up on the liquidity provider's balance

[6] Please see McConnell and Saretto (2010) for some examples of how floating max rates are set.

[7] "Prolonged disruption of the auction rate market could have negative impact on some ratings," Special Report, Moody's Investors Service, February 20, 2008.

sheet.

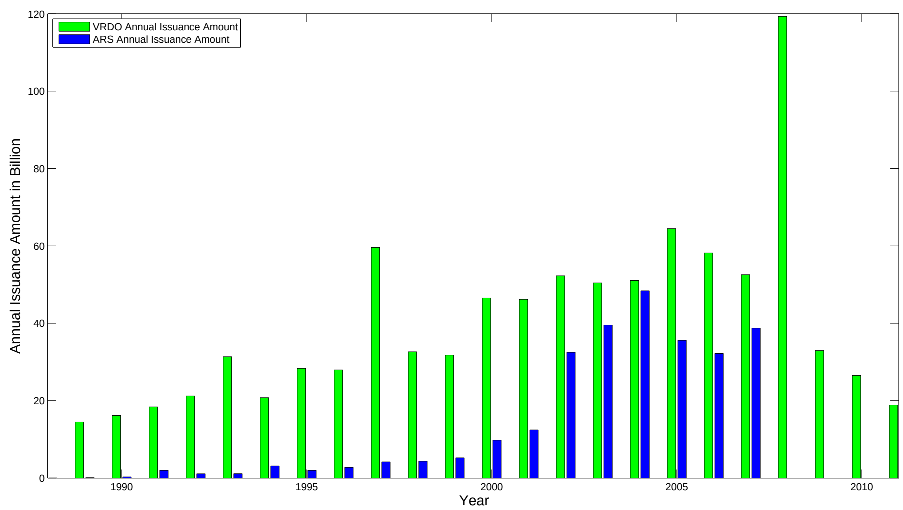
The liquidity facility is in the form of a *direct Letter of Credit* (LOC) under which the liquidity provider acts as the first source of payment of principal and interest, or a *standby* LOC under which the issuer is the first source of liquidity and the liquidity provider acts as a back-up, or a *Standby Bond Purchase Agreement* (SBPA) under which the VRDO instrument is insured by an investment-grade insurer and in the case of unsuccessful remarketing, the liquidity provider is obligated to buy the tendered bonds as long as the insurer maintains its investment grade rating. Regardless of which structure is used, the liquidity provider is the ultimate source of liquidity. As a result, the VRDO instrument carries short-term rating of its liquidity provider.

VRDOs are also typically sold with credit enhancement, which takes the form of a municipal bond insurance policy provided by some monoline bond insurers. The credit enhancement protects creditors and the liquidity provider from long-term credit risk. Therefore, the VRDO instrument carries long-term rating of its insurer, typically triple A.

2.2 The VRDO and ARS Markets and the Crisis in 2008

The VRDO and ARS markets are significant components of the $3.7 trillion municipal bond market, with sizes of about $200 billion and $500 billion in 2008 at their peak time, respectively. The markets were an attractive financing venue for municipal issuers because they allow for the issuance of long-term obligations using short-term interest rates that are typically lower than long-term interest rates. For investors, these securities were also attractive because they offered better returns than traditional money market investments. Both markets have existed since 1980s and had functioned well until the financial crisis broke out in 2007. In the aftermath of the financial crisis, the ARS market collapsed afterwards and there have been no new ARS issuance since 2008. Meanwhile, new issuance of VRDOs surged in 2008 as many existing ARS were converted into VRDOs. Figure 2 below plots the annual amount of issuance in both markets since 1988, calculated using SDC platinum.

Figure 2: VRDO and ARS Annual Issuance Amounts (in Billion)



The ARS market encountered significant problems in early 2008. Since mid-2007, the disruption in the subprime mortgage market spread to the monoline insurance market where several major municipal bond insurers (e.g., Ambac and MBIA) were downgraded because of their exposure to subprime mortgage debt. These downgrades resulted in increased selling pressure in ARS. On the other hand, the subprime mortgage meltdown also significantly strained balance sheets of auction agents (e.g., Citibank, Goldman Sachs, Lehman Brothers, UBS, Royal Bank of Canada and JP Morgan) to the extent that they decided not to intervene and let the auctions fail in mid-February 2008. Reportedly, about 60% to 80% of auctions failed in the second half of February in 2008.[8] The wave of auction failures drove up the ARS rate to as high as 6.6% around mid-February 2008 as shown in Figure 1. The sheer volume of

[8] See, "Breakdown of auction rate securities markets," congressional testimony by Leslie Norwood, September 18, 2008.

failed auctions and fear of future auction failures propelled more investors to run on ARS.

The run on ARS highlighted the implicitness of the liquidity provision in the ARS market: although in less tumultuous times prior to 2007, auction agents had almost always stepped in to buy some of these securities to help keep the market functioning, they have no contractual obligations to do so. During the financial crisis, major auction agents indeed chose no longer to be "buyers of last resort." By contrast, the VRDO market was not as much affected in early 2008 due to the explicit structure of liquidity facility.

However, later in 2008 the VRDO (as well as ARS) market experienced a run as a result of the bankruptcy of Lehman Brothers declared on September 15, 2008 and the subsequent panic in the market of money market mutual funds (e.g., runs on the Reserve Primary Fund that "broke the buck", and other money market mutual funds). Investors worried about whether banking institutions that explicitly provided liquidity facility would be able to meet their obligations. The run on VRDO is evident in the spike of 7.96% of the average VRDO rate on September 24, 2008, as shown in Figure 1.

The runs on ARS and VRDO in 2008 allow us to distinguish the differential effects of explicit and implicit liquidity provisions on the running decision of investors. In particular, we build a dynamic-debt-run model of the VRDO and ARS markets to illustrate why the ARS market became more susceptible to runs in early 2008 once investors started to recognize the implicitness of the liquidity provision. Furthermore, we also structurally estimate the model to assess the value of providing an explicit liquidity facility as in the case of VRDOs.

3 The Model

We develop a model of dynamic debt runs for the markets for VRDOs and ARS, based on He and Xiong (2012). The model contains several common features shared by VRDOs and ARS: a floating short-term interest rate and a pre-specified maximum interest rate. Moreover, the model captures a unique feature of VRDOs that ARS do not have: a liquidity backstop (facility), structured as imperfect credit lines, to support the tender option of VRDO investors. We use "floating-rate (municipal) bonds" or simply "bonds" to refer to both VRDOs and ARS when describing the setting that applies to both.

3.1 Asset

At time 0, a government-related entity (referred to as a municipality, hereafter) issues floating-rate municipal bonds (i.e., VRDOs or ARS) to borrow \$1 to finance a long-horizon project that generates cash flow at a constant rate r. At a random arrival time τ_ϕ according to a Poisson process with intensity $\phi > 0$, the project is terminated with a final payoff. The final payoff is the realization of a geometric Brownian motion process y_t at time τ_ϕ,

$$dy_t = y_t \left(\mu dt + \sigma dZ_t\right), \tag{1}$$

where $\{Z_t\}$ is a standard Brownian motion. The project's fundamental value under a discount rate ρ is determined as follows:

$$F\left(y_t\right) = E_t\left[\int_t^{\tau_\phi} e^{-\rho(s-t)} r ds + e^{-\rho\left(\tau_\phi - t\right)} y_{\tau_\phi}\right] = \frac{r}{\rho+\phi} + \frac{\phi}{\rho+\phi-\mu} y_t. \tag{2}$$

Due to tax exemption, the discount rate ρ equals the after-tax risk-free rate, that is,

$$\rho = r_f\left(1-\tau\right), \tag{3}$$

where r_f denotes the taxable Treasury yield and τ the marginal tax rate. The discount rate ρ is identical for all recditors.

3.2 VRDO/ARS Financing

The long-term project is financed by the issuance of VRDOs or ARS whose maturity coincides with the termination of the project. That is, the issued VRDOs or ARS have a long-term nominal maturity, equal to $1/\phi$ or the expected time until the project is terminated. Despite the long-term nominal maturity, VRDOs and ARS have been considered as short-term securities in practice, because of periodical (typically, weekly) remarketing or auctions through which creditors can sell their holdings (see Section 2). To capture the short-term nature of VRDO/ARS financing in a tractable way, we assume that there is a continuum of risk neutral creditors with measure 1, and each creditor decides to sell his bond holdings at a random time τ_δ which arrives following a Poisson process with intensity $\delta >> \phi > 0$. This assumption shares the spirit of the Calvo (1983) staggered-pricing model: at each time interval $[t, t + dt]$, a fixed fraction δdt of creditors arrive to make their rollover decision. For example, creditors may experience idiosyncratic private liquidity shocks such that their rollover decision making is uniformly spread out across time.

A coordination problem between current and future creditors arises in the model. This is because current creditors face a so-called rollover risk that they may suffer losses if future creditors choose not to roll over their debt. As a result, each creditor's rollover decision depends on the action of future creditors he anticipates. This dynamic nature of creditors' rollover decisions makes this model distinct from the static bank-run models (Diamond and Dybvig (1983)).

Another prominent feature associated with financing via VRDO or ARS is the floating interest rate. In the next section, we will discuss in detail how the interest rate is determined.

3.3 Runs, Liquidity Backstops, and Auction Failures

A run occurs if creditors decide not to roll over their debt. In the model, creditors may refuse to roll over their debt for fear of future distressed liquidations, but also for fear of auction failures in the ARS market that lacks a liquidity backstop, which are both triggered by a run by future creditors. The latter fear of auction failures due to the lack of a liquidity backstop is an innovative feature of our model. As we will show shortly, because one central distinction between VRDOs and ARS is whether or not a liquidity backstop exists, a run induces very different dynamics in these markets.

VRDOs are structured with an explicit liquidity backstop (i.e., a liquidity facility in the form of a letter of credit or stand-by purchase agreement) committed by a liquidity provider. Upon a run when the remarketing agent cannot find enough buyers, the liquidity provider is contractually obligated to provide liquidity and buy the bonds. However, the liquidity facility may not be perfectly reliable, even though it is explicitly committed. For example, the liquidity provider may becomes so severely financial distressed (e.g., Lehman Brothers) that it may fail to honor its liquidity commitment. To model the extent of unreliability of the liquidity facility, we assume that with probability $\theta\delta dt$, the committed liquidity support may fail, and, once it fails, the asset will be forced into premature liquidation, sold at a fraction α of its fundamental value (e.g., fire sale). That is, the liquidation value is

$$L(y_t) = \alpha F(y_t) = \frac{\alpha r}{\rho + \phi} + \frac{\alpha \phi}{\rho + \phi - \mu} y_t \equiv L + l y_t. \tag{4}$$

If the liquidation value is not enough to pay off all the creditors, a bankruptcy occurs. Therefore, a run in the future will expose creditors to possible bankruptcy loesses. In anticipation of the bankruptcy losses, creditors may refuse to roll over the debt earlier on.

By contrast, ARS are not structured with a liquidity backstop. As a result, ARS creditors face an additional risk of auction failures. Without a liquidity commitment, the auction agent can choose whether or not to participate in an auction. Upon a run when there are insufficient buyers, the auction agent has no contractual obligation to act as the residual bidder and the auction would fail if the agent decides not to step in. To capture this layer of uncertainty due to the lack of a liquidity backstop in the ARS market, we assume that upon a run, with probability $\kappa\delta dt$, the auction agent will not step in to intervene the market and the auctions will fail; with probability $1 - \kappa\delta dt$, the auction agent will intervene to keep the auctions functioning. For tractability, we further assume that once an auction fails, all the following auctions continue to fail. In the event of successful auctions, the market-clearing interest rate r_t prevails and premature liquidation occurs with probability $\theta\delta dt$. In the autarkic event of failed auctions, the max rate $\overline{r}$ is imposed and premature liquidation occurs with probability $\overline{\theta}\delta dt$.

3.4 Timeline

Figures 3A and 3B summarize the sequence of events in the model of VRDOs and ARS, respectively. All participants observe the fundamental y_t and the max rate $\overline{r}$. At the beginning, the (remarketing or auction) agent announces and commits to an interest rate formula $R(y_t; y_*)$ which may depend on certain (endogenously) determined parameter y_* which we will discuss shortly in the following section.

In the case of VRDOs (shown in Figure 3A), at each time t, a fraction δdt of creditors decide whether to roll over their debt or to run. They base their decision on the observation of the fundamental y_t and the interest rate r_t reset by the remarketing agent. If they decide to roll over, the game continues to the next instant. If they decide to run, the liquidity facility is drawn upon to purchase the tendered bonds, but the facility may fail with probability $\theta\delta dt$. If it fails, the game ends and the project is liquidated to pay off all the creditors. If it succeeds, the game continues to the next instant.

Figure 3: Models of VRDO and ARS

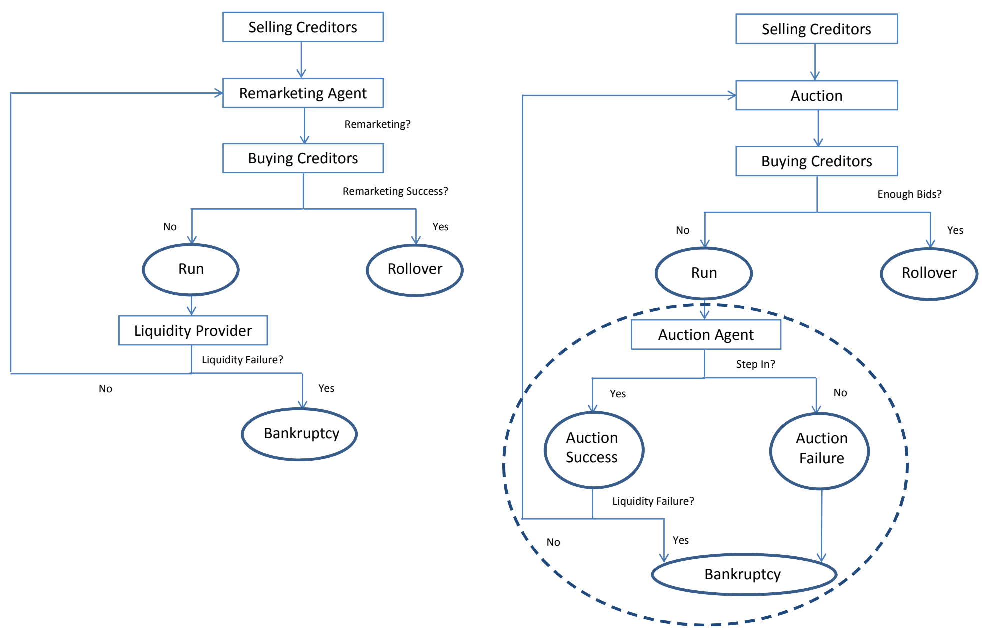


Figure 3A: VRDO

Figure 3B: ARS

The case of ARS, shown in Figure 3B, has a similar timeline as VRDOs, except that when the creditors decide to run, with probability $\kappa\delta dt$ the auction agent may decide not to intervene and then the auctions would continue to fail until the project fails eventually. This additional layer of uncertainty, highlighted by the flowchart within the dashed circle in Figure 3B, captures the central distinction between VRDOs and ARS in terms of the existence of a liquidity backstop.

