A dynamic model of optimal creditor dispersion

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Abstract

Firms often choose to raise capital from multiple creditors even though doing so may lead to inefficient liquidation caused by coordination failure. Potential coordination failure can, however, improve a firm's incentive to repay its debt, thus increasing its debt capacity. Given this trade-off between higher liquidation risk and enhanced pledgeability, it is important to understand how firms choose the number of creditors and how this decision changes over time. I build a dynamic rollover model to analyze these questions. Consistent with empirical findings, I show that firms optimally increase the number of creditors when they perform badly. Even though having more creditors increases the liquidation probability, allowing for potential coordination failure from multiple creditors is valuable. Policies that commit the creditors to ex post efficient coordination exacerbate rollover difficulty and the reduction in firm value ex ante. Finally, if the firm can renegotiate its debt very frequently, the extra pledgeability from multiple creditors diminishes. The model also generates empirical implications for the firm value, the interest rates, and the probabilities of liquidation, renegotiation, and default.

1 Introduction

Many firms borrow simultaneously from multiple creditors. Having multiple creditors brings the disadvantage of coordination problems, which in bad times make it harder for a firm to restructure its debt to avoid liquidation. In good times, however, these same coordination problems enhance pledgeability by making it harder for a firm to opportunistically hold up its creditors.

In this paper, I study the trade-off between these two forces—liquidation risk and enhanced pledgeability—for a firm that seeks to roll over its existing debt. In contrast to the existing literature,[1] I focus on the case in which a firm has insufficient internal resources to repay its outstanding debt. Instead, the firm must issue new debt to repay the maturing debt—that is, roll over its debt. This case is empirically relevant. In reality, 47% of the Compustat firms during fiscal years 2012 and 2013 have insufficient operating cash flow to repay their maturing debt and thereby have to rely on debt rollover.[2]

A firm's ability to roll over its debt is fundamentally a dynamic concept: the ability to roll over debt today depends on whether the firm's new creditors anticipate that they will be able to, in turn, roll over their debt in the future, which in turn depends on whether creditors anticipate that rollover will be possible even further in the future. I build a parsimonious dynamic model to analyze a firm's choice of the number of creditors in a rollover framework. Each period, a firm trades off the increase in liquidation risk with the enhanced pledgeability that a greater number of creditors engenders. Despite the model's parsimony, it is challenging to analyze and generates a rich set of predictions.

I use my model to make three main points. First, my model delivers predictions on how many creditors a firm has as well as when it decides to seek more creditors or consolidate the existing ones. I show that firms with higher growth prospects can support more creditors,

[1] For example, Berglöf and von Thadden (1994), Bolton and Scharfstein (1996), and Diamond (2004).

[2] I use the Compustat variables *total debt in current liability* (*DLC* — the total amount of short-term notes and the current portion of long-term debt that is due in one year) and *EBITDA* as the proxies for maturing debt and operating cash flow. For 47% of the firms, EBITDA is smaller than total debt in current liability.

which is consistent with cross-sectional empirical findings. In the time series, I show that firms increase the number of creditors when they perform badly and need to support a higher leverage, a point well illustrated by the following case. School Specialty Inc. is a distributor of classroom supplies that went bankrupt in 2013. Rick Barrett (2013) writes, “The [subprime] recession and cuts in public spending severely affected school budgets and hurt School Specialty” (para. 17). The company increased the pool of creditors and borrowed $64 million from a new lender, Bayside, in January 2012 after its current lenders refused to provide new loans to refinance its existing debt according to Dugan (2013 para. 1). The firm indeed survived one more year until it breached a loan covenant set forth by Bayside and filed for bankruptcy protection.

Second, I challenge the received wisdom that having multiple creditors and the resulting coordination problems are responsible for firms' difficulties in rolling over their debt. In the School Specialty case, Bayside's demand for a full repayment after School Specialty's covenant violation triggered its bankruptcy filing. It is easy to conclude that introducing the additional lender Bayside and its high priority prevented private debt restructuring that could have led to a more efficient resolution. Implicit in such views is the idea that the firm would have had an easier time if it had had fewer creditors. But this counterfactual ignores the fact that borrowing from more creditors is an endogenous choice made by the firm in the past. Without the decision of borrowing from more creditors the firm could have failed even earlier. To make a more meaningful comparison, I compare the expected liquidation probability and the firm value in my model to the ones in a counterfactual model in which the firm can borrow from only one creditor. I show that for a large range of fundamental values, firms with multiple creditors would have an even higher chance of liquidation and lower firm value, if they were forced to borrow from just one lender. An interesting policy implication is that ex post efficient reorganization processes, such as the automatic stay clause and Chapter 11 reorganization, that eliminate coordination failure among creditors may reduce a firm's ability to raise money ex-ante and result in lower welfare due to a more difficult debt rollover.

Finally, the model sheds light on how renegotiation frequency affects pledgeability. I show that in the limit when the firm can instantaneously renegotiate its debt, the enhanced pledgeability from more creditors becomes negligible. Although more creditors can indeed force more repayment, the source of this additional payout comes from the growth between two negotiation dates. With very frequent negotiation, per period growth vanishes, as does the additional debt capacity from having more creditors.

2 Related Literature

It has been well understood that having multiple creditors can cause coordination problems. Perhaps the most famous example is bank (creditor) run. Diamond and Dybvig (1983) show that in a static setting, socially inefficient bank run equilibria generally exist. Goldstein and Pauzner (2005) further characterize the probability of a bank run under a global game framework. He and Xiong (2012a) study the dynamic evolution of a panic-based run on staggered corporate debt.

If borrowing from multiple lenders is costly, then why do firms continue doing so? Berglöf and von Thadden (1994) claim that having multiple lenders specialize in lending at different maturities is a superior structure. The short-term creditors can impose externalities on the long-term creditors at the renegotiation stage, thereby increasing the ex post repayment incentives and in turn the ex ante efficiency. Following this line of thinking, Diamond (2004) demonstrates that when enforcing a debt contract is difficult, a single lender with a large stake in the firm has limited or no incentive to take ex post disciplinary actions against the firm, since such actions also hurt the lender himself. The firm, knowing that disciplinary actions are not credible, will misbehave ex ante. In the case of multiple creditors, the creditor who takes the action can claim against the whole firm, thereby hurting the other creditors. The improved incentive for lenders to be active ex post forces the borrowers to behave and thus increases the amount of money that can be raised. These papers share the key insight that potential coordination failure with multiple creditors disciplines the firm and can potentially improve the ex ante outcome. However, they take the variation in the number of creditors exogenously and therefore are silent on when firms endogenously change the number of creditors.

Bolton and Scharfstein (1996) further develop this idea and study the optimal choice between one and two creditors. The firms in their model can strategically default and renegotiate the debt even when they have the money to repay. The creditor(s), upon (either strategic or fundamental-based) default by the firm, can sell the project to an inefficient outsider. Under a multilateral bargaining setup, the benefit of having multiple creditors is to increase the collective bargaining power against the firm following a strategic default. In this case, the creditors can extract higher repayments from the firm. However, the cost of introducing a second creditor is that it lowers the expected payoff following a bad state, where this stronger collective bargaining power makes it less likely for the creditors to get an outside investor. Although all of these papers study the benefit brought by coordination failure from multiple creditors, they are all static (i.e. one-shot negotiation). My model shares the classic idea that having multiple lenders is a costly mechanism to induce correct behavior from the borrowers, but instead I focus the optimal number of creditors with a dynamic model. This dynamic feature is particularly important since firms usually do not have sufficient operating cash flow to pay back the maturing debt and must rely on repeatedly rollover.

Several other papers have also explicitly investigated the cost and benefit of having multiple creditors from various perspectives. Brunnermeier and Oehmke (2013) extend this idea by allowing the borrower to choose the maturity of the debt contract and explain why an excessively short maturity structure prevails in equilibrium, despite the increased rollover risks. Detragiache, Garella, and Guiso (2000) present a completely different trade-off. If banks can fail, then having multiple banking relationships is beneficial because financing is more robust in this case and will not fail unless all banks do. However, when all banks actually do fail, having more relationship banks is a stronger negative signal and therefore increases refinancing costs. Petersen and Rajan (1995) propose a model that illustrates how lenders' market power affects the quality of the financed firms and the cost of credit. They take the lenders' market power as an exogenous parameter. My paper endogenizes the variation of bargaining power by explicitly modeling the game between the firm and its creditors. Furthermore, their empirical studies in Petersen and Rajan (1994,1995) suggest that having more creditors is associated with a higher cost of credit in equilibrium, which is consistent with my model's prediction.