3.5 Parameter Restrictions

We impose a few parameter restrictions for the model to be meaningful. We keep the same parameter restrictions as in HX:

$$
\begin{aligned}
\mu &< \rho + \phi, && (5)\\
\phi &< \theta\delta\left(1 - L - l\right) + \kappa\delta\left(1 - U\left(1\right)\right), && (6)\\
\alpha &< \left[\frac{r}{\rho + \phi} + \frac{\phi}{\rho + \phi - \mu}\right]^{-1}. && (7)
\end{aligned}
$$

The first one of the above three restrictions imposes an upper bound on the growth rate of the fundamental to ensure the fundamental value is finite. The second restriction ensures that the parameter θ is sufficiently high so that bankruptcy becomes likely when some creditors choose to run. $U(\cdot)$ denotes the value function in the case of continued auction failures, which we derive shortly in the next section. The third restriction stipulates a sufficiently low premature recovery rate so that $L + l < 1$.

In addition, we impose the following restriction for the additional parameters in our model, namely, $\overline{r}$, Δ, and κ:

$$
\overline{r} > \rho + \Delta + \kappa\delta\left(1 - U\left(\frac{1-L}{l}\right)\right), \tag{8}
$$

$$
0 \leq \Delta < \frac{\eta_1\left(\rho + \phi\right)\left(\rho + \phi + \theta\delta\left(1 - L\right) - \overline{r}\right)}{\gamma_2\left(\rho + \phi + \delta\left(1 + \theta + \kappa\right)\right)}, \tag{9}
$$

$$
\begin{aligned}
\left(\eta_1 - 1\right)\left(K_2 + \overline{K}_7\right) + \eta_3\left(\eta_1 - \eta_3\right)U_1 &> 0, && \text{(10a)}\\
\left(\eta_1 - 1\right)\left(K_4 + \overline{K}_9\right) + \eta_3\left(\eta_1 - \eta_3\right)U_3\left(\left(1 - L\right)/l\right)^{\eta_3 - 1} &> 0, && \text{(10b)}
\end{aligned}
$$

where η_1 and γ_2 as well as other constants (e.g., K_2) are defined in Appendix A. The restriction (8) ensures that the max rate is sufficiently high for the model to be meaningful. The restriction (9) rules out the degenerate case where the liquidity component Δ is too large and it is thus always profitable to hold the bonds, implying that the equilibrium threshold y_* is zero. Note that $\rho+\phi+\theta\delta\left(1-L\right)$ is an endogenous upper bound on the max rate. Furthermore, this restriction also implies $U\left(0\right)<1$. Lastly, the restriction (10a,b) is needed for the model to be well-behaved. To simplify exposition, we assume $\overline{\theta}=1+\theta$ throughout the paper.

4 Equilibrium

We now turn to the characterization of monotone equilibriums in which creditors choose to roll over if and only if the fundamental is above a threshold. In this section, we first analyze an individual creditor's problem of optimal threshold choice. Then we study how the interest rate should be set in a monotone equilibrium. Lastly, we derive a unique symmetric monotone equilibrium in closed form in which the optimal threshold, denoted by y_*, is unique for all creditors and the equilibrium interest rate is set in a way that the debt is priced at par whenever $y_t \geq y_*$. We also discuss an extension of the model which is needed when we conceptualize and estimate the value of a liquidity backstop.

4.1 Value Functions

We derive the optimal rollover threshold y_* by solving an individual creditor's optimal rollover problem. Consider an individual creditor who is making his rollover decision. Suppose all the other creditors choose a rollover threshold y_* and the (remarketing or auction) agent resets the interest rate $r_t = R\left(y_t; y_*\right)$ based on the same threshold y_*. Denote by $V\left(y_t; y_*\right)$ the creditor's value function when auctions have been successful, and by $U\left(y_t\right)$ the value function when auctions have failed.

First, we determine the value function $V\left(y; y_*\right)$ when auctions have been successful. For each unit of debt, each creditor receives a stream of interest payments $R(y_t; y_*)$ until the earliest of the following four events occur. The first event occurs at stopping time τ_ϕ when the asset matures and the creditor gets a final payoff of $\min\left(1, y_{\tau_\phi}\right)$. The second event occurs at stopping time τ_δ when the creditor gets the opportunity to decide whether to roll over the debt. Whether or not the creditor decides to roll over depends on whether or not the continuation value $V\left(y_{\tau_\delta}; y_*\right)$ exceeds the one-dollar par value. The third and fourth events occur when the fundamental falls below other creditors' rollover threshold y_*: upon a run by other creditors, with probability $1_{\{y_t \le y_*\}}\kappa\delta dt$, the third event occurs at the stopping time τ_κ when auctions fail and the creditor will be stuck with the debt valued at $U\left(y_{\tau_\kappa}\right)$; with probability $1_{\{y_t \le y_*\}}\theta\delta dt$, the fourth event occurs at the stopping time τ_θ when the project is forced to premature liquidation with payoff $\min\left(1, L + l y_{\tau_\theta}\right)$. The stopping time $\tau \equiv \min\left\{\tau_\phi, \tau_\delta, \tau_\kappa, \tau_\theta\right\}$ is the minimum of these four stopping times, representing the earliest time when any of these four events occur. Due to risk neutrality, the value function $V(y; y_*)$ is given by

$$
\begin{aligned}
V\left(y_t; y_*\right) \quad = \quad & E_t\left[\int_t^\tau e^{-\rho(s-t)} R\left(y_s; y_*\right) ds + e^{-\rho(\tau-t)} \min\left(1, y_\tau\right) 1_{\left\{\tau=\tau_\phi\right\}}\right. \\
& + e^{-\rho(\tau-t)} \min\left(1, L + l y_\tau\right) 1_{\{\tau=\tau_\theta\}} + e^{-\rho(\tau-t)} U\left(y_t\right) 1_{\{\tau=\tau_\kappa\}} \\
& \left. + e^{-\rho(\tau-t)} \max_{\textit{rollover} \text{ or } \textit{run}} \left(V\left(y_\tau; y_*\right), 1\right) 1_{\{\tau=\tau_\delta\}}\right].
\end{aligned} \tag{11}
$$

The Hamilton-Jacobi-Bellman (HJB) equation is given below:

$$
\begin{aligned}
\rho V\left(y_t; y_*\right) \quad = \quad & \mu y_t V_y\left(y_t; y_*\right) + \frac{\sigma^2}{2} y_t^2 V_{yy}\left(y_t; y_*\right) + R\left(y_t; y_*\right) \\
& + \phi\left[\min\left(1, y_t\right) - V\left(y_t; y_*\right)\right] \\
& + \theta\delta 1_{\{y_t \le y_*\}}\left[\min\left(1, L + l y_t\right) - V\left(y_t; y_*\right)\right] \\
& + \kappa\delta 1_{\{y_t \le y_*\}}\left[U\left(y_t\right) - V\left(y_t; y_*\right)\right] \\
& + \delta \max_{\textit{rollover} \text{ or } \textit{run}} \left(0, 1 - V\left(y_t; y_*\right)\right).
\end{aligned} \tag{12}
$$

It shows that the creditor's required return on the left hand side, $\rho V(y_t; y_*)$, is equal to the expected increase in his continuation value as summarized by the terms on the right hand side. The creditor will choose to roll over the debt if and only if $V(y_t; y_*) \geq 1$. If we denote as $y'_* = \inf\{y_t : V(y_t; y_*) \geq 1\}$ the minimum fundamental value at which the continuation value is no less than 1, then y'_* is the creditor's optimal rollover threshold since $V(y'_*; y_*) = 1$ and $V(y; y_*) < 1$ for $y < y'_*$. In the symmetric equilibrium we consider below, each creditor's optimal threshold choice y'_* must coincide with other creditors' threshold y_*. Thus the optimality condition is

$$V(y_*; y_*) = 1.$$

Similarly, we can determine the value function $U(y_t)$ when auctions have continued to fail. Under the assumption that the auctions, once failed, would continue to fail, creditors' rollover decision becomes irrelevant and thus the value function $U(y_t)$ does not depend on their rollover threshold y_*. In this autarkic scenario, the max rate $\overline{r}$ is imposed until the asset matures at the stopping time τ_ϕ or the project is prematurely liquidated at the stopping time $\tau_{\overline{\theta}}$. As a result, the value function $U(y_t)$ is given by

$$\begin{aligned} U(y_t) \quad = \quad & E_t\left[\int_t^{\tau_\phi \wedge \tau_{\overline{\theta}}} e^{-\rho(s-t)}\overline{r}ds + e^{-\rho\left(\tau_\phi - t\right)}\min\left(1, y_{\tau_\phi}\right)1_{\left\{\tau_\phi \leq \tau_{\overline{\theta}}\right\}}\right. \\ & \left. + e^{-\rho\left(\tau_{\overline{\theta}} - t\right)}\min\left(1, L + l y_{\tau_{\overline{\theta}}}\right)1_{\left\{\tau_\phi > \tau_{\overline{\theta}}\right\}}\right]. \end{aligned} \tag{13}$$

In Lemma 1, we derive the value function in closed form as below, and prove that it is strictly monotonically increasing:

$$U(y_t) = \begin{cases} \overline{K}_1 + \overline{K}_2 y_t + U_1 y_t^{\eta_3}, & \text{if } y \in (0, 1] \\ \overline{K}_3 + \overline{K}_4 y_t + U_2 y_t^{-\gamma_3} + U_3 y_t^{\eta_3}, & \text{if } y \in (1, \frac{1-L}{l}] \\ \overline{K}_5 + U_4 y_t^{-\gamma_3}, & \text{if } y \in (\frac{1-L}{l}, \infty) \end{cases}, \tag{14}$$

where the coefficients $\overline{K}_1$, etc. are defined in Appendix A or in the proof of Lemma 1.

Lemma 1 $U(y)$ *is strictly monotonically increasing.*

To determine the value function $V(y; y_*)$, we need to spell out how the floating interest rate $r_t = R(y_t; y_*)$ is determined first, to which we will turn next.

4.2 Floating Interest Rate

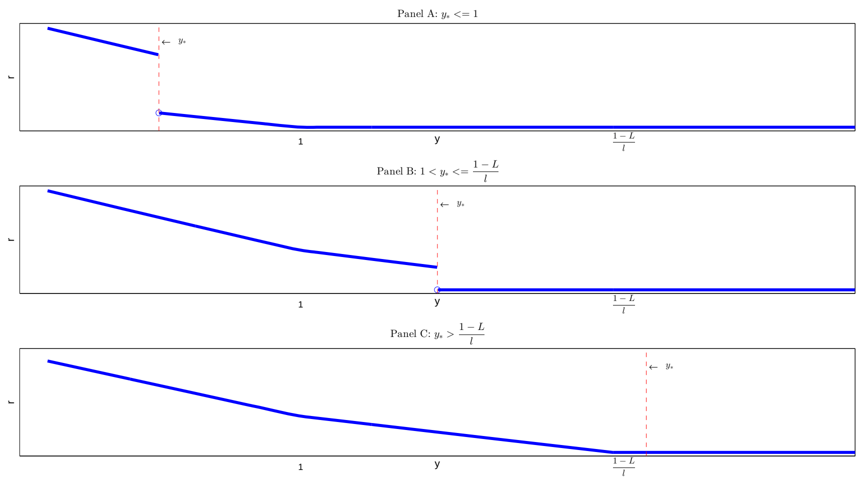
We now consider how the interest rate r_t is reset at each point of time. We first derive the unconstrained interest rate in the benchmark case where there is no interest rate cap, and then determine the constrained interest rate once an interest rate cap is imposed.

In the absence of a maximum interest rate, for any fixed threshold $y_* \geq 0$, the interest rate is unbounded and can be set arbitrarily high to ensure that VRDOs or ARS are priced at par. Based on the HJB equation (12), the value function is always equal to one under the following unconstrained interest rate $R^u(y_t; y_*)$

$$R^u(y_t; y_*) = \rho + \phi(1 - y_t)^+ + 1_{\{y_t \leq y_*\}}\left[\theta\delta(1 - [L + ly_t])^+ + \kappa\delta(1 - U(y_t))\right] \quad (15)$$

where $(x)^+$ denotes x if $x > 0$, or zero otherwise. The unconstrained interest rate schedule takes a different shape when y_* is in a different range: $y_* \leq 1$, $1 < y_* \leq \frac{1-L}{l}$, and $y_* > \frac{1-L}{l}$, as shown in Panels A, B, and C of Figure 4, respectively.

Figure 4: Unconstrained Interest Rate Schedule



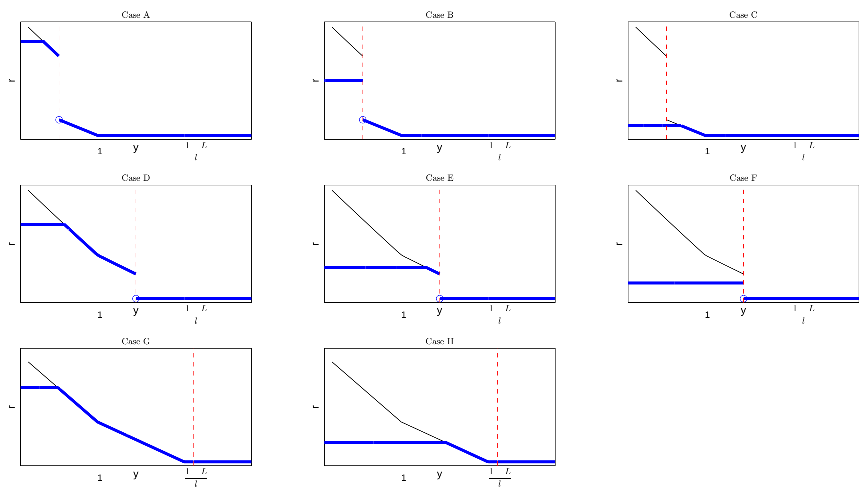
The unconstrained interest rate in Eq. (15) can be decomposed into three components: a risk-free component ρ, a component related to losses at maturity $\phi(1-y_t)^+$, and the last component associated with possible credit losses. Intuitively, the unconstrained interest rate decreases with the fundamental y_t. That is, creditors are generally paid by a higher interest rate when the fundamental deteriorates. As we will show shortly, such countercyclicality of the interest rate tends to alleviate runs. Furthermore, the unconstrained interest rate jumps to a higher value when the threshold y_* is reached from above. The upward jumps occur in these cases because of the possible losses incurred due to premature liquidation $(1-[L+ly_t])^+$ and auction failures $(1-U(y_t))$. However, in the absence of a maximum rate, the interest rate can freely adjust to guarantee the value of debt always equal to one. As a result, in equilibrium, creditors are indifferent between rolling over and running.