The effects of debt rollover and renegotiation on credit risk and debt prices have been studied from an asset-pricing perspective. Mella-Barral and Perraudin (1997) and Mella-Barral (1999) study the asset-pricing implications when the firm can renegotiate and service the troubled debt, rather than just defaulting directly as in Leland (1998). He and Xiong (2012b) investigate how creditors with different maturities strategically interact with each other when they decide whether or not to roll over the maturing debt. Similar to the work of Diamond (2004), the creditors' decisions not to roll over pose externalities on other incumbent creditors with claims not yet matured. Hege and Mella-Barral (2005) look at an economy in which a firm can exchange liquidation rights for coupon concessions on debt and study how that feature affects the credit risk premia as the number of creditors changes. These papers focus on pricing the debt claims given the possibility of renegotiation or rollover frictions, assuming the creditors' structure is exogenously fixed. My paper, on the other hand, focuses on the optimal choice creditor dispersion.

3 The Model

3.1 The Project and Financing

Time t is discrete and the discount rate is normalized to 1. A risk-neutral penniless entrepreneur starts a firm at $t = 0$ with a project.[3] The project requires an upfront investment I_0 and generates no cash flow except for a final liquidating dividend at a random project maturity. At each date, the project matures with probability π. The actual realization of the final dividend depends on a stochastic firm-specific state $\theta_t \in \{G(ood), B(ad)\}$ and the fundamental $Y_t = Y_0 \Pi_{1 \leq s \leq t} z_s$, where z_s are i.i.d positive random variables with continuous density $g(z)$. Assume $g(z)$ has a compact support $[\underline{z}, \overline{z}]$. The random variables θ_t and z_s are independent. Denote the mean $E(z_s) = \mu > 1$ and assume $\underline{z} < 1$. If the project matures when the state is good ($\theta_t = G$), the realized final dividend is Y_t; otherwise, if the state is bad ($\theta_t = B$), the dividend is 0. The state θ_t follows a Markov process with transition probability $p^\theta = Prob(\theta_{t+1} = \theta | \theta_t = \theta)$ (for $\theta = G, B$), which can be interpreted, for example, as the demand shock for the firm's output or the firm-specific productivity shock. To ensure that the project has a finite value, I impose the following parameter assumption:

$$(1 - \pi)\mu < 1. \tag{1}$$

Denote τ_π to be the random project maturity date. Then given the initial state θ_1 and fundamental Y_0, the expected value of the project's final dividend can be naturally defined as

$$E(\mathbf{1}_{\theta_{\tau_\pi} = G} Y_{\tau_\pi} | \theta_1, Y_0). \tag{2}$$

Lemma 1 *If the project is carried through to its random maturity τ_π, then its expected value*

[3] Note that I do not distinguish between the entrepreneur and the firm in the model and use the two terms interchangeably.

defined in (2) conditional on the current state θ and fundamental Y is given by

$$\begin{aligned} E(\mathbf{1}_{\theta_{\tau_\pi}=G} Y_{\tau_\pi} | G, Y) &= \tfrac{\pi[1-(1-\pi)\mu p^B]\mu}{[1-(1-\pi)\mu][1-(1-\pi)\mu(p^G+p^B-1)]} Y, \\ E(\mathbf{1}_{\theta_{\tau_\pi}=G} Y_{\tau_\pi} | B, Y) &= \tfrac{\pi(1-p^B)(1-\pi)\mu^2}{[1-(1-\pi)\mu][1-(1-\pi)\mu(p^G+p^B-1)]} Y. \end{aligned} \tag{3}$$

At any time t, the project can be liquidated prematurely for λY_t. The liquidation value is assumed to be independent of θ because it is possible to sell the project to other firms that are not subject to this firm-specific shock. The liquidation coefficient $\lambda \leq 1$ captures the inefficient separation of the project from its original developers. If liquidation is inefficient enough, i.e. $E(\mathbf{1}_{\theta_{\tau_\pi}=G} Y_{\tau_\pi} | B, Y) > \lambda Y$, then the project is always better off continuing even in the bad state. By (3), this is equivalent to

$$\lambda < \frac{\pi(1-p^B)(1-\pi)\mu^2}{[1-(1-\pi)\mu][1-(1-\pi)\mu(p^G+p^B-1)]}, \tag{4}$$

which I assume throughout the paper. Under this assumption, the values specified by (2) are indeed first best. Denote them by $V_{FB}^{\theta\star}(Y)$.

If the entrepreneur has enough cash to finance the up-front investment I_0, then the project is optimally carried through to its maturity and the first best firm value is realized. However, as I have assumed, the firm does not have (sufficient) cash to begin with. In addition, to highlight the rollover and pledgeability frictions, I assume that the firm can only issue one-period debt to *short-lived* creditors. Since the project does not generate any interim cash flow, the firm must repeatedly issue new debt to finance the payment to the maturing creditors. The detailed game between the entrepreneur and the creditors will be defined later following a formal introduction of the timeline.

3.2 Timeline

Figure 1 outlines the timeline and the evolution of the state variables. The firm enters period t with N_t incumbent creditors and a total promised face value F_t. The current state

θ_t and the previous fundamental Y_{t-1} are also publicly known. At period t, a new shock z_t (or equivalently Y_t) is realized, and then the project matures with probability π. If it matures, the game ends with a final dividend $Y_t \mathbf{1}_{\theta_t=G}$. Otherwise, the project continues to the *repayment stage*, and a new state θ_{t+1} is realized. The entrepreneur then has the following three options: (a) to voluntarily liquidate the project, (b) to make the promised repayment F_t, or (c) to initiate a repayment negotiation (described in the next subsection). If an agreement on the actual payment cannot be reached, the firm is forced into liquidation. Otherwise, if a repayment X_t is mutually accepted (in case (b) $X_t = F_t$ or in case (c) the negotiated amount), the firm enters the *refinancing stage* to raise *exactly* X_t from N_{t+1} identical creditors with an aggregate face value F_{t+1}. Both N_{t+1} and F_{t+1} are the firm's choice variables. Following a successful refinancing, the firm survives period t and the next period begins.

3.3 The Repayment Negotiation

At the repayment stage, the firm can choose to negotiate the payment (option (c) in the previous subsection). During a negotiation, the firm meets each creditor sequentially in a *random* order and makes a take-it-or-leave-it offer S_i to the ith creditor. Here, the index i reflects the realized random negotiation order. The offer history is public information. Each creditor, when it is his turn to negotiate, can either accept (A) the new promised payment or reject (R) the offer and exercise the liquidation right. If any creditor rejects the offer, the negotiation fails and the firm is liquidated. I assume that the rejecting creditor has priority over the liquidation proceeds and gets $\min(\frac{F_t}{N_t}, \lambda Y_t)$. The remaining creditors (who either previously accepted the offer or have not yet negotiated) get the remaining liquidation proceeds equally, $\min(\frac{F_t}{N_t}, \frac{1}{N_t - 1} \max(0, \lambda Y_t - \frac{F_t}{N_t}))$. If all N_t creditors accept the new offers, the firm then enters the refinancing stage and tries to borrow $X_t = \sum_{i=1}^{N_t} S_i$.

3.4 The Firm's Refinancing Decision

Since the project does not generate any interim cash flow, the firm has to finance the repayment X_t and roll over this obligation to the next period. The firm chooses N_{t+1} new creditors and offers them the same one-period debt contract with total face value F_{t+1} in exchange for cash X_t to honor the repayment to the N_t incumbent creditors. The new creditors simultaneously accept or reject the new debt offerings. If anyone rejects, the new creditors get a reservation payoff of 0 and the firm is liquidated. The N_t incumbent creditors equally share the liquidation proceeds up to the face value and each gets $\frac{1}{N_t}\min(F_t, \lambda Y_t)$. On the other hand, if all N_{t+1} new creditors accept the offer, the firm survives period t and the game moves on to period $t+1$.