In the presence of an interest rate cap $\overline{r}$, creditors will be under-compensated in bad states if $\overline{r}$ is imposed instead of the higher unconstrained interest rate. Therefore, if the interest rate is given as the lower value between $\overline{r}$ and $R^u(y_t; y_*)$, then creditors's continuation value is strictly less than 1 and thus always prefers to run, i.e., $y_* = \infty$. To avoid such a degenerate case and to keep tractability, throughout the rest of the paper, we adds a new component $\Delta \geq 0$ to the unconstrained interest rate, which we refer to as a "*liquidity premium*", and then impose the following interest rate schedule:

$$R(y_t; y_*) = \min\left(R^u(y_t; y_*) + \Delta, \overline{r}\right). \tag{16}$$

Depending on the values of Δ and $\overline{r}$, as shown in Figure 5, there are totally eight different cases (Case A, $\cdots$, Case H) where the constrained interest rate schedule takes a different functional form.

Figure 5: Constrained Interest Rate Schedule



The component Δ introduces a trade-off for creditors. On one hand, when the fundamental falls below the rollover threshold, creditors are undercompensated by the max rate, which is usually lower than what the market-required rate would have been had creditors decided to roll over the debt. On the other hand, when the fundamental remains above the threshold, creditors are overcompensated by an amount equal to Δ. Therefore, however small Δ is, creditors receive the benefit of overcompensation during tranquil times and trade off the benefit against the loss due to undercompensation during future run scenarios. As we prove in the next subsection, the trade-off guarantees uniqueness of the symmetric equilibrium where the threshold y_* is uniquely pinned down at which creditors are just indifferent between rolling over and running. Another possible interpretation of the component Δ is that it can also be an extra compensation demanded by the (auction or remarketing) agent for possible inventory risk or holding costs. Lastly and importantly, it is also related to the concept of the value of a liquidity backstop we introduce shortly in the next section. For the above reasons, we will refer to Δ as a "*liquidity premium.*"

4.3 Unique Threshold Equilibrium

We focus on symmetric monotone equilibria where all creditors in equilibrium will choose the same threshold y_* and the agent resets the interest rate based on y_*. The threshold y_* is defined as the minimum value at which $V(y_t; y_*) \geq 1$, i.e., $y_* = \min\{y_t : V(y_t; y_*) \geq 1\}$. When y_t falls below the threshold y_*, due to monotonicity of the value function, the decision to run is strictly preferable since $V(y; y_*) < 1$ for $y < y_*$. Theorem 1 below proves the existence of a unique symmetric monotone equilibrium.

Theorem 1 *There exists a unique symmetric monotone equilibrium in which the run threshold y_* is uniquely determined — each maturing creditor chooses to roll over his debt if $y_t > y_*$, and to run otherwise.*

Proof. See Appendix C. ■

5 Model Implications

In this section, we explore main implications of our model. The model has two key ingredients: a liquidity backstop and a floating interest rate. The rest of this section is devoted to understanding how these features affect equilibrium outcomes, in particular, the likelihood of runs. We first examine the role of a floating interest rate by focusing on the model of VRDOs where κ is set to zero to reflect the existence of an explicit liquidity backstop. We then turn to the model of ARS to examine the role of (lack of) a liquidity backstop where $\kappa > 0$ is positive to reflect the possibility of auction failures. Lastly, we formally define the value of a liquidity backstop, which will be estimated structurally in the next section.

5.1 Implications of Floating Interest Rate: The VRDO Model

To single out the role of a floating interest rate, we start with a special case where κ is set to be zero — the liquidity support, albeit imperfect, is explicit such that the liquidity provider has contractual obligation to honor its liquidity commitment. In this special case where $\kappa = 0$, the model reduces to the one of VRDOs.

The endogenous interest payment $\{r_t\}$ in this paper, a key departure from HX, affects the creditors' rollover decision in a profound way. For example, as the fundamental deteriorates, the interest rate increases in a manner so as to compensate creditors for credit losses. However, the magnitude of overall interest payments to creditors depends on two factors: the maximum interest rate $\overline{r}$ and the liquidity premium Δ.

We show in Proposition 1 below that the equilibrium rollover threshold y_* decreases with the maximum interest rate $\overline{r}$ or the liquidity premium Δ. The result that y_* decreases with the liquidity premium is very important when we define and measure the value of a liquidity backstop later. The intuition is straightforward: a higher maximum interest rate $\overline{r}$ or a higher liquidity premium Δ allows the interest rate to increase further in a severely adverse environment; therefore, it increases the expected interest income for creditors and they will roll over more frequently. In the extreme case where the maximum interest rate $\overline{r}$ is sufficiently high, then the run threshold is zero (i.e., $y_* = 0$), that is, the likelihood of runs is zero.

Proposition 1 *The equilibrium rollover threshold y_* decreases with the maximum interest rate $\overline{r}$ or the liquidity premium Δ. In particular, when Δ goes to 0, the equilibrium run threshold y_* tends to infinity.*

Proof. See Appendix C. ■

5.2 Implications of Liquidity Backstop: The ARS Model

Next, we examine how the lack of a liquidity backstop in the ARS market affects equilibrium outcomes. To examine the role of (lack of) a liquidity backstop in isolation, we assume away the floating interest rate. In particular, we assume that the interest rate is always fixed as $\overline{r}$, the max rate, regardless of auction success or failure. This simplified model is very similar to the one in HX, except that there is an additional risk of auction failures.

In Proposition 2 below, we prove that when the max rate is low enough, increasing κ from zero to a positive value makes creditors more likely to run. Intuitively, a low enough max rate leads to a very low continuation value $U(y)$ in the event of failed auctions and thus, ex ante, creditors choose to run more often.

Proposition 2 *If $\overline{r}$ is sufficiently low, the equilibrium rollover threshold y_* increases as the arrival intensity of auction failures κ increases from zero; that is,*

$$\left.\frac{dy_*}{d\kappa}\right|_{\kappa=0} > 0.$$

Proof. See Appendix C. ■

Proposition 2 illustrates how the lack of a liquidity backstop may exacerbate runs, which provides an explanation for the turmoil in the ARS market in early 2008 when investors started to factor in the possibility of auctions failures. As we show below, the destabilizing effect of the lack of liquidity backstops results from a new type of externality. The running decision of current creditors accelerates the issuer's default probability and may also trigger auction failures. Therefore, their decision to run affects payoffs of future creditors. Table 1 summarizes the current and future creditors' payoffs in different scenarios depending on whether the current creditors run or not.

Table 1: Run-Induced Externalities

Choice of current creditors	Run			Rollover
Liquidity Provision	NO	YES		
		Failed	Survived	Survived
Probability	$\kappa\delta dt$	$\theta\delta dt$	$1-\kappa\delta dt-\theta\delta dt$	1
Payoff of current creditors	$U(y)$	$L(y)$	1	$V(y)$
Payoff of future creditors	$U(y)$	$L(y)$	$V(y)$	$V(y)$

From Table 1, we can see that the current creditors will choose to run if and only if $1 \cdot (1-\kappa\delta dt-\theta\delta dt)+L(y)\cdot\theta\delta dt+U(y)\cdot\kappa\delta dt > V(y)\cdot 1$, or $V(y)<1$ after ignoring higher order terms. Furthermore, because of the lack of a committed liquidity facility, a run on ARS may lead to auction failure when the auction agent stops providing liquidity, which imposes an additional implicit cost on future maturing creditors. Specifically, a run by the current creditors reduce the future creditors' value function by

$$
\begin{aligned}
& V(y) - [V(y) \cdot (1 - \kappa\delta dt - \theta\delta dt) + L(y) \cdot \theta\delta dt + U(y) \cdot \kappa\delta dt] \\
= \; & \underbrace{[V(y) - L(y)]\theta\delta dt}_{\text{cost due to default loss}} + \underbrace{[V(y) - U(y)]\,\kappa\delta dt}_{\text{cost due to auction failure}}.
\end{aligned}
$$

Besides the implicit cost of default loss as studied in HX, a run in our model also induces an additional cost in the event of auction failure. This additional externality, absent in the VRDO market, makes the ARS market more susceptible to runs: in anticipation of possible auction failures and the associated losses as a result of runs by future creditors, the current creditors have less incentives to roll over their debt.

5.3 The Value of a Liquidity Backstop

We are now in position to define the value of a liquidity backstop. From our earlier discussion (Section 2), prior to the crisis, ARS were considered to have the same explicit liquidity backstops as VRDOs. Put differently, before 2007 investors perceived the probability of auction failures to be negligible, i.e., $\kappa = 0$ for both ARS and VRDOs. However, the wave of auction failures in 2008 during the crisis revealed the implicit nature of liquidity support in the ARS market, and investors started to realize that $\kappa > 0$. As proved in Proposition 2, an increase in κ increases the run threshold y_*. Our estimation results reported in the following section confirms that after the crisis broke out, the estimated run threshold $y_*^{ARS}(\Delta, \kappa)$ in the ARS market is substantially higher than $y_*^{VRDO}(\Delta, 0)$ in the VRDO market. Note that we explicitly type the argument (Δ, κ) of the run threshold (in the case of VRDOs, κ is always zero).

To define the value of a liquidity backstop, let us consider the following thought experiment. On one hand, an ARS issuer can pay a certain fee per annum to purchase a liquidity backstop from a liquidity provider, and effectively reduce the run threshold y_*^{ARS} to the same level as $y_*^{VROD}(\Delta, 0)$. On the other hand, the ARS issuer can raise the level of interest rate by a constant amount $\Gamma > 0$, and thus effectively increases Δ to $\Delta + \Gamma$. According to Proposition 1, a higher liquidity premium $\Delta + \Gamma$ induces a lower run threshold. From the perspective of the (risk neutral) issuer, the two methods are equivalent as long as the fee to purchase a liquidity backstop is the same as Γ. Therefore, the increment in the ARS rate Γ measures the value of a liquidity backstop, satisfying

$$y_*^{ARS}(\Delta + \Gamma, \kappa) = y_*^{VRDO}(\Delta, 0). \tag{17}$$

In the next section, we use the historical data to estimate Γ based on the estimated thresholds $y_*^{ARS}(\Delta, \kappa)$ and $y_*^{VRDO}(\Delta, 0)$.

6 Calibration

The markets for VRDOs and ARS provide an ideal laboratory for us to identify the value of a liquidity backstop. The identification benefits from the structural change in the belief of ARS investors since the wave of auction failures in mid-February 2008. We first describe the data and calibration of the parameters, and then report calibration results.

6.1 Data

The weekly data of 1-week tax-exempt VRDO and ARS rates are obtained directly from the Securities Industry and Financial Markets Association (SIFMA) website.[9] The historical data for the VRDO rate is available for the period from May 22, 1991 to October 24, 2012, while the ARS historical rate is only available for a shorter

[9] The website's URL is http://archives.sifma.org/swapdata.html.

period from May 31, 2006 to December 30, 2009. We also obtain the 1-week Treasury repo rate from Bloomberg for the same sample period.

For the purpose of calibration, we obtain information about characteristics of VRDOs or ARS (e.g., the max rate) from the Municipal Securities Rulemaking Board (MSRB)'s SHORT database from its inception date of April 1, 2009 through November 8, 2012. The SHORT database has been built from the Short-term Obligation Rate Transparency (SHORT) System and the Real-Time Transaction Reporting System (RTRS), which the MSRB launched in early 2009 to collect and disseminate interest rates and important descriptive information about these ARS and VRDOs. The SHORT database provides a centralized source of information about municipal ARS and VRDOs that was previously unavailable. Starting from May 2011, MSRB rules require VRDO remarketing agents to report to the MSRB the aggregate amount of par value of "bank bonds" and bonds held by investors or remarketing agents. There are 20,547 distinct VRDOs in the SHORT database during our sample period. We focus on the VRDOs with weekly interest resets, which accounts for 90.7% of the whole sample (i.e., 18,630).

The SHORT database does not contain maturity information. Therefore, we merge it with the Mergent Municipal Bond database to collect information on maturities.

6.2 Calibration

There are ten primitive model parameters in the model: $\overline{r}, \phi, \rho, \delta, \theta, \kappa, L, l, \mu, \sigma, \Delta$. The contractual maximum interest rate $\overline{r}$ is calibrated to be 12% using the SHORT database. Among the 18,630 VRDOs with weekly interest resets in the SHORT database, 53.42% of them have the max rate of 12%, 26.13% of them have the max rate of 10%, and 10.37% of them have the max rate of 15%. The weighted average of these three rates is 11.76%. Therefore, we set $\overline{r}$ as 12%. The average debt maturity of our merged VRDO sample from the SHORT and Mergent databases is 25.2 years (and the median is 25.96 years). We therefore set $1/\phi$, the expected asset maturity, to 25 based on the assumption that the average maturity coincides with the average asset maturity; that is, $\phi = 0.04$. The tax-adjusted risk-free rate ρ is set to the average value of the tax-adjusted repo rate, or $\rho = 0.0195$, during the sample period between 1991 and 2012 using a tax rate of 40% following Longstaff (2011).

The parameter δ represents the arrival intensity of creditors who make the running decision. In the model, once a run occurs, the proportion of creditors who decide not to roll over the debt is $\delta\Delta t$, where we set $\Delta t = 7/365.25$ to reflect the weekly frequency of the interest rate reset for the constituent VRDOs/ARS in the SIFMA indexes. In reality, VRDO/ARS creditors come to the remarketing or auction agent to buy or sell the securities on the interest rate reset dates. A run is considered to occur if a significant number of creditors decide to not roll over the debt. As a result, we set $\delta = 12$, meaning that on average creditors make the running decision on a monthly basis, and upon a run, about $\delta\Delta t = 23\%$ of the securities are not rolled over.

The default intensity θ is set to 0.0278 so that the average time from a run to eventual bankruptcy is equal to $1/(\theta\delta) = 3$ years, which is roughly in line with the bankruptcy experience of Jefferson County, AL (Woodley (2012)). Furthermore, we set $\kappa = 0.0159$, under which the fraction of auctions that have failed within the 14-week window between November 14, 2007 and February 20, 2008 is about 5%.[10]

The parameters L and l is calibrated based on the formula $L = \frac{\alpha r}{\rho+\phi}$ and $l = \frac{\alpha\phi}{\rho+\phi-\mu}$. We set $\alpha = 90\%$ to reflect relatively high recovery rates for municipal bonds.[11]

[10] By definition, following a run, a fraction $(\kappa\delta dt)$ of auctions will fail in the first week, or $(1-\kappa\delta dt)$ of auctions will surive the first week. Similarly, among the ARS whose auctions succeeded in the first week, a fraction of them, $(1-\kappa\delta dt)^2$, will continue to surve in the second week, $\cdots$. The cumulative fraction of auctions that have failed with N weeks equals $1-(1-\kappa\delta dt)^N$. Plugging in $\kappa = 0.01159$, $\delta = 12$, $dt = 7/365$ leads to a failure rate of 5% in a 14-week window.

[11] The recovery rate of municipal bonds is not readily available given municipal bankruptcy is rare. See, for instance, Coval and Stafford (2007) for the estimates of the recovery rate for stocks; and Andrade and Kaplan (1998), Hennessy and Whited (2007) , Ellul, Jotikashira, and Lundblad (2010),

Furthermore, we set r to be equal to the average VRDO interest rate in the data, i.e. $r = 0.024$. That is, the average rate at which the project generates cash flow is close to the average interest rate the issuer pays to creditors. We further assume $\mu = 0$ to have zero expected growth rate in the levels of the fundamental. Given the values of α, r, and μ, the parameters governing the recovery rate of the asset in the worse case scenario are calibrated as: $(L, l) = (36.3\%, 60.5\%)$. Lastly, the volatility σ is calibrated to fit the volatility of the historical VRDO rate in the pre-crisis period, or $\sigma = 0.143$. The last structural parameter Δ is left uncalibrated. In our numerical experiments, we consider a variety of realistic values for Δ. The calibrated parameter values are reported in Table 2.

Table 2: Calibrated Parameter Values

Variable	Symbol	VRDO	ARS
max rate	$\overline{r}$	0.12	same
avg. maturity	$1/\phi$	25	same
avg. duration	$1/\delta$	1/12	same
tax-adj. riskless rate	ρ	0.02	same
drift	μ	0	same
volatility	σ	0.143	same
recovery rate	(L, l)	$(0.36, 0.61)$	same
default intensity	θ	0.0278	same
auction failure intensity	κ	0	0.0159

for the estimates of the recovery rate for corporate bonds.

6.3 Results

We apply the above estimation methodology to the SIFMA historical interest rate indexes for the VRDO and ARS markets. The VRDO historical data ranges between May 22, 1991 and October 24, 2012, while the ARS data ranges between May 31, 2006 and December 30, 2009 when SIFMA stopped producing the index for the ARS market. As shown in Figure 1, the ARS rate had largely moved in lockstep with the VRDO rate until November 14, 2007, and have diverged since then. One interpretation of this behavior is that market participants considered the possibility of auction failures as remote and viewed both ARS and VRDO as almost identical securities. In fact, auction failures were very rare prior to 2007 and ARS were often marketed by broker-dealers as "cash equivalents." As a result, we set κ to zero for the pre-crisis period when using the ARS data. Specifically, in our structural estimation, we restrict κ to zero for the whole sample periods for both VRDOs and ARS, except for the period between November 14, 2007 and December 30, 2009 for ARS when we allow for a positive κ to reflect a possible structural change in investors' belief.

Table 3: Value of a Liquidity Backstop and Equilibrium Thresholds

Δ	y_*^{VRDO}	y_*^{ARS}	Γ
1	0.65	0.80	38
5	0.62	0.77	43
10	0.59	0.75	45
20	0.55	0.71	50
30	0.50	0.68	54
40	0.45	0.65	59
50	0.40	0.61	64

The calibration results are reported in Table 3. We consider a wide range of values for the liquidity component Δ from 1 to 50 basis points. A value of Δ equal to 50 basis points accounts for about 21% of the average VRDO interest rate (i.e., 2.4%), or 17% of the average ARS interest rate (i.e., 3.02%) in the data. Note that the wave of auction failures in early 2008 revealed the fact that the liquidity support by auction agents is only implicit and may fail when it is needed most.

The estimation results also suggest that floating interest rates tend to mitigate runs in both markets. In fact, if we restrict the interest rate to be the constant r, the model in the VRDO case reduces to the HX model with a fixed interest rate, and the implied run threshold $y_*^{HX} = 0.87$ is typically higher than the one for VRDOs y_*^{VRDO}. It is worthwhile to point out that the constant rate r is calibrated to be equal to the average VRDO interest rate; hence, VRDO investors receive almost the same interest rate payments on average as those in the HX model, and tend to run less frequently exactly because the floating interest rates are reset higher in bad times.

The positive κ, capturing the lack of a liquidity backstops in the ARS market, implies a higher rollover threshold for ARS investors: $y_*^{ARS} > y_*^{VRDO}$, as shown in the third column of Table 3. As we discussed in Section 5.2, the fear of getting stuck when future auctions fail propels ARS creditors more likely to run, ex ante, relative to VRDO creditors. The higher rollover threshold for ARS creditors reflects the lack of a liquidity backstop in the ARS market. On the other hand, from Proposition 1, we know that the rollover threshold y_* decreases with the liquidity/risk premium Δ. Therefore, to measure the value of a liquidity backstop Γ, we should find out how much the interest rate need to be increased so that the ARS rollover threshold can be reduced to the same level of the VRDO rollover threshold. The required increase Γ in the interest rate is a measure of the value of a liquidity backstop. Mathematically, we express the rollover thresholds $y_*^{ARS}(\Delta, \kappa)$ and $y_*^{VRDO}(\Delta, 0)$ to denote their dependence on Δ, and define the value of a liquidity backstop $\Gamma > 0$ as the solution to the following equation:

$$y_*^{ARS}(\Delta + \Gamma, \kappa) = y_*^{VRDO}(\Delta, 0).$$

The last column of Table 3 reports the valuation of a liquidity backstop, ranging from 38 to 64 basis points. In present value (i.e., $\Gamma/(\rho + \phi)$), a liquidity backstop is evaluated to be about 6.4% to 10.8% of the par value. Therefore, the implied value (or cost) of providing liquidity backstops for the ARS market is about $12.8 to $21.5 billion for the $200 billion ARS market at the peak level before its collapse.

7 Concluding Remarks

In this paper, we develop a model of dynamic debt runs in the markets for ARS and VRDOs. Not only does the model capture several common features of these markets (e.g., floating interest rate and an interest rate cap), it also captures distinctive characteristics, such as the explicit (implicit) liquidity provision in the VRDO (ARS) market. Based on the calibrated model, we show that the value of a liquidity backstop is worth about 40-60 basis points per annum.

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Appendix A: Notation

We denote by $-\gamma_i$ and η_i two real roots of the quadratic equation $\frac{1}{2}\sigma^2 x(x-1)+\mu x-(\rho+\phi+\delta_i)=0$, $i=1,2,3$,

$$\begin{aligned}
-\gamma_i &= -\frac{\mu-\frac{1}{2}\sigma^2+\sqrt{\left(\frac{1}{2}\sigma^2-\mu\right)^2+2\sigma^2\left[\rho+\phi+\delta_i\right]}}{\sigma^2}<0, \\
\eta_i &= -\frac{\mu-\frac{1}{2}\sigma^2-\sqrt{\left(\frac{1}{2}\sigma^2-\mu\right)^2+2\sigma^2\left[\rho+\phi+\delta_i\right]}}{\sigma^2}>0.
\end{aligned}$$

where $\delta_1=\delta(1+\theta+\kappa)$, $\delta_2=0$, $\delta_3=\overline{\theta}\delta$.

The following notation is used in determining equilibrium threshold

$$\begin{array}{ll}
K_1=\frac{\overline{r}+\delta(1+\theta L)}{\rho+\phi+\delta(1+\theta+\kappa)} & \overline{K}_1=\frac{\overline{r}+\overline{\theta}\delta L}{\rho+\phi+\overline{\theta}\delta} \\
K_2=\frac{\phi+\theta\delta l}{\rho+\phi+\delta(1+\theta+\kappa)-\mu} & \overline{K}_2=\frac{\phi+\overline{\theta}\delta l}{\rho+\phi+\overline{\theta}\delta-\mu} \\
K_3=\frac{\overline{r}+\phi+\delta(1+\theta L)}{\rho+\phi+\delta(1+\theta+\kappa)} & \overline{K}_3=\frac{\overline{r}+\phi+\overline{\theta}\delta L}{\rho+\phi+\overline{\theta}\delta} \\
K_4=\frac{\theta\delta l}{\rho+\phi+\delta(1+\theta+\kappa)-\mu} & \overline{K}_4=\frac{\overline{\theta}\delta l}{\rho+\phi+\overline{\theta}\delta-\mu} \\
K_5=\frac{\overline{r}}{\rho+\phi} & \overline{K}_5=\frac{\overline{r}+\phi+\overline{\theta}\delta}{\rho+\phi+\overline{\theta}\delta} \\
K_6=\frac{\phi}{\rho+\phi-\mu} & \overline{K}_6=\kappa\delta\frac{\overline{K}_1}{\rho+\phi+\delta(1+\theta+\kappa)} \\
K_7=\frac{\overline{\rho}+\phi+\delta(1+\theta+\kappa)}{\rho+\phi+\delta(1+\theta+\kappa)} & \overline{K}_7=\kappa\delta\frac{\overline{K}_2}{\rho+\phi+\delta(1+\theta+\kappa)-\mu} \\
K_8=\frac{\overline{\rho}+\phi}{\rho+\phi} & \overline{K}_8=\kappa\delta\frac{\overline{K}_3}{\rho+\phi+\delta(1+\theta+\kappa)} \\
K_9=\frac{\overline{r}+\phi+\delta(1+\theta)}{\rho+\phi+\delta(1+\theta+\kappa)} & \overline{K}_9=\kappa\delta\frac{\overline{K}_4}{\rho+\phi+\delta(1+\theta+\kappa)-\mu} \\
K_{10}=\frac{\overline{r}+\phi}{\rho+\phi} & \overline{K}_{10}=\kappa\delta\frac{\overline{K}_5}{\rho+\phi+\delta(1+\theta+\kappa)}
\end{array}$$

Appendix B: Proofs

Proof of Lemma 1. The HJB equation for $U(y)$ is the following

$$\begin{aligned}
\rho U(y_t) &= \mu y_t U_y(y_t)+\frac{\sigma^2}{2}y_t^2 U_{yy}(y_t)+\overline{r} \\
&\quad +\phi\left[\min(1,y_t)-U(y_t)\right]+\overline{\theta}\delta\left[\min(1,L+ly_t)-U(y_t)\right].
\end{aligned}$$

Depending on the value of y, the HJB equation can be re-expressed as

$$\left(\rho+\phi+\overline{\theta}\delta\right)U-\mu y U_y-\frac{\sigma^2}{2}y^2U_{yy}=\begin{cases}\overline{r}+\phi y+\overline{\theta}\delta(L+ly), \text{ if } y\in(0,1]; \\ \overline{r}+\phi+\overline{\theta}\delta(L+ly), \text{ if } y\in(1,\frac{1-L}{l}]; \\ \overline{r}+\phi+\overline{\theta}\delta, \text{ if } y\in(\frac{1-L}{l},\infty).\end{cases}$$

Therefore, the solution has the functional form in Eq. (14). We determine the unknown coefficients $U_1, \cdots, U_4$ from the value-matching and smooth-pasting conditions:

$$
\begin{aligned}
U_1 &= \left(\overline{K}_3 + \overline{K}_4\right) - \left(\overline{K}_1 + \overline{K}_2\right) + U_2 + U_3, \\
U_2 &= -\frac{\eta_3 \left(\overline{K}_3 - \overline{K}_1\right) + (\eta_3 - 1)\left(\overline{K}_4 - \overline{K}_2\right)}{\eta_3 + \gamma_3}, \\
U_3 &= \frac{-\gamma_3 \left(\overline{K}_3 - \overline{K}_5\right) - (\gamma_3 + 1)\, \overline{K}_4 \left(\frac{1-L}{l}\right)}{(\eta_3 + \gamma_3)\left(\frac{1-L}{l}\right)^{\eta_3}}, \\
U_4 &= -\frac{\overline{K}_4 \left(\frac{1-L}{l}\right) - \gamma_3 U_2 \left(\frac{1-L}{l}\right)^{-\gamma_3} + \eta_3 U_3 \left(\frac{1-L}{l}\right)^{\eta_3}}{\gamma_3 \left(\frac{1-L}{l}\right)^{-\gamma_3}}.
\end{aligned}
$$

To prove the monotonicity of $U(y)$, we first prove that $U_i < 0$, for $i = 1, \cdots, 4$. Substituting the expressions of $\overline{K}_1, \cdots, \overline{K}_5$ into U_1, U_2, U_3, we have

$$
\begin{aligned}
U_1 &= \frac{\phi + \overline{\theta}\delta (1-L) \left(\frac{1-L}{l}\right)^{-\eta_3}}{(\eta_3 + \gamma_3)} \left[\frac{\gamma_3}{\rho + \phi + \overline{\theta}\delta} - \frac{\gamma_3 + 1}{\rho + \phi + \overline{\theta}\delta - \mu}\right] < 0; \\
U_2 &= -\frac{\phi}{\eta_3 + \gamma_3} \left[\frac{\eta_3}{\rho + \phi + \overline{\theta}\delta} - \frac{\eta_3 - 1}{\rho + \phi + \overline{\theta}\delta - \mu}\right] < 0; \\
U_3 &= \frac{\overline{\theta}\delta (1-L)}{(\eta_3 + \gamma_3)\left(\frac{1-L}{l}\right)^{\eta_3}} \left[\frac{\gamma_3}{\rho + \phi + \overline{\theta}\delta} - \frac{\gamma_3 + 1}{\rho + \phi + \overline{\theta}\delta - \mu}\right] < 0.
\end{aligned}
$$

Lastly, from the above expression of U_3 and the result $U_2 < 0$, we have

$$
\gamma_3 \left(\frac{1-L}{l}\right)^{-\gamma_3} U_4 = -\left[\frac{(\eta_3 - 1)\, \overline{\theta}\delta (1-L)}{(\eta_3 + \gamma_3)\left(\rho + \phi + \overline{\theta}\delta - \mu\right)} - \gamma_3 U_2 \left(\frac{1-L}{l}\right)^{-\gamma_3}\right] < 0.
$$

Next, we prove $U(y)$ is monotonically increasing for $y > 0$. Note that $U'(y) = U_4 (-\gamma_3)\, y^{-\gamma_3 - 1} > 0$ for $y > \frac{1-L}{l}$ since $U_4 < 0$. Therefore, we only need to establish the monotonicity for $0 < y \leq \frac{1-L}{l}$. We prove it for the cases of $0 < y \leq 1$ and $1 < y \leq \frac{1-L}{l}$, respectively. For $0 < y \leq 1$, because $\left(\frac{1-L}{l}\right)^{-\eta_3} < \frac{l}{1-L}$ (or equivalently,