3.5 Terminal Payoffs, Markov Strategies, and Equilibrium Definition

The entrepreneur is long-lived and the creditors live for one period. The game ends at date t if one of the following events occurs: (a) the project matures, (b) the negotiating creditor forces liquidation, (c) the entrepreneur voluntarily liquidates the project, or (d) the refinancing offer is rejected. If (a) the project matures, each incumbent creditor (living from period t to $t+1$) gets $-\frac{X_{t-1}}{N_t} + \frac{1}{N_t}\min(\mathbf{1}_{\theta_t=G}Y_t, F_t)$ and the entrepreneur gets the remaining $\max(0, \mathbf{1}_{\theta_t=G}Y_t - F_t)$, where X_{t-1} is the amount of total repayment made by the firm in the previous period (or total funds borrowed from the current incumbent creditors). The first term $-\frac{X_{t-1}}{N_t}$ in the creditors' payoff captures the up-front cash lending in the previous period. If (b) one of the creditors forces liquidation, then the liquidating creditor gets $-\frac{X_{t-1}}{N_t} + \min(\frac{F_t}{N_t}, \lambda Y_t)$, every remaining creditor receives $-\frac{X_{t-1}}{N_t} + \min(\frac{F_t}{N_t}, \frac{1}{N_t - 1}\max(0, \lambda Y_t - \frac{F_t}{N_t}))$, and the residual liquidation proceeds $\max(0, \lambda Y_t - F_t)$ go to the entrepreneur. If (c) the entrepreneur liquidates the project or (d) the refinancing offer is rejected, then each incumbent creditor receives $-\frac{X_{t-1}}{N_t} + \frac{1}{N_t}\min(\lambda Y_t, F_t)$ and the entrepreneur gets $\max(0, \lambda Y_t - F_t)$. Finally, if the firm survives each period, then the ith (i represents the realized negotiation order) short-lived incumbent creditor receives $-\frac{X_{t-1}}{N_t} + S_i$ in the case of having a negotiation and $-\frac{X_{t-1}}{N_t} + \frac{F_t}{N_t}$ otherwise.

A pure Markov strategy profile includes the following items. The firm has a negotiation strategy for the ith creditor $S_i^{\theta_{t+1}}(\sum_{j<i} S_j, F_t, Y_t, N_t) \in \mathbb{R}_+$ as a function of the total negotiated repayment in this period untill now $\sum_{j<i} S_j$, the originally promised face value F_t, the current fundamental Y_t, the realized next period state θ_{t+1}, and the number of incumbent creditors N_t. With a slight abuse of notation, $S_0^{\theta_{t+1}}(F_t, Y_t, N_t) \in \{L, F\}$ denotes a voluntary liquidation or a full repayment of F_t. The firm also has a set of financing strategies to choose the new number of creditors $N_+^{\theta_{t+1}}(X_t, F_t, Y_t) \in \mathbb{N}$ and the total face value $F_+^{\theta_{t+1}}(X_t, F_t, Y_t) \in \mathbb{R}_+$, as functions of the required financing amount X_t, the originally promised face value F_t, the fundamental Y_t, and the state θ_{t+1}. In addition, in each period, the ith incumbent creditor has an acceptance strategy after receiving an offer S_i: $s_i^{\theta_{t+1}}(\sum_{j<i} S_j, S_i, F_t, Y_t, N_t) \in \{A, R\}$. Finally, given any refinancing offer (F_+, N_+), the new creditors have acceptance strategies: $r_i^{\theta_{t+1}}(X_t, F_+, Y_t, N_+) \in \{A, R\}$ for all $i \leq N_+$.

In this paper, I focus on the Markov perfect equilibria, meaning the strategy profiles described above that are subgame perfect.

Remark 1: Rather than taking a contract design approach, as for example in Berglöf and von Thadden (1994), Bolton and Scharfstein (1996), and Diamond (2004), I instead assume that only standard debt contracts are possible. I make this assumption because, unlike the other papers, I take the cross-externality among investors as given and investigate how firms choose exposure to this friction dynamically. In addition, I do not allow the firm to save. However, as will be discussed in section 6, I do not expect the possibility of savings to change the firm's choice of number of creditor.

Remark 2: One interpretation of the priority structure is that the rejecting creditor can partially liquidate the project to secure as much of the originally promised amount as possible. The project, however, is fundamentally impaired and will be forced into a full liquidation before the next creditor negotiates, in which case all other creditors share the remaining liquidation proceeds equally. Note that the results do not depend on the specific priority structure. As long as the liquidating creditor has some priority over the proceeds which can hurt other creditors, the story remains valid. The key economic force here is that the creditors can pose externalities on each other as in Berglöf and von Thadden (1994), Diamond (2004) and Brunnermeier, and Oehmke (2013). As the number of creditors increases, each one of them can pose a larger externality on others by forcing a liquidation. Such an externality provides the creditors with stronger incentives to commit to an ex post liquidation and hence creates a stronger incentive for the firm to repay as well. The cost, on the other hand, is an early termination of the project as a result of coordination failure when the firm is in distress.

4 Equilibrium Construction

In this section, I explicitly construct an equilibrium. Before doing so, I introduce several key variables including debt value, debt capacity, and the total firm value.

4.1 Debt Value, Debt Capacity, and Firm Value

Given any strategy profile, I can define the total value of debt claims $D_{N_t}^{\theta_t}(F_t, Y_{t-1})$ at the beginning of each period. Here I keep the time indices to make the evolution of the state variables transparent.

$$\begin{aligned} D_{N_t}^{\theta_t}(F_t, Y_{t-1}) \quad = \quad & E\{\pi \min(F_t, Y_{t-1}z_t)\mathbf{1}_{\theta_t=G} + (1-\pi) \\ & [(\Pi_{i\leq N_t}\mathbf{1}_{s_i^{\theta_{t+1}}=A})\mathbf{1}_{S_0^{\theta_{t+1}}\neq L}(\Pi_{i\leq N_+^{\theta_{t+1}}}\mathbf{1}_{r_i^{\theta_{t+1}}=A})X_t+ \\ & [1-(\Pi_{i\leq N_t}\mathbf{1}_{s_i^{\theta_{t+1}}=A})\mathbf{1}_{S_0^{\theta_{t+1}}\neq L}(\Pi_{i\leq N_+^{\theta_{t+1}}}\mathbf{1}_{r_i^{\theta_{t+1}}=A})]\min(F_t, \lambda Y_{t-1}z_t)]\} \end{aligned} \tag{5}$$

The expectation is taken over the random variables z_t and θ_{t+1}. In the future, when the time indices are omitted, I use θ' to denote the next period state θ_{t+1}. The first term captures the payout to the debt holders upon project maturity, which happens with probability π. If the project does not mature, then the total (possibly negotiated) repayment X_t is honored if every player chooses not to liquidate. Otherwise, if anyone liquidates the project (rejects the offer), then the liquidation payoff is distributed. Let τ_L be the stopping time when any player chooses liquidation depending, which can potentially depend on the entire history. Define $\tau_S = \min(\tau_\pi, \tau_L)$ to be the time when the game ends. The total value of the firm at the beginning of each period is

$$V_{N_t}^{\theta_t}(F_t, Y_{t-1}) = E\{\mathbf{1}_{\tau_S=\tau_\pi}\mathbf{1}_{\theta_{\tau_\pi}=G}Y_{\tau_\pi} + \mathbf{1}_{\tau_S=\tau_L}\lambda Y_{\tau_L}\},$$

which can be expressed recursively as:

$$\begin{aligned} V_{N_t}^{\theta_t}(F_t, Y_{t-1}) \quad = \quad & E\{\pi Y_{t-1}z_t\mathbf{1}_{\theta_t=G} + (1-\pi) \\ & [(\Pi_{i\le N_t}\mathbf{1}_{s_i^{\theta_{t+1}}=A})\mathbf{1}_{S_0^{\theta_{t+1}}\neq L}(\Pi_{i\le N_+^{\theta_{t+1}}}\mathbf{1}_{r_i^{\theta_{t+1}}=A})V_{N_+^{\theta_{t+1}}}^{\theta_{t+1}}(F_+^{\theta_{t+1}}, Y_{t-1}z_t) \\ & +[1-(\Pi_{i\le N_t}\mathbf{1}_{s_i^{\theta_{t+1}}=A})\mathbf{1}_{S_0^{\theta_{t+1}}\neq L}(\Pi_{i\le N_+^{\theta_{t+1}}}\mathbf{1}_{r_i^{\theta_{t+1}}=A})]\lambda Y_{t-1}z_t]\} \end{aligned} \tag{6}$$

It is convenient to define the debt capacity from N creditors as follows:

$$DC_{N_t}^{\theta_t}(Y_{t-1}) \equiv \max_{F_t} D_{N_t}^{\theta_t}(F_t, Y_{t-1}) \tag{7}$$

and the total debt capacity as

$$DC^{\theta_t}(Y_{t-1}) \equiv \max_{N_t} DC_{N_t}^{\theta_t}(Y_{t-1}). \tag{8}$$

Finally, define $\overline{F}_{N_t}^{\theta_t}$ to be the face value that maximizes the value of debt, given the number of creditors N_t, fundamental Y_{t-1}, and state θ_t. If several values of F deliver this maximum, $\overline{F}_{N_t}^{\theta_t}$ is the smallest one:

$$\overline{F}_{N_t}^{\theta_t}(Y_{t-1}) \equiv \min[\arg\max_{F_t} D_N^{\theta}(F_t, Y_{t-1})]. \tag{9}$$

As will be transparent in the next subsection, the entrepreneur has no incentive to pick a new face value $F_+ > \overline{F}_{N_+}$ because doing so would weakly reduce the firm value.

4.2 Equilibrium Characterization

Proposition 1 is the main result that characterizes the equilibrium strategies and the value functions.

Proposition 1 *Consider the following strategies:*

1. *The entrepreneur always makes an offer*

$$S_i^{\theta\star} = \min(\frac{F}{N}, \lambda Y) \tag{10}$$

   to the ith creditor.

2. *The entrepreneur's financing strategy $N_+^{\theta\star}(X, F, Y)$ and $F_+^{\theta\star}(X, F, Y)$ solves*

$$\begin{aligned} &\max\nolimits_{N_+} V^{\theta}_{N_+}(F_+, Y) \\ \textit{s.t.} \quad &F_+ \textit{ is the smallest solution to } \; D^{\theta}_{N_+}(F_+, Y) = X. \end{aligned} \tag{11}$$

   If there is no combination of (N_+, F_+) such that (11) holds, then the firm chooses $N_+^{\theta\star} = 1$ and $F_+^{\theta\star} = 0$.[4]

3. *The ith creditor accepts the offer S_i (i.e., $s_i^{\theta\star}(\sum_{j<i} S_j, S_i, F, Y, N) = A$) if and only if $S_i \geq \min(\frac{F}{N}, \lambda Y)$ and*

$$\sum_{j<i} S_j + S_i + \sum_{j>i} S_i^{\theta\star} \leq DC^{\theta}(Y). \tag{12}$$

4. *The potential new creditors accept the financing offers $r_i^{\theta\star}(X, F_+, Y, N_+) = A$ if and only if $D^{\theta}_{N_+}(F_+, Y) \geq X$.*

[4]In fact, in this case, the financing strategy can be arbitrary because it will be rejected by the potential new creditors.

Under the proposed strategies, for any state $\theta = G, B$ and any number of creditors N,

1. *the value of debt $D_N^\theta(F, Y)$ is continuous and homogeneous of degree one (HD1) in (F, Y);*

2. *the minimum face value that achieves the N creditor debt capacity from (9) is linear in Y, i.e., $\overline{F}_N^\theta(Y) = \overline{f}_N^\theta Y$ for some constant $\overline{f}_N^\theta$;*

3. *the debt capacity from N creditors from (7) is linear in Y, i.e., $DC_N^\theta(Y) = \kappa_N^\theta Y$ for some constant κ_N^θ.*

Define

$$\kappa^\theta \equiv \max_N \kappa_N^\theta. \tag{13}$$

If $\min(\kappa^G, \kappa^B) > \lambda$, then the proposed strategies indeed constitute a subgame perfect equilibrium. In addition, the firm's value function $V_N^\theta(F, Y)$ satisfies

1. *$V_N^\theta(F, Y) \geq \kappa_N^\theta Y$, when $F \leq \overline{F}_N^\theta(Y)$;*

2. *$V_N^\theta(F, Y)$ is continuous, HD1 in (F, Y), weakly decreasing in F, and increasing in Y.*

The proof takes a guess and verify approach, with the full version in the appendix. However, outlining the procedures to establish the equilibrium is still helpful. The key to this construction lies in finding a consistent (κ^G, κ^B) that dictates the debt capacities in (8). Given a linear conjecture, the equilibrium strategies imply that rollover in state θ is possible only when the total offered repayment is feasible:

$$\min(F, N\lambda Y) \leq DC^\theta(Y) = \kappa^\theta Y. \tag{14}$$

With the continuation region explicitly expressed, the debt value (5) be rewritten as

$$\begin{aligned} D_N^\theta(F, Y) \quad = \quad & E\{\pi \min(F, Yz)\mathbf{1}_{\theta=G} + (1-\pi) \\ & [\mathbf{1}_{\min(F, N\lambda Yz) \leq \kappa^{\theta'} Yz} \min(F, N\lambda Yz) + \mathbf{1}_{\min(F, N\lambda Yz) > \kappa^{\theta'} Yz} \min(F, \lambda Yz)]\}, \end{aligned} \tag{15}$$

where the expectation is taken over z and θ'. It is easy to see that this is HD1 in (F, Y). Linearity of debt capacities is then just a simple corollary of HD1 with the coefficient:

$$\begin{aligned} \kappa_N^\theta \quad = \quad & \max_f E\{\pi \min(f, z)\mathbf{1}_{\theta=G} + (1-\pi) \\ & [\mathbf{1}_{\min(F,N\lambda z)\leq\kappa^{\theta'}z} \min(f, N\lambda z) + \mathbf{1}_{\min(F,N\lambda z)>\kappa^{\theta'}z} \min(f, \lambda z)]\} \end{aligned} \tag{16}$$

Clearly (16) depends on the initial conjecture of (κ^G, κ^B), and it has to arrive at the same debt capacity in equilibrium by equating (13). Any guess of (κ^G, κ^B) that survives this procedure is consistent and can be supported in an equilibrium.

If creditors expect a low debt capacity tomorrow, then the pledgeable amount today decreases today as in (16), resulting in a lower debt capacity today. Therefore, the self-fulfilling feature could result in multiple equilibria. It can be shown that the equilibrium is unique conditional on a fixed choice of κ^θ and the results of this paper do not depend on which κ^θ I choose.

Despite the potential multiplicity, the existence of any equilibrium is not obvious at all. This is because the right-hand side in (16) as a function of κ^θ is not continuous. For example, when $\kappa^\theta \geq \lambda N$, it is always possible to roll over. However, as soon as κ^θ decreases to just below λN, there is a nontrivial chance that the firm will be liquidated, which hurts the ex-ante borrowing capacity discontinuously. Fortunately, despite the discontinuity of (16), the right-hand side is still order preserving and Tarski's fixed point theorem guarantees a solution.

With a consistent conjecture of κ^θ held fixed, the firm's value function (6) reduces to the following dynamic programming problem:

$$\begin{aligned} V_N^\theta(F, Y) \quad = \quad & E\{\pi Y z \mathbf{1}_{\theta=G} + (1-\pi) \\ & [\mathbf{1}_{\min(F,N\lambda Yz)\leq\kappa^{\theta'}Yz} V_{N_+}^{\theta'}(F_+, Yz) + \mathbf{1}_{\min(F,N\lambda Yz)>\kappa^{\theta'}Yz}\lambda Y z]\} \end{aligned} \tag{17}$$

where F_+ is the minimum solution to

$$D^{\theta'}_{N_+}(F_+, Y) = \min(F, N\lambda Y z). \tag{18}$$

Establishing continuity in V^θ_N is challenging, since a small change in (F, Y) can result in a discontinuous change in the minimum solution F_+. Therefore, the constraint correspondence $(F, Y) \mapsto \{(N_+, F_+)|$ s.t. (18) holds$\}$ is discontinuous and the standard theorem of maximum does not apply. Even so, one can show that the value function in equilibrium is indeed continuous. After proving the properties of the value functions V^θ_N, it is relatively straightforward to verify that the constructed strategy profile is indeed optimal.

Despite the complicated construction and verification, the equilibrium is quite intuitive. The entrepreneur has all the bargaining power, so he just needs to *credibly* offer each creditor his liquidation payoff $\min(\frac{F}{N}, \lambda Y)$ as in (10). On the other hand, for an incumbent creditor to accept an offer S_n, it must be weakly higher than the liquidation payoff. In addition, condition (12) implies that the offer must be credible in the sense that following the proposed strategies, the total repayment can be financed.

The cost and benefit of having more creditors are immediately transparent in (14) and (15). With a higher N, the left-hand side of (14) weakly increases, causing a weakly higher chance of liquidation. On the other hand, having more creditors lowers the stake of an individual creditor relative to the whole firm and therefore effectively grants creditors higher bargaining power. The total actual repayment conditional on rollover in (15) weakly increases, as does the pledgeability.