$\left(\frac{1-L}{l}\right)^{\eta_3-1} > 1)$

$$\begin{aligned}
& U'(y) = \overline{K}_2 + \eta_3 U_1 y^{\eta_3-1} \\
\geq\ & \overline{K}_2 + \eta_3 U_1 \\
=\ & \frac{\phi + \overline{\theta}\delta l}{\rho + \phi + \overline{\theta}\delta - \mu} + \eta_3 \frac{\phi + \overline{\theta}\delta\,(1-L)\left(\frac{1-L}{l}\right)^{-\eta_3}}{(\eta_3 + \gamma_3)} \left[\frac{\gamma_3}{\rho + \phi + \overline{\theta}\delta} - \frac{\gamma_3 + 1}{\rho + \phi + \overline{\theta}\delta - \mu}\right] \\
>\ & \frac{\phi + \overline{\theta}\delta l}{\rho + \phi + \overline{\theta}\delta - \mu} + \eta_3 \frac{\phi + \overline{\theta}\delta l}{(\eta_3 + \gamma_3)} \left[\frac{\gamma_3}{\rho + \phi + \overline{\theta}\delta} - \frac{\gamma_3 + 1}{\rho + \phi + \overline{\theta}\delta - \mu}\right] \\
=\ & \frac{\gamma_3\left(\phi + \overline{\theta}\delta l\right)}{\eta_3 + \gamma_3} \left[\frac{\eta_3}{\rho + \phi + \overline{\theta}\delta} - \frac{\eta_3 - 1}{\rho + \phi + \overline{\theta}\delta - \mu}\right] \\
>\ & 0;
\end{aligned}$$

and for $1 < y \leq \frac{1-L}{l}$,

$$\begin{aligned}
& U'(y) = \overline{K}_4 + (-\gamma_3)\, U_2 y^{-\gamma_3-1} + \eta_3 U_3 y^{\eta_3-1} \\
>\ & \overline{K}_4 - \gamma_3 U_2 + \eta_3 U_3 \left(\frac{1-L}{l}\right)^{\eta_3-1} \\
=\ & \frac{\overline{\theta}\delta l}{\rho + \phi + \overline{\theta}\delta - \mu} + \gamma_3 \frac{\phi}{\eta_3 + \gamma_3} \left[\frac{\eta_3}{\rho + \phi + \overline{\theta}\delta} - \frac{\eta_3 - 1}{\rho + \phi + \overline{\theta}\delta - \mu}\right] \\
& + \eta_3 \frac{\overline{\theta}\delta l}{(\eta_3 + \gamma_3)} \left[\frac{\gamma_3}{\rho + \phi + \overline{\theta}\delta} - \frac{\gamma_3 + 1}{\rho + \phi + \overline{\theta}\delta - \mu}\right] \\
=\ & \frac{\gamma_3\left(\phi + \overline{\theta}\delta l\right)}{\eta_3 + \gamma_3} \left[\frac{\eta_3}{\rho + \phi + \overline{\theta}\delta} - \frac{\eta_3 - 1}{\rho + \phi + \overline{\theta}\delta - \mu}\right] \\
>\ & 0.
\end{aligned}$$

■

The following lemmas are needed.

Lemma 2 *For η_i and γ_i, $i = 1, 2, 3$, defined in Appendix A, the following statements are true: (i)*

$$\frac{\eta_i \gamma_i}{\rho + \phi + \delta_i} = \frac{(\eta_i - 1)(\gamma_i + 1)}{\rho + \phi + \delta_i - \mu} = \frac{2}{\sigma^2},$$

(ii) Under the restriction $\rho + \phi > \mu$,

$$\chi_i \equiv \frac{\eta_i}{\rho + \phi + \delta_i} - \frac{\eta_i - 1}{\rho + \phi + \delta_i - \mu} > 0.$$

Proof of Lemma 2. (i) Note that $\eta_i\gamma_i = \frac{2}{\sigma^2}(\rho + \phi + \delta_i)$ and $\eta_i - \gamma_i = \frac{2}{\sigma^2}\left(\mu - \frac{1}{2}\sigma^2\right)$.

Thus, $\frac{\eta_i \gamma_i}{\rho+\phi+\delta_i} = \frac{2}{\sigma^2}$ and $\frac{(\eta_i-1)(\gamma_i+1)}{\rho+\phi+\delta_i-\mu} = \frac{\frac{2}{\sigma^2}(\rho+\phi+\delta_i) - \frac{2}{\sigma^2}\left(\mu-\frac{1}{2}\sigma^2\right)-1}{\rho+\phi+\delta_i-\mu} = \frac{2}{\sigma^2}$.

(ii) It is equivalent to proving the following inequality

$$\frac{\rho+\phi+\delta_i}{\mu} > \eta_i = \frac{\sqrt{\left(\frac{1}{2}\sigma^2-\mu\right)^2 + 2\sigma^2\left[\rho+\phi+\delta_i\right]} - \left(\mu - \frac{1}{2}\sigma^2\right)}{\sigma^2}$$

which holds as long as $\rho+\phi+\delta_i > \mu$. Therefore, under the restriction $\rho+\phi > \mu$, the statement indeed holds since $\delta_i \geq 0$, for $i = 1, 2, 3$. ■

Lemma 3 *Under the restrictions on parameter values, the function $W(y)$ is strictly increasing.*

Proof of Lemma 3. Because there are eight different cases and in each different case the function $W(y)$ takes a different form. Below we prove the monotonicity of $W(y)$ in each of the ten cases.

Define

$$y_{**}^C = \frac{\overline{\rho}+\phi-\overline{r}}{\phi} < 1,$$

and y_{**}^A, y_{**}^E as solutions to the following equations:

$$\begin{aligned}
\overline{r} &= \overline{\rho} + \phi\left(1-y_{**}^A\right) + \theta\delta\left(1-\left[L+ly_{**}^A\right]\right) + \kappa\delta\left(1-\left[\overline{K}_1+\overline{K}_2 y_{**}^A + U_1\left(y_{**}^A\right)^{\eta_3}\right]\right), \\
\overline{r} &= \overline{\rho} + \theta\delta\left(1-\left[L+ly_{**}^E\right]\right) + \kappa\delta\left(1-\left[\overline{K}_3+\overline{K}_4 y_{**}^E + U_2\left(y_{**}^E\right)^{-\gamma_3} + U_3\left(y_{**}^E\right)^{\eta_3}\right]\right).
\end{aligned}$$

(i) In Case A where $\overline{r} > \overline{\rho} + \phi(1-y) + \theta\delta(1-[L+ly]) + \kappa\delta(1-U(y))$, the function $W(y) = W_A(y)$, for $y \in (0,1]$

$$W_A(y) = \frac{\eta_1 K_7 + \gamma_2 K_8}{\eta_1+\gamma_2} + \frac{\eta_1+\gamma_1}{\eta_1+\gamma_2} A_2 y^{-\gamma_1}$$

where

$$A_2 = \frac{\eta_1\left(K_1+\overline{K}_6-K_7\right) + (\eta_1-1)\left(K_2+\overline{K}_7\right)y_{**}^A + (\eta_1-\eta_3)U_1\left(y_{**}^A\right)^{\eta_3}}{(\eta_1+\gamma_1)\left(y_{**}^A\right)^{-\gamma_1}}.$$

To prove $W_A(y)$ is strictly increasing, we only need to prove $A_2 < 0$. Substituting the expression of r_{**}^{A} into the above equation, we have

$$\begin{aligned}
& (\eta_1+\gamma_1)\left(y_{**}^{A}\right)^{-\gamma_1} A_2 \\
= \; & \eta_1 \frac{\overline{r}-\left[\overline{\rho}+\phi+\theta\delta\left(1-L\right)+\kappa\delta\left(1-\overline{K}_1\right)\right]}{\rho+\phi+\delta\left(1+\theta+\kappa\right)} \\
& +(\eta_1-1)\frac{\left(\phi+\theta\delta l+\kappa\delta\overline{K}_2\right)y_{**}^{A}}{\rho+\phi+\delta\left(1+\theta+\kappa\right)-\mu}+(\eta_1-\eta_3)U_1\left(y_{**}^{A}\right)^{\eta_3} \\
= \; & -\left(\phi+\theta\delta l+\kappa\delta\overline{K}_2\right)y_{**}^{A}\chi_1+\left[\eta_1-\eta_3-\frac{\kappa\delta\eta_1}{\rho+\phi+\delta\left(1+\theta+\kappa\right)}\right]U_1\left(y_{**}^{A}\right)^{\eta_3}.
\end{aligned}$$

From Condition (10)a and Lemma 2, we have

$$\frac{(\eta_1-1)\left(\phi+\theta\delta l+\kappa\delta\overline{K}_2\right)}{\rho+\phi+\delta\left(1+\theta+\kappa\right)-\mu}=\left(\phi+\theta\delta l+\kappa\delta\overline{K}_2\right)\gamma_1\chi_1>-\eta_3\left(\eta_1-\eta_3\right)U_1,$$

and, note that $\eta_1-\gamma_1=\eta_3-\gamma_3$ and $\frac{\eta_1\gamma_1}{(1+\theta+\kappa)}=\frac{\eta_3\gamma_3}{\rho+\phi+\overline{\theta}\delta}$,

$$\begin{aligned}
& \eta_3\left(\eta_1-\eta_3\right)+\gamma_1\left(\eta_1-\eta_3-\frac{\kappa\delta\eta_1}{\rho+\phi+\delta\left(1+\theta+\kappa\right)}\right) \\
= \; & \eta_3\left(\eta_1-\gamma_1-\eta_3\right)+\frac{\left(\rho+\phi+\delta\left(1+\theta\right)\right)\eta_1\gamma_1}{\rho+\phi+\delta\left(1+\theta+\kappa\right)} \\
= \; & -\eta_3\gamma_3+\frac{\left(\rho+\phi+\delta\left(1+\theta\right)\right)\eta_1\gamma_1}{\rho+\phi+\delta\left(1+\theta+\kappa\right)} \\
= \; & 0.
\end{aligned}$$

Therefore, since $0<\left(y_{**}^{A}\right)^{\eta_3}<y_{**}^{A}<1$, we have

$$\begin{aligned}
& (\eta_1+\gamma_1)\left(y_{**}^{A}\right)^{-\gamma_1} A_2 \\
< \; & \left[-\left(\phi+\theta\delta l+\kappa\delta\overline{K}_2\right)\chi_1+\left(\eta_1-\eta_3-\frac{\kappa\delta\eta_1}{\rho+\phi+\delta\left(1+\theta+\kappa\right)}\right)U_1\right]\left(y_{**}^{A}\right)^{\eta_3} \\
< \; & \left[\eta_3\left(\eta_1-\eta_3\right)+\gamma_1\left(\eta_1-\eta_3-\frac{\kappa\delta\eta_1}{\rho+\phi+\delta\left(1+\theta+\kappa\right)}\right)\right]U_1\frac{\left(y_{**}^{A}\right)^{\eta_3}}{\gamma_1} \\
= \; & 0.
\end{aligned}$$

(ii) In Case B where $\overline{\rho}+\phi\left(1-y\right)\leq\overline{r}\leq\overline{\rho}+\phi\left(1-y\right)+\theta\delta\left(1-\left[L+ly\right]\right)+\kappa\delta\left(1-U\left(y\right)\right)$, the function $W\left(y\right)=W_B\left(y\right)$, for $y\in(0,1]$

$$W_B\left(y\right)=\frac{\eta_1\left(K_1+\overline{K}_6\right)+\gamma_2K_8+\left(\eta_1-1\right)\left(K_2+\overline{K}_7\right)y}{\eta_1+\gamma_2}+\frac{\left(\eta_1-\eta_3\right)U_1y^{\eta_3}}{\eta_1+\gamma_2}.$$

Under Condition (10)a and $y \leq 1$, we have

$$\begin{aligned}
& \left(\eta_1+\gamma_2\right) W_B'(y) = \left(\eta_1-1\right)\left(K_2+\overline{K}_7\right)+\eta_3\left(\eta_1-\eta_3\right) U_1 y^{\eta_3-1} \\
\geq\ & \left(\eta_1-1\right)\left(K_2+\overline{K}_7\right)+\eta_3\left(\eta_1-\eta_3\right) U_1 \\
>\ & 0.
\end{aligned}$$

(iii) In Case C where $\overline{r}<\overline{\rho}+\phi(1-y)<\overline{\rho}+\phi$, the function $W(y)=W_C(y)$, for $y \in (0,1]$

$$\begin{aligned}
& W_C(y) \\
=\ & \frac{\gamma_2 K_5+\eta_1\left(K_1+\overline{K}_6\right)+\left(\gamma_2+1\right) K_6 y+\left(\eta_1-1\right)\left(K_2+\overline{K}_7\right) y}{\left(\eta_1+\gamma_2\right)} \\
& +\frac{\eta_1-\eta_3}{\eta_1+\gamma_2} U_1 y^{\eta_3}+\frac{\gamma_2\left(K_8-K_5\right)-\left(\gamma_2+1\right) K_6 y_{**}^C}{\left(\eta_1+\gamma_2\right)\left(y_{**}^C\right)^{\eta_2}} y^{\eta_2}.
\end{aligned}$$

When $\overline{r}<\overline{\rho}+\phi$, it is true that

$$\gamma_2\left(K_8-K_5\right)-\left(\gamma_2+1\right) K_6 y_{**}^C=(\overline{\rho}+\phi-\overline{r})\left[\frac{\gamma_2}{\rho+\phi}-\frac{\gamma_2+1}{\rho+\phi-\mu}\right]<0.$$

Furthermore, since $y<y_{**}^C<1$ and $U_1<0$, we have that under Condition (10)a,

$$\begin{aligned}
& \left(\eta_1+\gamma_2\right) W_C'(y) \\
=\ & \left(\eta_1-1\right)\left(K_2+\overline{K}_7\right)+\left(\gamma_2+1\right) K_6+\eta_3\left(\eta_1-\eta_3\right) U_1 y^{\eta_3-1} \\
& +\left[\gamma_2\left(K_8-K_5\right)-\left(\gamma_2+1\right) K_6 y_{**}^C\right] \eta_2 \frac{y^{\eta_2-1}}{\left(y_{**}^C\right)^{\eta_2}} \\
>\ & \left(\eta_1-1\right)\left(K_2+\overline{K}_7\right)+\left(\gamma_2+1\right) K_6+\eta_3\left(\eta_1-\eta_3\right) U_1\left(y_{**}^C\right)^{\eta_3-1} \\
& +\left[\gamma_2\left(K_8-K_5\right)-\left(\gamma_2+1\right) K_6 y_{**}^C\right] \frac{\eta_2}{y_{**}^C} \\
=\ & \left(\eta_1-1\right)\left(K_2+\overline{K}_7\right)+\eta_3\left(\eta_1-\eta_3\right) U_1\left(y_{**}^C\right)^{\eta_3-1} \\
>\ & \left(\eta_1-1\right)\left(K_2+\overline{K}_7\right)+\eta_3\left(\eta_1-\eta_3\right) U_1 \\
>\ & 0.
\end{aligned}$$

(iv) In Case D where $\overline{r} \geq \overline{\rho}+\theta\delta(1-L-l)+\kappa\delta(1-U(1))$, the function $W(y)=W_D(y)=W_A(y)$, for $y \in \left(1, \frac{1-L}{l}\right)$. The proof in (i) applies here too.