4.3 Creditor Capacity and Safe Number of Creditors

Even though I do not pose any assumption on the transition probability p^θ, the debt capacities in the two states $\theta = G, B$ can be ordered in equilibrium.

Lemma 2 *The debt capacity is strictly higher in the good state, i.e., $\kappa^G > \kappa^B$.*

To understand this relationship, one needs to remember the recursive nature of debt rollover. The maximum amount that the firm can borrow now depends on the maximum amount that the firm can borrow in the next period. Suppose that the firm is in a bad state now. With probability π, the firm dies without any payout. If the firm has a weakly higher debt capacity in the bad state, then the actual refinanceable repayment in the next period is bounded by the debt capacity in the bad state $\kappa^B Y_{t+1}$. The expected maximum repayment, however, is insufficient to support the debt capacity today $(1-\pi)E_t(\kappa^B Y_{t+1}) = (1-\pi)\mu\kappa^B Y_t < \kappa^B Y_t$. Therefore, the firm has no chance of repaying $\kappa^B Y_t$ in the bad state. In other words, in order to finance the debt capacity in a bad state, the firm must rely on the possibility a good state realization in the next period and utilize that higher borrowing capacity.

Given this lemma,[5] we can conveniently define

$$\overline{N} \equiv [\frac{\max(\kappa^G, \kappa^B)}{\lambda}] + 1 = [\frac{\kappa^G}{\lambda}] + 1. \tag{19}$$

When the number of creditors becomes large, namely $N > \overline{N}$, the equilibrium no longer depends on the number of creditors N. This is because when (19) holds, the liquidation threat becomes credible in both states $\theta = G, B$, and creditors reject any offer less than the full repayment of F. The firm then always repays the original face value whenever possible. The debt and the total firm values from (15) and (17) become

$$\begin{aligned} D_N^\theta(F,Y) &= E\{\pi \min(F, Yz)\mathbf{1}_{\theta=G} + (1-\pi) \\ &\quad [\mathbf{1}_{F\le\kappa^{\theta'}Yz}F + \mathbf{1}_{F>\kappa^{\theta'}Yz}\min(F,\lambda Yz)]\} \end{aligned}$$

and

$$\begin{aligned} V_N^\theta(F,Y) &= E\{\pi Yz\mathbf{1}_{\theta=G} + (1-\pi) \\ &\quad [\mathbf{1}_{F\le\kappa^{\theta'}Yz}V^{\theta'}_{N_+^{\theta\star}}(F_+^{\theta'\star}, Yz) + \mathbf{1}_{F>\kappa^{\theta'}Yz}\lambda Yz]\} \end{aligned}$$

[5] The bracket denotes the floor function: $[a]$= the largest natural number weakly smaller than a.

with the corresponding condition (11) replaced by $D^{\theta'}_{N_+}(F_+, Y) = F$. Since both the rollover (liquidation) region and the payoffs are independent of N, the firm's and the creditors' problems are no longer sensitive to N. I refer to $\overline{N}$ defined in (19) as the *creditor capacity* in the future. Without loss of generality, we can limit the firm's choice of the number of creditors to weakly below $\overline{N}$. This finite bound turns out to be a key piece in proving the general existence of the value functions V^{θ}_N in proposition 1.

Similarly, I define the *safe number of creditors*:

$$\underline{N} \equiv [\frac{\kappa^B}{\lambda}]. \tag{20}$$

When the number of creditors is lower than $\underline{N}$, condition (14) always holds, meaning that rollover is always possible. In this case, having more creditors enhances pledgeability without an immediate risk of liquidation. Despite this seemingly costless benefit, as we will see shortly, this does not imply that the firm always prefers to have at least $\underline{N}$ creditors.

5 Key Trade-offs and Empirical Predictions

Only in this section, I study the comparative statics of the exogenous changes in number of creditors. To do so, I change the incumbent number of creditors as if it is a parameter and keep the equilibrium continuation strategies. In other words, I study the outcome of a one shot deviation of the number of creditors in equilibrium. This exercise highlights the trade off between pledgeability and the liquidation risk that the firm faces when choosing creditor dispersion. Many empirical predictions can be carried through in equilibrium, whereas others may be reversed by the firm's selection effect. This topic will be discussed in section 6.3.

5.1 Pledgeability

5.1.1 Value of Debt, Debt Capacity, and Interest Rate

Having multiple creditors has two offsetting effects on the value of debt. On the one hand, the entrepreneur's payout incentive increases with more creditors, which in turn raises the value of debt for any given face value. On the other hand, having more creditors reduces the ex-post financial flexibility that leads to more liquidation, which in turn hurts the debt value. In some cases, however, an increase in the number of creditors has no effect on the liquidation probability, so the debt value increases.

Proposition 2 *Suppose that one of the following three conditions holds: (a)$\underline{N} \geq N_2 > N_1$, (b)$\overline{N} > N_2 > N_1 > \underline{N}$, or (c)$N_2 > N_1 = 1$. Then,*

1. *for any face value F and fundamental Y, the value of debt $D^\theta_{N_2}(F, Y) \geq D^\theta_{N_1}(F, Y)$,*
2. *as an immediate consequence of 1, the debt capacity is higher with more creditors, i.e., $\kappa^\theta_{N_2} \geq \kappa^\theta_{N_1}$,*
3. *also as an immediate consequence of 1, the required interest rate is lower with more creditors: for any θ and $X \leq DC^\theta_{N_1}(Y)$, let F^θ_k $(k = N_1, N_2)$ be the minimum solution to $X = D^\theta_k(F^\theta_k, Y)$. Then the solutions exist and $F^\theta_{N_2} \leq F^\theta_{N_1}$.*

The three cases in proposition 2 are quite transparent. In case (a), as discussed following equation (20), rollover is always possible even in the bad state. Therefore the incumbent creditors pose no liquidation risk. In case (b), the firm is liquidated only in the bad state when the creditors cannot be paid in full. Case (c) is a little different. It states that the value of debt is the worst when there is just one creditor. It is the worst because with a single creditor, the actual repayment is just the liquidation payoff $\min(F, \lambda Y)$ regardless of whether or not rollover is possible.[6] The debt capacity is attained when $F \to \infty$:

$$\kappa^\theta_1 = [\pi \mathbf{1}_{\theta=G} + (1-\pi)\lambda]\mu. \tag{21}$$

[6]The actual repayment is conditional on the project not maturing.

It is easy to see from (15) that multiple creditors can at least secure a repayment of the liquidation value. Thus, having multiple creditors always weakly improves pledgeability.

In general, the benefit of having more creditors is the enhanced pledgeability, which lowers the required interest rate proxied by $\frac{F_N^\theta}{X}$ in proposition 2. The cost, as will become more clear in the next subsection, is a higher liquidation probability. Note here that one should not expect the negative correlation between the number of creditors and the interest rates to hold *in equilibrium.* I will postpone this discussion until section 6. As a preview, in equilibrium, the firms choose more creditors when they do badly. In these cases, their debts are more likely to default so their creditors demand higher interest rates. Therefore, having more creditors is associated with poorer performance, which in turn causes higher interest rates.

5.1.2 Probability of Renegotiation and Default

I call it *renegotiation* whenever the firm *successfully* rolls over with an actual repayment that is strictly less than the promised face value. This occurs when $N\lambda Yz < \min(F, \kappa^\theta Yz)$ and the firm continues by repaying each creditor the liquidation value λYz. Similarly, I call it *default* whenever the creditors do not receive the full repayment F. Mathematically, default means $F > \min(\kappa^\theta, N\lambda)Yz$ when the project does not mature. In addition, the firm also defaults if the project matures and yet the final cash flow is insufficient to repay the creditors in full, namely, $Yz\mathbf{1}_{\theta=G} < F$. By definition, renegotiation is a special case of default. Notice that a firm can renegotiate or default multiple times over its life cycle. To avoid any confounding effect, I denote τ_R and τ_D to be the *first* time that the firm renegotiates or defaults and let

$$R_N^{\theta,T}(F,Y) \quad = \quad Prob(\tau_R \leq T \text{ and } \tau_R \leq \tau_\pi) \tag{22}$$

$$DFT_N^{\theta,T}(F,Y) \quad = \quad Prob(\tau_D \leq T \text{ and } \tau_D \leq \tau_\pi) \tag{23}$$

be the probability that firm does so at least once during the next $T \leq \infty$ periods before or when the project matures at τ_π. The probabilities of renegotiation and default must satisfy the following recursive formulation:

$$R_N^{\theta,T}(F,Y) \;=\; (1-\pi)E[R_{N_+^{\theta\star}}^{\theta',T-1}(F_+^{\theta'\star}, Yz)\mathbf{1}_{F\leq\min(\kappa^{\theta'},N\lambda)Yz} + \mathbf{1}_{N\lambda Yz<\min(F,\kappa^{\theta'}Yz)}] \tag{24}$$

and

$$\begin{aligned} DFT_N^{\theta,T}(F,Y) \;=\;& \pi[Prob(Yz<F)\mathbf{1}_{\theta=G} + \mathbf{1}_{\theta=B}] + (1-\pi) \\ & E[DFT_{N_+^{\theta\star}}^{\theta',T-1}(F_+^{\theta'\star}, Yz)\mathbf{1}_{F\leq\min(\kappa^{\theta'},N\lambda)Yz} + \mathbf{1}_{F>\min(\kappa^{\theta'},N\lambda)Yz}] \end{aligned} \tag{25}$$