(v) In Case E where $\overline{\rho}+\theta\delta(1-L-ly)+\kappa\delta(1-U(y))<\overline{r}<\overline{\rho}+\theta\delta(1-L-l)+$

$\kappa\delta\left(1-U\left(1\right)\right)$, the function $W\left(y\right)=W_E\left(y\right)$ for $1<y_{**}^E<y<\frac{1-L}{l}$

$$W_E\left(y\right)=\frac{\eta_1 K_7+\gamma_2 K_8}{\eta_1+\gamma_2}+\frac{\eta_1+\gamma_1}{\eta_1+\gamma_2}E_4 y^{-\gamma_1},$$

where

$$\begin{aligned}
E_4 &= E_2+\frac{\left(y_{**}^E\right)^{\gamma_1}}{\left(\eta_1+\gamma_1\right)}\left[\begin{array}{c}\eta_1\left(K_3+\overline{K}_8-K_7\right)+\left(\eta_1-1\right)\left(K_4+\overline{K}_9\right)y_{**}^E\\ +\left(\eta_1+\gamma_3\right)\kappa U_2\left(y_{**}^E\right)^{-\gamma_3}+\left(\eta_1-\eta_3\right)\kappa U_3\left(y_{**}^E\right)^{\eta_3}\end{array}\right],\\
E_2 &= \frac{1}{\eta_1+\gamma_1}\left[\begin{array}{c}\eta_1\left(K_1+\overline{K}_6-K_3-\overline{K}_8\right)-\left(\eta_1-1\right)\left(K_4+\overline{K}_9-K_2-\overline{K}_7\right)\\ +\left(\eta_1-\eta_3\right)\kappa\left(U_1-U_3\right)-\left(\eta_1+\gamma_3\right)\kappa U_2\end{array}\right].
\end{aligned}$$

To prove the increasing monotonicity of $W_E\left(y\right)$, we only need to prove the coefficient of $y^{-\gamma_1}$ in $W_E\left(y\right)$ is negative, or $E_4<0$. It is straightforward to verify that $E_2=0$. As a result,

$$\begin{aligned}
& \left(\eta_1+\gamma_1\right)E_4\left(y_{**}^E\right)^{-\left(\gamma_1+1\right)}\\
= & -\left(\theta\delta l+\kappa\delta\overline{K}_4\right)\chi_1+\left(\eta_1+\gamma_3-\frac{\kappa\delta\eta_1}{\rho+\phi+\delta\left(1+\theta+\kappa\right)}\right)U_2\left(y_{**}^E\right)^{-\left(\gamma_3+1\right)}\\
& +\left(\eta_1-\eta_3-\frac{\kappa\delta\eta_1}{\rho+\phi+\delta\left(1+\theta+\kappa\right)}\right)U_3\left(y_{**}^E\right)^{\eta_3-1}\\
< & \left[-\left(\theta\delta l+\kappa\delta\overline{K}_4\right)\chi_1+\left(\eta_1-\eta_3-\frac{\kappa\delta\eta_1}{\rho+\phi+\delta\left(1+\theta+\kappa\right)}\right)U_3\left(\frac{1-L}{l}\right)^{\eta_3-1}\right]\left(\frac{y_{**}^E}{\left(1-L\right)/l}\right)^{\eta_3-1}\\
< & \ 0.
\end{aligned}$$

where we used the following fact based on a similar argument as in in the proof of (i) that under Condition (10)b

$$\begin{aligned}
& -\left(\theta\delta l+\kappa\delta\overline{K}_4\right)\chi_1+\left(\eta_1-\eta_3-\frac{\kappa\delta\eta_1}{\rho+\phi+\delta\left(1+\theta+\kappa\right)}\right)U_3\left(\frac{1-L}{l}\right)^{\eta_3-1}\\
< & \ \frac{1}{\gamma_1}\left[\eta_3\left(\eta_1-\eta_3\right)+\gamma_1\left(\eta_1-\eta_3-\frac{\kappa\delta\eta_1}{\rho+\phi+\delta\left(1+\theta+\kappa\right)}\right)\right]U_3\left(\frac{1-L}{l}\right)^{\eta_3-1}\\
= & \ 0.
\end{aligned}$$

(vi) In Case F where $\overline{r}\leq\overline{\rho}+\theta\delta\left(1-L-ly\right)+\kappa\delta\left(1-U\left(y\right)\right)$, the function $W\left(y\right)=W_F\left(y\right)$ for $y\in\left(1,\frac{1-L}{l}\right)$

$$W_F\left(y\right)=\frac{\left[\begin{array}{c}\eta_1\left(K_3+\overline{K}_8\right)+\gamma_2 K_8+\left(\eta_1-1\right)\left(K_4+\overline{K}_9\right)y\\ +\left(\eta_1+\gamma_3\right)U_2 y^{-\gamma_3}+\left(\eta_1-\eta_3\right)U_3 y^{\eta_3}\end{array}\right]}{\left(\eta_1+\gamma_2\right)}+\frac{\eta_1+\gamma_1}{\eta_1+\gamma_2}F_2 y^{-\gamma_1}$$

where $F_2 = E_2 = 0$. Under Condition (10)b, we have

$$\begin{aligned}
& (\eta_1 + \gamma_2) W_F'(y) \\
= \;& (\eta_1 - 1)\left(K_4 + \overline{K}_9\right) - \gamma_3(\eta_1 + \gamma_3) U_2 y^{-\gamma_3 - 1} + \eta_3(\eta_1 - \eta_3) U_3 y^{\eta_3 - 1} \\
> \;& \left[(\eta_1 - 1)\left(K_4 + \overline{K}_9\right) + \eta_3(\eta_1 - \eta_3) U_3 \left(\frac{1-L}{l}\right)^{\eta_3 - 1}\right]\left(\frac{y}{(1-L)/l}\right)^{\eta_3 - 1} \\
> \;& 0.
\end{aligned}$$

(vii) In Case G where $\overline{r} \geq \overline{\rho} + \theta\delta(1 - L - l) + \kappa\delta(1 - U(1))$, the function $W(y) = W_G(y) = W_A(y)$ for $y \geq \frac{1-L}{l}$. The proof of (i) applies here too.

(viii) In Case H where $\overline{\rho} + \kappa\delta\left(1 - U\left(\frac{1-L}{l}\right)\right) \leq \overline{r} < \overline{\rho} + \theta\delta(1 - L - l) + \kappa\delta(1 - U(1))$, the function $W(y) = W_H(y) = W_E(y)$ for $y \geq \frac{1-L}{l}$. The proof of (v) applies here too. ■

Proof of Theorem 1. The equilibrium threshold y_* is determined by the condition $V(y_*; y_*) = 1$. Define $W(y_*) \equiv V(y_*; y_*)$. Here we prove that there always exists a unique y_* such that $W(y_*) = 1$. To simplify notation, we replace y_* by y and express $W(y_*)$ as $W(y)$ throughout the proof. It is easy to show that under the parameter restriction (9), $W_C(0) < W_B(0) < 1$, $W_A(\infty) > 1$, and $W_E(\infty) > 1$.

Denote by $y_{**} = \max\{y : R(y; y_*) = \overline{r}\}$ the maximum fundamental value that is associated with the max rate. That is, the constraint of the max rate is binding if and only if $y \leq y_{**}$. It is straightforward to see that in Case B or Case F, y_{**} coincides with y_* (i.e., $y_{**} = y_*$), and in Case C, $y_{**}^C \equiv \frac{\overline{\rho} + \phi - \overline{r}}{\phi} \leq 1$. For the other cases, y_{**} is determined by $f(y_{**}) = 0$ where the function $f(\cdot)$ is defined as

$$f(y) = \overline{\rho} + \phi(1 - y)^+ + \theta\delta(1 - L - ly)^+ + \kappa\delta(1 - U(y)) - \overline{r}.$$

Then from Lemma 1, $f(y)$ is continuous and strictly decreasing. Furthermore, under the parameter restrictions (8) and (9), we have $f(0) > 0$ and $f\left(\frac{1-L}{l}\right) \leq 0.$, implying that $f(y_{**}) = 0$ has a unique solution $y_{**} \in (0, \frac{1-L}{l}]$. It is straightforward to check that $W_B\left(y_{**}^C\right) = W_C\left(y_{**}^C\right)$, $W_A(y_{**}) = W_B(y_{**})$, $W_F(y_{**}) = W_E(y_{**})$, and $W_B(1) = W_F(1)$.

We now prove the existence of the unique threshold y_* by considering all the possible max rates $\overline{r}$. Under the restriction (6), $\overline{\rho} + \phi < \overline{\rho} + \theta\delta(1 - L - l) + \kappa\delta(1 - U(1))$. There are three possibilities.

(i) Consider the possibility where $\overline{r} \geq \overline{\rho} + \theta\delta(1 - L - l) + \kappa\delta(1 - U(1))$, implying $f(1) \leq 0$ and $y_{**} \in (0, 1]$. Based on the strict monotonicity of W_A and W_B, as well as $y_{**} \leq 1$, we have

$$W_B(0) < W_A(y_{**}) = W_B(y_{**}) \leq W_A(1) < W_A(\infty).$$

If $W_A(1) < 1$, then Case D or Case G holds (note $W_A(\infty) > 1$) where $W_A(y) = 1$ has a unique root $y > 1$, depending on whether $W_A\left(\frac{1-L}{l}\right) \geq 1$ or not. Otherwise, if $W_A(1) \geq 1$, depending on whether $W_A(y_{**}) = W_B(y_{**}) < 1$ or not, either Case A holds where $W_A(y) = 1$ has a unique root $y \in (y_{**}, 1]$, or Case B holds where $W_B(y) = 1$ has a unique root $y \in (0, y_{**}]$.

(ii) Consider the possibility where $\overline{\rho} + \phi \leq \overline{r} < \overline{\rho} + \theta\delta(1 - L - l) + \kappa\delta(1 - U(1))$, implying $f(1) > 0$ and $y_{**} \in (1, \frac{1-L}{l}]$. Based on the strict monotonicity of W_B, W_E, and W_F, and $y_{**} > 1$, we know

$$W_B(1) = W_F(1) < W_F(y_{**}) = W_E(y_{**}).$$

If $W_E(y_{**}) < 1$, then Case E or Case H holds (note $W_E(\infty) > 1$) where $W_E(y) = 1$ has a unique root $y > y_{**}$, depending on whether $W_E\left(\frac{1-L}{l}\right) \geq 1$ or not. Otherwise, if $W_E(y_{**}) = W_F(y_{**}) \geq 1$, depending on whether $W_B(1) = W_F(1) < 1$ or not, either Case F holds where $W_F(y) = 1$ has a unique root $y \in (1, y_{**}]$, or Case B holds where $W_B(y) = 1$ has a unique root $y \in (0, 1]$.

(iii) Consider the possibility where $\overline{r} < \overline{\rho} + \phi$, implying $0 < y_{**}^C \leq 1$ and $y_{**} \in (1, \frac{1-L}{l}]$. Based on the strict monotonicity of W_B and W_F, as well as $y_{**} > 1$, we have

$$W_C(0) < W_B\left(y_{**}^C\right) = W_C\left(y_{**}^C\right) < W_B(1) = W_F(1) < W_F(y_{**}) = W_E(y_{**}).$$

If $W_C\left(y_{**}^C\right) \geq 1$, then Case C holds (note $W_C(0) < 1$) where $W_C(y) = 1$ has a unique solution $y \in (0, y_{**}^C]$. Otherwise, if $W_C\left(y_{**}^C\right) < 1$, by the same argument used in Possibility (ii), we can prove that Case B holds if $W_B(1) \geq 1$, or Case E or Case H holds if $W_B(1) < 1$ and $W_E(y_{**}) < 1$, or Case F holds if $W_B(1) < 1$ and $W_E(y_{**}) \geq 1$. ■

Proof of Proposition 1. (i) We first prove $\frac{dy_*}{d\overline{r}} < 0$. By the implicit function theorem, $\frac{dy_*}{d\overline{r}} = -\frac{\partial W/\partial \overline{r}}{\partial W/\partial y_*}$. We have shown in Lemma 3 that $\partial W/\partial y > 0$. Therefore, we only need to show that $\partial W/\partial \overline{r} > 0$ for each of functions $W_A(y), \cdots, W_H(y)$ in order to prove the claim. From Lemma 2, we have

$$\begin{aligned}
\frac{\partial W_A(y)}{\partial \overline{r}} &= \frac{y^{-\gamma_1}}{(\eta_1 + \gamma_2)} \frac{\partial \left[\eta_1 (K_1 - K_7)\left(y_{**}^A\right)^{\gamma_1} + (\eta_1 - 1) K_2 \left(y_{**}^A\right)^{\gamma_1 + 1}\right]}{\partial \overline{r}} \\
&= \frac{(\gamma_1 + 1) y^{-\gamma_1}\left(y_{**}^A\right)^{\gamma_1}}{(\eta_1 + \gamma_2)} \left[\frac{\eta_1}{\rho + \phi + \delta(1 + \theta)} - \frac{\eta_1 - 1}{\rho + \phi + \delta(1 + \theta) - \mu}\right] \\
&> 0.
\end{aligned}$$

When $\overline{r} < \overline{\rho} + \phi$, for $y < y_{**}^{C}$, from Lemma 2, we have

$$\begin{aligned}
& \frac{\partial W_C(y)}{\partial \overline{r}} = \frac{\partial \left[\frac{\eta_1 K_1 + \gamma_2 K_5}{\eta_1 + \gamma_2} + \frac{\gamma_2 (K_8 - K_5) - (\gamma_2 + 1) K_6 y_{**}^{C}}{(\eta_1 + \gamma_2)(y_{**}^{C})^{\eta_2}} y^{\eta_2} \right]}{\partial \overline{r}} \\
= \;& \frac{1}{\eta_1 + \gamma_2} \left[\frac{\eta_1}{\rho + \phi + \delta(1+\theta)} + \frac{\gamma_2}{\rho + \phi} \right] + \frac{(\eta_2 - 1) y^{\eta_2} \left(y_{**}^{C}\right)^{-\eta_2}}{\eta_1 + \gamma_2} \left[\frac{\gamma_2}{\rho + \phi} - \frac{\gamma_2 + 1}{\rho + \phi - \mu} \right] \\
> \;& \frac{1}{\eta_1 + \gamma_2} \left[\frac{\eta_1}{\rho + \phi + \delta(1+\theta)} + \frac{\gamma_2}{\rho + \phi} \right] + \frac{\eta_2 - 1}{\eta_1 + \gamma_2} \left[\frac{\gamma_2}{\rho + \phi} - \frac{\gamma_2 + 1}{\rho + \phi - \mu} \right] \\
= \;& \frac{1}{\eta_1 + \gamma_2} \frac{\eta_1}{\rho + \phi + \delta(1+\theta)} \\
> \;& 0.
\end{aligned}$$

When $\overline{r} < \overline{\rho} + \theta\delta(1 - L - l)$, from Lemma 2, we have

$$\begin{aligned}
& \frac{\partial W_E(y)}{\partial \overline{r}} = \frac{y^{-\gamma_1}}{\eta_1 + \gamma_2} \frac{\partial \left[\eta_1 (K_3 - K_7) \left(y_{**}^{E}\right)^{\gamma_1} + (\eta_1 - 1) K_4 \left(y_{**}^{E}\right)^{1+\gamma_1} \right]}{\partial \overline{r}} \\
= \;& \frac{(\gamma_1 + 1) y^{-\gamma_1} \left(y_{**}^{E}\right)^{\gamma_1}}{\eta_1 + \gamma_2} \left[\frac{\eta_1}{\rho + \phi + \delta(1+\theta)} - \frac{(\eta_1 - 1)}{\rho + \phi + \delta(1+\theta) - \mu} \right] \\
> \;& 0.
\end{aligned}$$

Lastly, $\frac{\partial W_B(y)}{\partial \overline{r}} = \frac{\partial W_F(y)}{\partial \overline{r}} = \frac{\eta_1}{\eta_1 + \gamma_2} \frac{1}{\rho + \phi + \delta(1+\theta)} > 0$.