The expression (24) is not difficult to understand. With probability $1-\pi$, the firm enters the repayment stage. Renegotiation occurs if $N\lambda Yz < \min(F, \kappa^\theta Yz)$; otherwise, if rollover is possible with a full repayment F, the continuation probability of renegotiation in the next $T-1$ periods is calculated by using the equilibrium refinancing strategies for the next period number of creditors $N_+^{\theta\star}$ and face value $F_+^{\theta\star}$. The expression (25) can be similarly interpreted.

If the project continues without a renegotiation or default, the creditors are paid F in full regardless of the number of creditors N. Therefore, the continuation probabilities $R_{N_+^{\theta\star}}^{\theta,T-1}(F_+^{\theta\star}, Yz)$ in (24) and $DFT_{N_+^{\theta\star}}^{\theta,T-1}(F_+^{\theta\star}, Yz)$ in (25) are independent of N as well. On the other hand, as N increases, the region in which the firm makes the full repayment widens, since $F \leq \min(\kappa^\theta, N\lambda)Yz$ is more likely to hold. Intuitively, more creditors collectively have more bargaining power and provide a higher immediate incentive for the firm to pay back its debt. This effect reduces both the probability of renegotiation and default. The result is summarized in the following proposition.

Proposition 3 *The probabilities of renegotiation and default are lower with more creditors, i.e.,* $R_{N_2}^{\theta,T}(F,Y) \leq R_{N_1}^{\theta,T}(F,Y)$ *and* $DFT_{N_2}^{\theta,T}(F,Y) \leq DFT_{N_1}^{\theta,T}(F,Y)$*, for all* $N_2 > N_1$*,* θ*,* F*, and* Y*.*

Proposition 3 is another way to demonstrate the pledgeability channel. Having more creditors provides a better repayment incentive and therefore reduces the probability that the firm willingly or unwillingly cuts debt repayment.

5.2 Liquidation Risk

5.2.1 Probability of Liquidation

Recall that τ_L and τ_π are the random times of liquidation and project maturity. Define

$$L_{N_t}^{\theta_t,T}(F_t, Y_{t-1}) = Prob(\tau_L \leq t+T \text{ and } \tau_L < \tau_\pi), \tag{26}$$

at the beginning of period period t, to be the expected probability of liquidation in the next $T \leq \infty$ periods before the project matures. Since liquidation occurs if and only if (14) is violated, the liquidation probability L must satisfy the recursive formulation:

$$L_N^{\theta,T}(F,Y) \quad = \quad (1-\pi)E[L_{N_+^{\theta\star}}^{\theta',T-1}(F_+^{\theta'\star}, Yz)\mathbf{1}_{\min(F,N\lambda Yz)\leq\kappa^{\theta'}Yz} + \mathbf{1}_{\min(F,N\lambda Yz)>\kappa^{\theta'}Yz}] \tag{27}$$

With probability $1-\pi$, the firm enters the repayment stage. A failed negotiation results in an immediate liquidation; otherwise, the continuation probability of liquidation in the next $T-1$ periods is calculated by using the equilibrium refinancing strategies for the next period number of creditors $N_+^{\theta\star}$ and face value $F_+^{\theta\star}$.

A direct consequence of having more creditors is that the immediate liquidation probability

$$L_N^{\theta,1}(F,Y) \quad = \quad (1-\pi)E[P(\min(F,N\lambda Yz) > \kappa^{\theta'}Yz)]$$

increases because the rollover condition (14) is less likely to hold with a bigger N. I state this simple result as a lemma.

Lemma 3 *The one-period-ahead liquidation probability increases with the number of creditors, i.e.,* $L_{N_2}^{\theta,1}(F,Y) \geq L_{N_1}^{\theta,1}(F,Y)$ *for all* $N_2 > N_1$, θ, F, *and* Y.

Lemma 3 highlights the cost of having more creditors arising from a higher chance of an immediate liquidation. It is also helpful to compare lemma 3 with a seemingly contradictory result proposition 3. Fundamentally unlike liquidation, renegotiation and default as I defined in subsection 5.1.2 pose no direct welfare loss, since they do not lead to an inefficient termination of the project. Instead, they (oppositely) reflect the entrepreneur's endogenous commitment level. With more creditors, the entrepreneur commits to make (more) repayment at the cost of a more likely ex post liquidation.

One can interpret renegotiation or default as financial distress and liquidation as a *costly* outcome (for example, failed private debt restructuring). Under this interpretation, the results in this subsection state that with more creditors, the firm ex ante is less likely to end up in distress. Once it is in distress, however, the creditors are less likely to strike a deal. This prediction is confirmed by Gilson, John, and Lang (1990), who find that financially distressed firms with more creditors are less likely to turn around and emerge from a private debt restructuring.

5.2.2 Firm Value

Having more creditors in general reduces the total firm value. An immediate consequence of having more creditors is a greater liquidation risk in the next period. The long-run effect is the higher actual repayment which permanently increases the future liquidation probability. Both effects lower the firm value.

Proposition 4 *The firm value is lower with more creditors:* $V^{\theta}_{N_1}(F, Y) \geq V^{\theta}_{N_2}(F, Y)$ *for any* θ, F, Y, *and* $N_1 < N_2$.

Note that, from proposition 2, the value of debt is in general higher with more creditors for any given face value. Therefore, the conclusion is a joint statement about both higher *market leverage* ($\frac{D^{\theta}_{N_i}}{Y}$) and more creditors. To focus on the net effect of creditor dispersion on the firm value, one can hold the value of debt constant. Recall that this is exactly the firm's refinancing problem (11). The next section analyzes this choice explicitly.

6 Creditor Dynamics

The dynamics associated with the number of creditors is determined by the firm's refinancing problem (11). As we have seen from the previous section, with more new creditors N_+, the benefit is a potentially lower refinancing cost F_+, as in proposition 2. On the other hand, the cost is a higher immediate liquidation threat in the next period, as in lemma 3. The firm optimally chooses N_+ by balancing the cost and benefit. Unfortunately, for a discrete choice problem like this one, an analytical solution is typically not available. However, all the numerical experiments that I have calculated unanimously show that the cost of having more creditors *always* outweighs the benefit. The firm chooses more creditors only when borrowing the required level of repayment from fewer creditors is infeasible.

6.1 A Numerical Example

In this subsection, I explicitly describe a numerical example based on the following parameter choices: the per period shock to the final dividend process $z \sim uniform(0.63, 1.83)$, the probabilities of the states staying unchanged $(p^G, p^B) = (0.8, 0.3)$, the per period probability of the project maturing $\pi = 0.2$, the liquidation coefficient $\lambda = 1$, and the required up-front investment $I_0 = 1$. Even with the choice of $\lambda = 1$, liquidation is still inefficient since the future growth opportunities are lost. The key equilibrium variables, debt capacities, are calculated to be $(\kappa_1^G, \kappa_2^G, \kappa_1^B, \kappa_2^B) = (1.23, 1.273, 0.984, 1.022)$ and $\kappa^\theta = \kappa_2^\theta$. Under this parameterization, the creditor capacity $\overline{N} = 2$, and therefore the relevant choice for the new creditors N_+ is between 1 and 2. The numerical example is not designed to match any data, and the qualitative features of this example are robust to parameter and distribution choices.

Figure 2 plots the total firm value normalized by fundamental $(\frac{V_N^\theta(F,Y)}{Y})$ against the normalized value of the debt $(\frac{D_N^\theta(F,Y)}{Y})$ or equivalently the amount that has to be borrowed $\frac{X}{Y}$ in problem (11). The solid (dashed) line is the firm value with a single creditor when the fundamental $\theta = G$ ($\theta = B$). The dotted (dash-dotted) line is the firm value with two creditors when the fundamental $\theta = G$ ($\theta = B$). The thick solid segments can be supported only by two creditors (the lower curves). A quick observation is that when the value of debt is low, the firm values for one and two creditors converge. This is because the firm has to honor the promised face value regardless of the fundamental realization and the number of lenders.[7] Thus, the choice of the number of creditors has no impact on the firm value. As the value of debt increases, the two lines diverge and, when both are feasible, the single creditor case always delivers a higher firm value. This pattern suggests that the cost of inefficient liquidation is greater than the benefit of interest reduction (lower continuation face value F_+). However, since the curves end on the x-axis at κ_N^θ,[8] the lower curves for two creditor cases indeed extend farther than their single creditor counterparts. This means that when the firm needs to borrow beyond its single creditor debt capacities, it has to seek two creditors.

Figure 3 is a typical sample path of the firm. Areas are shaded when the state is bad. The solid (dashed) line denotes the exogenous fundamental process Y_{t-1} (the face value process F_t determined in equilibrium). I use bold segments when the firm chooses two creditors. The values plotted at each period t are the state variables entering this period: number of creditors N_t, the promised face value F_t, state θ_t, and fundamental process Y_{t-1}. Finally, the dotted bars plot the interest rates $\frac{F_t}{D_{N_t}^{\theta_t}(F_t, Y_{t-1})}$ during each period.

The firm starts by borrowing the required investment $I_0 = 1$ from one creditor in a good state with an interest rate of 9% (a level of 1.09 in the plot). During period 1, the fundamental drops to 0.7. With a single creditor, the firm negotiates the actual payment down to the liquidation value 0.7 and issues new debt with a face value of 0.76 and an interest rate of 9% to finance the repayment. During period 2, the fundamental keeps deteriorating to 0.49 and the state θ_3 switches to bad. The firm again negotiates the actual payment down to 0.49. In a bad state, however, the firm must refinance this payment from two creditors,

[7]This case is possible since per period shock z has a compact support. So when $\frac{F}{Y}$ is sufficiently small such that $\frac{F}{Y\underline{z}} \leq 1$, the firm will repay F in the next period in order to continue.

[8]This is because $\frac{D_N^\theta(F,Y)}{Y} \leq \kappa_N^\theta$ by definition.

because the debt capacity with a single creditor is insufficient to cover the liquidation value. The interest rate soars to 63%. The firm enters period 3 with face value $F_3 = 0.8$. During period 3, even though the state θ is still bad, the fundamental dramatically improves and the firm is able to make the promised repayment 0.8 and roll over the debt with a single creditor. The required interest rate reduces to 49%. What happens during period 4 is very similar to period 1. The state θ returns to good and the firm pays out and refinances the liquidation value by borrowing from one creditor at an interest rate of 9%. On period 5, the fundamental continues to improve to 1.23, and the firm can even issue risk free debt to finance the 0.77 debt obligation. This is possible since even if the project matures with the lowest shock realization $z = \underline{z} = 0.63$, the full value of the debt can still be repaid.[9] Period 6 and 7 are similar to period 2 and 3: the state switches to bad, the financing costs for the firm increases and two creditors are eventually required. Finally, during period 8, the state θ returns to good and the realized fundamental improves to 1.14. Even so, the borrowing capacity is only $1.14 \times 1.27 = 1.45$, which is not high enough to cover the promised amount of 1.55 to the two creditors. The firm is then liquidated.

The first noticeable feature in figure 3 is that the firm switches to two creditors only in the bad state $\theta = B$ when the fundamental deteriorates and consolidates back to a single creditor structure when its performance improves. In the model, the firm is never liquidated with a single incumbent creditor. Therefore, the extra pledgeability from two creditors is costly, and the firm uses it only as a last line of defense. Second, the interest rates are higher in general with more creditors. Why does this not contradict with proposition 2, which states that having more creditors reduces interest rates? Even though an exogenous increase in the number of creditors may increase pledgeability and lower the interest rate, once the number of creditors is endogenized in equilibrium, the firm only chooses to have more creditors when higher debt capacity is needed, which occurs in worse states and causes higher interest rates. Empirically, Petersen and Rajan (1994) find that companies with more

[9]To be specific, $1.23 \times 0.63 > 0.77$.

banking relationships also have higher cost of credit.

6.2 When Do Firms Choose More Creditors?

Although dynamic programming discrete choice models generally do not deliver analytical tractability, I can provide a sufficient condition under which the firm increases the number of creditors. This result inherits the idea from the previous subsection that the firm has to borrow from more creditors when its debt capacity with fewer creditors is insufficient. The next result argues that one of these scenarios is the case in which the firm has performed poorly in the past. Here, I keep the time subscripts to avoid any confusion.

Proposition 5 *Suppose that the realized fundamental is low $F_t \geq N_t \lambda Y_t$, the state is bad $\theta_{t+1} = B$, and rollover is possible $N_t \lambda \leq \kappa^B$. Then the continuation number of creditors must strictly increase, $N_+^{B\star}(N_t \lambda Y_t, F_t, Y_t) > N_t$.*

Providing the proof here is worthwhile. Since $N_t \lambda Y_t \leq \kappa^B Y_t$ and $N_t \lambda Y_t \leq F_t$, the firm can roll over by paying the liquidation value to each creditor, totaling $N_t \lambda Y_t$. Because the realized repayment to each creditor at period $t+1$ is at most $\min\{\frac{F_{t+1}}{N_{t+1}}, \lambda Y_{t+1}\} \leq \lambda Y_{t+1}$, the debt capacity in the bad state with N_{t+1} creditors is bounded by

$$\kappa^B_{N_{t+1}} Y_t \leq (1-\pi) N_{t+1} E(\lambda Y_{t+1}) = (1-\pi)\mu(N_{t+1}\lambda Y_t).$$

Since $(1-\pi)\mu < 1$ by assumption (1), $\kappa^B_{N_{t+1}} Y_t < N_{t+1} \lambda Y_t$. The firm chooses a continuation number of creditors N_+ at least to finance the required repayment $N_t \lambda Y_t$. Thus,

$$N_t \lambda Y_t \leq \kappa^B_{N_+} Y_t < N_+ \lambda Y_t.$$

Therefore, $N_+ > N_t$.

The intuition here is straightforward. For each individual creditor, the pledgeable amount is at most the expected liquidation value. In the bad state, with probability π, the firm dies without payout in the next period. The assumption $(1-\pi)\mu < 1$ implies that the expected liquidation value tomorrow is less than the liquidation value today. So for each liquidation value that the firm has to pledge today, it must seek more than one creditor on average. Therefore, the number of creditors must strictly increase. This result has also been empirically documented by Farinha and Santos (2002), who show that firms are more likely to abandon a single creditor structure when the performance measures are worse.[10]

6.3 Empirical Predictions Revisited

Recall that section 5 focused on the comparative statics of the number of creditors on interest rates, the probabilities of liquidation, renegotiation, and default, and the firm value. Now I discuss the corresponding implications in equilibrium, taking into account that the firm chooses more creditors when it is in bad shape. As we have seen in subsection 6.1, the implication of proposition 2 on the interest rate is reversed. The equilibrium selection effect dominates the pledgeability effect, resulting in higher interest rates associated with more creditors. However, as the direction predicted by proposition 4, the firm value is still lower with more creditors in equilibrium. The selection effect that links more creditors with bad performance reinforces the comparative static result in proposition 4. By the same reasoning, the liquidation probability jumps up with more creditors in equilibrium.

6.3.1 Growth and the Number of Creditors

When the per period shock to fundamental z_t on average improves, the future of the firm becomes more promising. This situation has several effects. A direct effect is that the firm has a higher liquidation value on average in the next period, which increases the bargaining position of the creditors. Second, the firm in the next period is more likely to have the resources to make the promised repayment or survive a negotiation. Both effects improve the debt value as well as the debt capacity, and more creditors can be supported.

[10] The performance measures include liquidity, cash flow, leverage and so on.