(ii) Next, we prove $\frac{dy_*}{d\Delta} < 0$. By the similar argument as in (i), we only need to show that $\partial W / \partial \Delta > 0$ for each of functions $W_A(y), \cdots, W_H(y)$. From Lemma 2, we have

$$\begin{aligned}
& \frac{\partial W_A(y)}{\partial \Delta} \\
= \;& \frac{\eta_1}{\eta_1 + \gamma_2} \frac{1}{\rho + \phi + \delta(1+\theta)} + \frac{\gamma_2}{\eta_1 + \gamma_2} \frac{1}{\rho + \phi} \\
& - \frac{(\gamma_1 + 1) y^{-\gamma_1} \left(y_{**}^{A}\right)^{\gamma_1}}{(\eta_1 + \gamma_2)} \left(\frac{\eta_1}{\rho + \phi + \delta(1+\theta)} - \frac{\eta_1 - 1}{\rho + \phi + \delta(1+\theta) - \mu} \right) \\
> \;& \frac{\eta_1}{\eta_1 + \gamma_2} \frac{1}{\rho + \phi + \delta(1+\theta)} + \frac{\gamma_2}{\eta_1 + \gamma_2} \frac{1}{\rho + \phi} \\
& - \frac{\gamma_1 + 1}{\eta_1 + \gamma_2} \left(\frac{\eta_1}{\rho + \phi + \delta(1+\theta)} - \frac{\eta_1 - 1}{\rho + \phi + \delta(1+\theta) - \mu} \right) \\
= \;& \frac{\gamma_2}{\eta_1 + \gamma_2} \frac{1}{\rho + \phi} \\
> \;& 0;
\end{aligned}$$

and when $\overline{r} < \overline{\rho} + \phi$, for $y < y_{**}^{C}$, $\frac{\partial W_C(y)}{\partial \Delta} = \frac{(\eta_2 - 1) y^{\eta_2} \left(y_{**}^{C}\right)^{-\eta_2}}{\eta_1 + \gamma_2} \left(\frac{\gamma_2 + 1}{\rho + \phi - \mu} - \frac{\gamma_2}{\rho + \phi} \right) > 0$; and furthermore, when $\overline{r} < \overline{\rho} + \theta\delta\left(1 - L - l\right)$, we have

$$
\begin{aligned}
& \frac{\partial W_E\left(y\right)}{\partial \Delta} \\
= \; & \frac{\eta_1}{\eta_1 + \gamma_2} \frac{1}{\rho + \phi + \delta\left(1 + \theta\right)} + \frac{\gamma_2}{\eta_1 + \gamma_2} \frac{1}{\rho + \phi} \\
& - \frac{\left(\gamma_1 + 1\right) y^{-\gamma_1} \left(y_{**}^{E}\right)^{\gamma_1}}{\eta_1 + \gamma_2} \left[\frac{\eta_1}{\rho + \phi + \delta\left(1 + \theta\right)} - \frac{\left(\eta_1 - 1\right)}{\rho + \phi + \delta\left(1 + \theta\right) - \mu} \right] \\
> \; & \frac{\eta_1}{\eta_1 + \gamma_2} \frac{1}{\rho + \phi + \delta\left(1 + \theta\right)} + \frac{\gamma_2}{\eta_1 + \gamma_2} \frac{1}{\rho + \phi} \\
& - \frac{\gamma_1 + 1}{\eta_1 + \gamma_2} \left[\frac{\eta_1}{\rho + \phi + \delta\left(1 + \theta\right)} - \frac{\left(\eta_1 - 1\right)}{\rho + \phi + \delta\left(1 + \theta\right) - \mu} \right] \\
= \; & \frac{\gamma_2}{\eta_1 + \gamma_2} \frac{1}{\rho + \phi} \\
> \; & 0.
\end{aligned}
$$

Lastly, $\frac{\partial W_B(y)}{\partial \Delta} = \frac{\partial W_F(y)}{\partial \Delta} = \frac{\gamma_2}{\eta_1 + \gamma_2} \frac{1}{\rho + \phi} > 0$.

(iii) Lastly, we prove by contradiction that $\underset{\Delta \to 0}{lim} y_* = \infty$. Suppose $lim_{\overline{\rho} \searrow \rho} y_* = y_0 < \infty$. Consider two cases: $\overline{r} > \rho + \theta\delta\left(1 - L - l\right)$ and $\overline{r} \leq \rho + \theta\delta\left(1 - L - l\right)$.

(iii-A) If $\overline{r} > \rho + \theta\delta\left(1 - L - l\right)$, then $lim_{\overline{\rho} \searrow \rho} W_A\left(y_*\right) = 1 = lim_{\overline{\rho} \searrow \rho} W_A\left(y_0\right)$. However, note that when $\overline{\rho}$ tends to ρ, K_7 and K_8 tends to 1. Therefore, we have

$$
\begin{aligned}
& lim_{\overline{\rho} \searrow \rho} W_A\left(y_0\right) \\
= \; & 1 + lim_{\overline{\rho} \searrow \rho} \frac{\eta_1\left(K_7 - 1\right) + \gamma_2\left(K_8 - 1\right)}{\eta_1 + \gamma_2} + \frac{\eta_1\left(K_1 - K_7\right) + \left(\eta_1 - 1\right) K_2 y_{**}^{A}}{\left(\eta_1 + \gamma_2\right)\left(y_{**}^{A}\right)^{-\gamma_1}} y_0^{-\gamma_1} \\
= \; & 1 + \frac{\eta_1\left(K_1 - K_7\right) + \left(\eta_1 - 1\right) K_2 y_{**}^{A}}{\left(\eta_1 + \gamma_2\right)\left(y_{**}^{A}\right)^{-\gamma_1}} y_0^{-\gamma_1} < 1,
\end{aligned}
$$

which is a contradiction.

(iii-B) If $\overline{r} \leq \rho + \theta\delta(1 - L - l)$, then $lim_{\overline{\rho}\searrow\rho} W_B(1) < 1$ because

$$\begin{aligned}
W_B(1) - 1 &= \frac{\eta_1 (K_1 - 1) + \gamma_2 (K_8 - 1) + (\eta_1 - 1) K_2}{\eta_1 + \gamma_2} \\
\rightarrow & \quad \frac{\eta_1}{\eta_1 + \gamma_2} \left(\frac{\overline{r} + \delta(1 + \theta L)}{\rho + \phi + \delta(1 + \theta)} - 1 \right) + \frac{\eta_1 - 1}{\eta_1 + \gamma_2} \frac{\phi + \theta\delta l}{\rho + \phi + \delta(1 + \theta) - \mu} \\
= & \quad -\frac{\eta_1}{\eta_1 + \gamma_2} \left(\frac{(\rho + \theta\delta(1 - L - l)) - \overline{r}}{\rho + \phi + \delta(1 + \theta)} \right) \\
& \quad -\frac{\phi + \theta\delta l}{\eta_1 + \gamma_2} \left(\frac{\eta_1}{\rho + \phi + \delta(1 + \theta)} - \frac{\eta_1 - 1}{\rho + \phi + \delta(1 + \theta) - \mu} \right) \\
< & \quad 0.
\end{aligned}$$

Similarly as before, we can prove that in this case, $lim_{\overline{\rho}\searrow\rho} y_* = \infty$. We can prove it by contradiction. Suppose $lim_{\overline{\rho}\searrow\rho} y_* = y_0 < \infty$. Then $lim_{\overline{\rho}\searrow\rho} W_E(y_*) = 1 = lim_{\overline{\rho}\searrow\rho} W_E(y_0)$. However,

$$lim_{\overline{\rho}\searrow\rho} W_E(y_0) = 1 + \left[\frac{\eta_1 + \gamma_1}{\eta_1 + \gamma_2} E_2 + \frac{\eta_1 (K_3 - K_7) + (\eta_1 - 1) K_4 y_{**}^E}{(\eta_1 + \gamma_2)(y_{**}^E)^{-\gamma_1}} \right] y_0^{-\gamma_1} < 1,$$

which is a contradiction. ■

Proof of Proposition 2. There are only three possibilities: $y_* < 1$, $1 \leq y_* \leq \frac{1-L}{l}$, and $y_* > \frac{1-L}{l}$:

1. If $y_* \leq 1$, then the value function is given by

$$V(y; y_*) = \begin{cases} \left(K_1 + \overline{K}_6\right) + \left(K_2 + \overline{K}_7\right) y + U_1 y^{\eta_3} + A_1 y^{\eta_1}, & \text{if } y \in (0, y_*] \\ K_5 + K_6 y + A_2 y^{-\gamma_2} + A_3 y^{\eta_2}, & \text{if } y \in (y_*, 1] \\ K_{10} + A_4 y^{-\gamma_2}, & \text{if } y \in (1, \infty) \end{cases}$$

2. If $1 < y_* \leq \frac{1-L}{l}$, then the value function is given by

$$V(y; y_*) = \begin{cases} \left(K_1 + \overline{K}_6\right) + \left(K_2 + \overline{K}_7\right) y + U_1 y^{\eta_3} + B_1 y^{\eta_1}, & \text{if } y \in (0, 1] \\ \left[\begin{array}{c} \left(K_3 + \overline{K}_8\right) + \left(K_4 + \overline{K}_9\right) y + U_2 y^{-\gamma_3} \\ + U_3 y^{\eta_3} + B_2 y^{-\gamma_1} + B_3 y^{\eta_1} \end{array} \right], & \text{if } y \in (1, y_*] \\ K_{10} + B_4 y^{-\gamma_2}, & \text{if } y \in (y_*, \infty) \end{cases}$$

3. If $y_* > \frac{1-L}{l}$, then the value function is given by

$$V(y; y_*) = \begin{cases} \left(K_1 + \overline{K}_6\right) + \left(K_2 + \overline{K}_7\right) y + U_1 y^{\eta_3} + C_1 y^{\eta_1}, & \text{if } y \in (0, 1] \\ \begin{bmatrix} \left(K_3 + \overline{K}_8\right) + \left(K_4 + \overline{K}_9\right) y + U_2 y^{-\gamma_3} \\ +U_3 y^{\eta_3} + C_2 y^{-\gamma_1} + C_3 y^{\eta_1} \end{bmatrix}, & \text{if } y \in (1, \frac{1-L}{l}] \\ \left(K_9 + \overline{K}_{10}\right) + U_4 y^{-\gamma_3} + C_4 y^{-\gamma_1} + C_5 y^{\eta_1}, & \text{if } y \in (\frac{1-L}{l}, y_*] \\ K_{10} + C_6 y^{-\gamma_2}, & \text{if } y \in (y_*, \infty) \end{cases}$$

where the unknown coefficients $A_1, \cdots, C_6$ are determined through the value matching and smooth pasting conditions.

Similarly as the proof of Proposition 1, to prove $\frac{dy_*}{d\kappa} > 0$, we only need to prove $\frac{\partial W(y)}{\partial \kappa} < 0$ for Case A, Case B, and Case C.

For simplicity, we only provide the proof for Case A: $\frac{\partial W_A(y)}{\partial \kappa} < 0$ for $0 < y \leq 1$. The proof for the other two cases is similar. In Case A,

$$\begin{aligned} & W_A(y) \\ = & \frac{\eta_1 \left(K_1 + \overline{K}_6\right) + \gamma_2 K_5}{\eta_1 + \gamma_2} + \frac{(\eta_1 - 1)\left(K_2 + \overline{K}_7\right) + (1 + \gamma_2) K_6}{\eta_1 + \gamma_2} y \\ & + \frac{\gamma_2 (K_{10} - K_5) - (1 + \gamma_2) K_6}{\eta_1 + \gamma_2} y^{\eta_2} + \frac{\eta_1 - \eta_3}{\eta_1 + \gamma_2} U_1 y^{\eta_3} \\ = & I_A + II_A y + III_A y^{\eta_2} + IV_A y^{\eta_3} \end{aligned}$$

Note that $\left.\frac{\partial IV_A}{\partial \kappa}\right|_{\kappa=0} = \frac{U_1}{\eta_1 + \gamma_2} \left.\frac{\partial \eta_1}{\partial \kappa}\right|_{\kappa=0} < 0$. Below we consider the first three terms. First, it is straightforward to show that

$$\begin{aligned} \frac{\partial \left(K_1 + \overline{K}_6\right)}{\partial \kappa} &= \frac{\partial \left(K_3 + \overline{K}_8\right)}{\partial \kappa} = -\frac{\delta^2 (1 - L)}{\left(\rho + \phi + \delta (1 + \theta + \kappa)\right)^2}, \\ \frac{\partial \left(K_2 + \overline{K}_7\right)}{\partial \kappa} &= \frac{\partial \left(K_4 + \overline{K}_9\right)}{\partial \kappa} = \frac{\delta^2 l}{\left(\rho + \phi + \delta (1 + \theta + \kappa) - \mu\right)^2}, \\ \frac{\partial \left(K_9 + \overline{K}_{10}\right)}{\partial \kappa} &= 0. \end{aligned}$$

Therefore,

$$
\begin{aligned}
& \left.\frac{\partial W_A(y)}{\partial \kappa}\right|_{\kappa=0} \\
< & \left[\frac{\gamma_2\left(K_1+K_2 y\right)}{\left(\eta_1+\gamma_2\right)^2}-\frac{\gamma_2\left(K_{10}-K_5\right)-\left(1+\gamma_2\right) K_6}{\left(\eta_1+\gamma_2\right)^2} y^{\eta_2}\right] \frac{\partial \eta_1}{\partial \kappa} \\
& +\frac{\eta_1}{\eta_1+\gamma_2}\left(-\frac{\delta^2(1-L)}{(\rho+\phi+\delta(1+\theta))^2}+\frac{\delta^2 l}{(\rho+\phi+\delta(1+\theta)-\mu)^2} y\right) .
\end{aligned}
$$

Because $\gamma_2\left(K_{10}-K_5\right)-\left(1+\gamma_2\right) K_6=\phi\left(\frac{\gamma_2}{\rho+\phi}-\frac{1+\gamma_2}{\rho+\phi-\mu}\right)<0$ and $y \leq 1$, we have

$$
\begin{aligned}
& \left.\frac{\partial W_A(y)}{\partial \kappa}\right|_{\kappa=0} \\
\leq & \frac{\gamma_2\left(K_1+K_2\right)-\left(\gamma_2\left(K_{10}-K_5\right)-\left(1+\gamma_2\right) K_6\right)}{\left(\eta_1+\gamma_2\right)^2} \frac{\partial \eta_1}{\partial \kappa} \\
& +\frac{\eta_1}{\eta_1+\gamma_2}\left(\frac{\delta^2 l}{(\rho+\phi+\delta(1+\theta)-\mu)^2}-\frac{\delta^2(1-L)}{(\rho+\phi+\delta(1+\theta))^2}\right) .
\end{aligned}
$$

Therefore, for $\left.\frac{\partial W_A(y)}{\partial \kappa}\right|_{\kappa=0}<0$, it is sufficient to have

$$
\begin{aligned}
K_1+K_2 < & \frac{\gamma_2\left(K_{10}-K_5\right)-\left(1+\gamma_2\right) K_6}{\gamma_2} \\
& +\frac{\eta_1\left(\eta_1+\gamma_2\right) \delta^2}{\gamma_2}\left(\frac{1-L}{(\rho+\phi+\delta(1+\theta))^2}-\frac{l}{(\rho+\phi+\delta(1+\theta)-\mu)^2}\right),
\end{aligned}
$$

which imposes an upper bound on $\bar{r}$.

In Case B,

$$
\begin{aligned}
W_B(y) = & \frac{\eta_1\left(K_3+\overline{K}_8\right)+\gamma_2 K_{10}}{\eta_1+\gamma_2}+\frac{\left(\eta_1-1\right)\left(K_4+\overline{K}_9\right)}{\eta_1+\gamma_2} y \\
& +\frac{\eta_1\left(\overline{K}_3-\overline{K}_1\right)+\left(\eta_1-1\right)\left(\overline{K}_4-\overline{K}_2\right)}{\eta_1+\gamma_2} y^{-\gamma_1} \\
& +\frac{\left(\eta_1+\gamma_3\right) U_2}{\eta_1+\gamma_2} y^{-\gamma_3}+\frac{\left(\eta_1-\eta_3\right) U_3}{\eta_1+\gamma_2} y^{\eta_3} \\
= & I_B+I I_B y+I I I_B y^{-\gamma_1}+I V_B y^{-\gamma_3}+V_B y^{\eta_3}
\end{aligned}
$$

$$
\begin{aligned}
& \left.\frac{\partial W_B}{\partial \kappa}\right|_{\kappa=0} \\
= & \frac{\partial\left(\frac{\eta_1}{\eta_1+\gamma_2}\right)}{\partial \kappa}\left(K_3+K_4 y+\left(\overline{K}_3-\overline{K}_1+\overline{K}_4-\overline{K}_2\right) y^{-\gamma_1}\right) \\
& +\frac{\eta_1}{\eta_1+\gamma_2} \frac{\partial\left(K_3+\overline{K}_8\right)}{\partial \kappa}+\frac{\eta_1-1}{\eta_1+\gamma_2} \frac{\partial\left(K_4+\overline{K}_9\right)}{\partial \kappa} y \\
& -\frac{\eta_1\left(\overline{K}_3-\overline{K}_1\right)+\left(\eta_1-1\right)\left(\overline{K}_4-\overline{K}_2\right)}{\eta_1+\gamma_2} \log (y) \frac{\partial \gamma_1}{\partial \kappa} y^{-\gamma_1} \\
= & \frac{\gamma_2}{\left(\eta_1+\gamma_2\right)^2} \frac{\partial \eta_1}{\partial \kappa}\left(K_3+K_4 y+\left(\overline{K}_3-\overline{K}_1+\overline{K}_4-\overline{K}_2\right) y^{-\gamma_1}\right) \\
& -\delta^2 \frac{\eta_1}{\eta_1+\gamma_2} \frac{1-L}{(\rho+\phi+\delta(1+\theta+\kappa))^2}+\delta^2 \frac{\eta_1-1}{\eta_1+\gamma_2} \frac{l}{(\rho+\phi+\delta(1+\theta+\kappa)-\mu)^2} y \\
& -\frac{\eta_1\left(\overline{K}_3-\overline{K}_1\right)+\left(\eta_1-1\right)\left(\overline{K}_4-\overline{K}_2\right)}{\eta_1+\gamma_2} \log (y) y^{-\gamma_1} \frac{\partial \eta_1}{\partial \kappa} \\
< & \frac{\gamma_2}{\left(\eta_1+\gamma_2\right)^2} \frac{\partial \eta_1}{\partial \kappa}\left(K_3+K_4 \frac{1-L}{l}+\left(\overline{K}_3-\overline{K}_1+\overline{K}_4-\overline{K}_2\right)\left(\frac{1-L}{l}\right)^{-\gamma_1}\right) \\
& -\delta^2 \frac{\eta_1}{\eta_1+\gamma_2} \frac{1-L}{(\rho+\phi+\delta(1+\theta+\kappa))^2}+\delta^2 \frac{\eta_1-1}{\eta_1+\gamma_2} \frac{1-L}{(\rho+\phi+\delta(1+\theta+\kappa)-\mu)^2}
\end{aligned}
$$

where we used the fact $1<y \leq \frac{1-L}{l}$,

$$
\begin{aligned}
\overline{K}_3-\overline{K}_1+\overline{K}_4-\overline{K}_2 & =\frac{\phi}{\rho+\phi+\overline{\theta} \delta}-\frac{\phi}{\rho+\phi+\overline{\theta} \delta-\mu}<0 \\
\eta_1\left(\overline{K}_3-\overline{K}_1\right)+\left(\eta_1-1\right)\left(\overline{K}_4-\overline{K}_2\right) & =\phi\left(\frac{\eta_1}{\rho+\phi+\overline{\theta} \delta}-\frac{\eta_1-1}{\rho+\phi+\overline{\theta} \delta-\mu}\right)>0
\end{aligned}
$$

Therefore, for $\left.\frac{\partial W_B(y)}{\partial \kappa}\right|_{\kappa=0}<0$, it is sufficient to have

$$
\frac{\eta_1}{(\rho+\phi+\delta(1+\theta))^2}>\frac{\eta_1-1}{(\rho+\phi+\delta(1+\theta)-\mu)^2}
$$

and

$$
\begin{aligned}
& K_3 + K_4 \frac{1-L}{l} \\
< & \left(\frac{\phi}{\rho+\phi+\overline{\theta}\delta-\mu} - \frac{\phi}{\rho+\phi+\overline{\theta}\delta}\right)\left(\frac{1-L}{l}\right)^{-\gamma_1} \\
& + \frac{\delta^2 (1-L)}{\frac{\partial \eta_1}{\partial \kappa}} \frac{\eta_1+\gamma_2}{\gamma_2}\left(\frac{\eta_1}{\left(\rho+\phi+\delta(1+\theta)\right)^2} - \frac{\eta_1 - 1}{\left(\rho+\phi+\delta(1+\theta)-\mu\right)^2}\right)
\end{aligned}
$$

which imposes an upper bound on μ and $\overline{r}$.

In Case C,

$$
\begin{aligned}
W_C(y) &= \frac{\eta_1\left(K_9+\overline{K}_{10}\right)+\gamma_2 K_{10}}{(\eta_1+\gamma_2)} \\
&\quad + \left[-\frac{\eta_1\left(K_3+\overline{K}_8-K_1-\overline{K}_6\right)+(\eta_1-1)\left(K_4+\overline{K}_9-K_2-\overline{K}_7\right)}{\eta_1+\gamma_2}\right] y^{-\gamma_1} \\
&\quad + \frac{\eta_1\left(K_3+\overline{K}_8-K_9-\overline{K}_{10}\right)+(\eta_1-1)\left(K_4+\overline{K}_9\right)\left(\frac{1-L}{l}\right)}{(\eta_1+\gamma_2)}\left(\frac{y}{(1-L)/l}\right)^{-\gamma_1} \\
&\quad + \frac{(\eta_1+\gamma_3)U_4}{(\eta_1+\gamma_2)} y^{-\gamma_3} - \frac{(\eta_1+\gamma_3)U_2+(\eta_1-\eta_3)(U_3-U_1)}{\eta_1+\gamma_2} y^{-\gamma_1} \\
&\quad + \frac{(\eta_1+\gamma_3)(U_2-U_4)\left(\frac{1-L}{l}\right)^{-\gamma_3}+(\eta_1-\eta_3)U_3\left(\frac{1-L}{l}\right)^{\eta_3}}{(\eta_1+\gamma_2)\left(\frac{1-L}{l}\right)^{-\gamma_1}} y^{-\gamma_1} \\
&= I_C + II_C y^{-\gamma_1} + III_C\left(\frac{y}{(1-L)/l}\right)^{-\gamma_1} + IV_C
\end{aligned}
$$

where we used the facts

$$
\frac{\partial\left(K_4+\overline{K}_9-K_2-\overline{K}_7\right)}{\partial\kappa} = \frac{\partial\left(K_3+\overline{K}_8-K_1-\overline{K}_6\right)}{\partial\kappa} = \frac{\partial\left(K_9+\overline{K}_{10}\right)}{\partial\kappa} = 0
$$

$$
\begin{aligned}
\eta_1(K_3-K_1)+(\eta_1-1)(K_4-K_2) &= \phi\chi_1 > 0 \\
\eta_1(K_3-K_9)+(\eta_1-1)K_4\left(\frac{1-L}{l}\right) &= -\theta\delta(1-L)\chi_1 < 0
\end{aligned}
$$

$$
\begin{aligned}
K_1 - K_3 + K_2 - K_4 &= \frac{\phi}{\rho + \phi + \delta\left(1 + \theta + \kappa\right) - \mu} - \frac{\phi}{\rho + \phi + \delta\left(1 + \theta + \kappa\right)} > 0 \\
K_3 - K_9 + K_4 \left(\frac{1 - L}{l}\right) &= \frac{\theta\delta\left(1 - L\right)}{\rho + \phi + \delta\left(1 + \theta + \kappa\right) - \mu} - \frac{\theta\delta\left(1 - L\right)}{\rho + \phi + \delta\left(1 + \theta + \kappa\right)} > 0
\end{aligned}
$$

and

$$
\begin{aligned}
& \eta_1 \frac{\partial\left(K_3 + \overline{K}_8\right)}{\partial \kappa} + \left(\eta_1 - 1\right) \frac{\partial\left(K_4 + \overline{K}_9\right)}{\partial \kappa} \left(\frac{1 - L}{l}\right) \\
= \; & -\delta^2 \left(1 - L\right) \left(\frac{\eta_1}{\left(\rho + \phi + \delta\left(1 + \theta + \kappa\right)\right)^2} - \frac{\eta_1 - 1}{\left(\rho + \phi + \delta\left(1 + \theta + \kappa\right) - \mu\right)^2}\right)
\end{aligned}
$$

thus

$$
\begin{aligned}
& \left.\frac{\partial W_C}{\partial \kappa}\right|_{\kappa=0} \\
< \; & \frac{\partial\left(\frac{\eta_1}{\eta_1 + \gamma_2}\right)}{\partial \kappa} \left(K_9 + \left(K_1 - K_3 + K_2 - K_4\right)\left(\frac{1 - L}{l}\right)^{-\gamma_1} + \left(K_3 - K_9 + K_4\left(\frac{1 - L}{l}\right)\right)\right) \\
& -\delta^2 \left(1 - L\right) \frac{\frac{\eta_1}{\left(\rho+\phi+\delta(1+\theta+\kappa)\right)^2} - \frac{\eta_1 - 1}{\left(\rho+\phi+\delta(1+\theta+\kappa)-\mu\right)^2}}{\left(\eta_1 + \gamma_2\right)} \left(\frac{y}{\left(1 - L\right)/l}\right)^{-\gamma_1}
\end{aligned}
$$

Therefore, for $\left.\frac{\partial W_C(y)}{\partial \kappa}\right|_{\kappa=0} < 0$, it is sufficient to have

$$
\frac{\eta_1}{\left(\rho + \phi + \delta\left(1 + \theta + \kappa\right)\right)^2} > \frac{\eta_1 - 1}{\left(\rho + \phi + \delta\left(1 + \theta + \kappa\right) - \mu\right)^2}
$$

and

$$
\begin{aligned}
& \frac{\gamma_2}{\left(\eta_1 + \gamma_2\right)^2} \frac{\partial \eta_1}{\partial \kappa} \left(K_9 + \left(K_1 - K_3 + K_2 - K_4\right)\left(\frac{1 - L}{l}\right)^{-\gamma_1} + \left(K_3 - K_9 + K_4\left(\frac{1 - L}{l}\right)\right)\right) \\
< \; & \delta^2 \left(1 - L\right) \frac{\frac{\eta_1}{\left(\rho+\phi+\delta(1+\theta+\kappa)\right)^2} - \frac{\eta_1 - 1}{\left(\rho+\phi+\delta(1+\theta+\kappa)-\mu\right)^2}}{\left(\eta_1 + \gamma_2\right)} \left(\frac{y}{\left(1 - L\right)/l}\right)^{-\gamma_1}
\end{aligned}
$$

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