Proposition 6 *Suppose g_i $(i = a, b)$ are two density functions for z, and g_a first-order stochastically dominates g_b. Then for any equilibrium under g_b, there exists an equilibrium under g_a such that $\kappa^{\theta,a} \geq \kappa^{\theta,b}$, where $\kappa^{\theta,i}$ are the corresponding debt capacity coefficients.*[11] *In addition, the creditor capacity and the safe number of creditors are both higher under g_a, i.e., $\overline{N^a} \geq \overline{N^b}$ and $\underline{N^a} \geq \underline{N^b}$.*

Since first-order stochastic dominance implies that the average growth rate is higher, a direct implication is that firms with higher growth rates can be associated with more creditors. This is consistent with the empirical evidence documented by Farinha and Santos (2002), who find that firms with a better growth perspective, as measured by sales growth, tend to have more creditors.

7 The Value of Coordination Failure

7.1 Ex Post Efficient Policies

Coordination failure among creditors reduces the financial flexibility that the firm needs during a crisis. Quite often, firms in distress or even default are more valuable as going concerns than they are being liquidated piecemeal. In fact, because of the coordination problem among creditors, many policies are designed to reduce or eliminate liquidation. For example, the *automatic stay* clause, which halts creditors' actions to claim a debtor's assets, and *Chapter 11 reorganization*, which promotes a constructive renegotiation with all creditors collectively, both fall into this category. If the policies indeed eliminate all ex post coordination failure and force multiple creditors to negotiate the debt as one, then I show that such policies cause ex ante higher chances of liquidation and lower firm values.

Committing to an ex post efficient negotiation is equivalent to a counterfactual model in which the firm can borrow *only* from one creditor. With one creditor, the firm at most

[11] The opposite direction holds too. That is, for any equilibrium under g_a, there exists an equilibrium under g_b such that $\kappa^{\theta,a} \geq \kappa^{\theta,b}$.

repays the liquidation value if the project does not mature, independent of the firm's ability to switch to multiple creditors. Therefore, it is easy to see that the debt capacities are still (κ_1^N, κ_1^B) given by (21) in this counterfactual case.

In the bad state, the debt capacity is $\kappa_1^B = (1-\pi)\mu\lambda < \lambda$, so when the realized fundamental Y is sufficiently weak ($F > \lambda Y$), repayment negotiation fails because the firm cannot credibly pledge the liquidation payoff $\min(F, \lambda Y) = \lambda Y > \kappa_1^B Y$. Therefore, the single-creditor counterfactual case has effectively no room for negotiation, when the state is bad. On the contrary, in the true model if the firm is allowed to have multiple creditors, it can pledge at least λY (in fact, $\kappa^B Y$), so a single creditor never liquidates. Therefore, the expected probability of liquidation $L_1^{\theta,T}(F, Y)$ is lower for the true model compared with the counterfactual one.

Using the same example as in section 6, figure 4 plots the expected probability of liquidation $L_1^{\theta,\infty}(F, Y)$ against the expected value of the debt conditional on the current state $\theta = G$ (top panel) and $\theta = B$ (bottom panel). The solid (dashed) line is the liquidation probability with a single creditor (two creditors) in the full model. The dotted line is for the counterfactual model in which the number of creditors is exogenously fixed at one.

As figure 4 illustrates, having two creditors generally means a higher liquidation probability than having a single creditor in the true model because of the following two adverse effects. The short-term effect is a higher probability of an immediate liquidation in the next period, captured by lemma 3. The long-term effect is that more creditors can secure a bigger repayment, which requires a larger continuation face value, which in turn causes a higher liquidation probability in the future. Even so, the option of having two creditors is still beneficial in the sense that it uniformly reduces the firm's liquidation probability with a single creditor compared with the counterfactual. The possibility of having multiple creditors in the future and supporting a higher debt level prevent an even sooner liquidation when the firm initially gets into trouble.

Firm values tell a similar story. Although establishing strict inequalities in a dynamic programming framework requires some work, the economics behind it is intuitive. Without the costly mechanism to support a higher leverage by more creditors, the firm fails even sooner, lowering its value.

Proposition 7 *Let $V_{CF}^{\theta}(F, Y)$ be the firm value in the counterfactual world. Then for any $F > 0$, $V_{CF}^{\theta}(F, Y) < V_1^{\theta}(F, Y)$, and for any $N > 1$, there exists a nonempty set $\mathbb{F}$ (may depend on N) such that $V_{CF}^{\theta}(F, Y) < V_N^{\theta}(F, Y)$ for all $F \in \mathbb{F}$.*

In this economy, since the creditors always break even, the total value of the firm is a welfare criterion. As predicted by proposition 7, eliminating the possibility of a coordination failure is socially inefficient. More interestingly, the result suggests that mistakenly sticking with a single creditor may be even more inefficient than having the firm mistakenly end up with multiple creditors. This comparision between two types of mistakes is also confirmed by the liquidation probability. In figure 4, for a substantial range of fundamental values, the liquidation probability with *two creditors* in the true model is strictly lower compared with the *single* creditor counterfactual.

These findings raise caution regarding ex post efficient procedures such as automatic stay clause and Chapter 11 reorganization. These policies can be somewhat viewed as a commitment that the creditors will accept ex post efficient offers. Although eliminating ex post inefficiency, the policies also prevent the firm from utilizing enhanced pledgeability in the future. As we have seen, such policies lead to more likely liquidation, lower firm value and lower welfare ex ante.

7.2 Collateral

Collateral is typically viewed as a means of securing a creditor's position. It alleviates the ex post coordination problem because the liquidating creditor can no longer pose externalities on the secured creditors. In the extreme case, if all positions are secured, then no ex post coordination failure exists. In the model this case is equivalent to the previous counterfactual model in which the firm is exogenously restricted to borrowing from only one creditor. All results in the previous subsection still hold, with the striking prediction that firms that issue collateralized debts are more likely to be liquidated and have lower values. In addition, counterintuitively collateralized debt also leads to lower borrowing capacity compared with an uncollateralized instrument that is subject to ex post coordination failure.

8 Renegotiation Frequency

How does renegotiation frequency affect the equilibrium outcome? Since renegotiation happens each time the debt matures, it is equivalent to the debt maturity in the model. To highlight the economic intuition, I simplify the model such that the shock $z_t = \mu$ is a constant and the transition matrix is symmetric $p^G = p^B = p$. Instead of shrinking the debt maturity directly, I keep a stationary structure of one-period debt and extend the expected project duration. Letting

$$\hat{\pi} = \frac{\pi}{T}, \tag{28}$$

the expected project duration becomes $E(\tau_{\hat{\pi}}) = \frac{T}{\pi} = TE(\tau_\pi)$, T times longer than in the original model. This structure effectively shrinks the debt maturity to $\frac{1}{T}$ period under the original calendar time. I then pick the new growth rate $\hat{\mu}$ and the switching probability $\hat{p}$ to match the first best firm values as defined in (3):

$$\hat{V}^{\theta\star}_{FB}(Y|\hat{\mu}, \hat{\pi}, \hat{p}) = V^{\theta\star}_{FB}(Y|\mu, \pi, p) \tag{29}$$

for both $\theta = G, B$, so the firm quality is unaffected by the change in timescale. The following lemma confirms that the proposed modifications are natural in the following sense. First, the parameters of the game after the timescale change are well defined. Second, when the period length is very small, the (probabilities of) changes in the state variables are also very small.

Lemma 4 *The new set of parameters after the timescale change* $\hat{\pi} = \frac{\pi}{T} \in (0,1)$, $\hat{\mu} = \frac{T\mu}{T\mu - \mu + 1} > 1$, *and* $\hat{p} = \frac{T-1+p(1-\pi)}{T-\pi} \in (0,1)$ *are well defined. In addition, as the effective debt maturity goes to* 0, *i.e., when* $T \to \infty$, *the new parameters satisfy* $\hat{\pi} = \frac{\pi}{T} \to 0$, $\hat{\mu} \to 1$, $\hat{p} \to 0$, *and* $(1-\hat{\pi})\hat{\mu} < 1$.

From (21), an immediate implication of lemma 4 is $\hat{\kappa}_1^\theta \to \lambda$. The next result characterizes the debt capacity under the new timescale. The key feature is that with more frequent negotiation, in the limit, the pledgeable amount in the bad state $\theta = B$ approaches the liquidation value. Recall from the example in section 6 and proposition 5 that the debt capacity in the bad state is crucial for when the firm increases the number of creditors. Therefore, the benefit of having multiple creditors becomes negligible as the firm renegotiates more frequently. I denote the variables with hats as the ones after the timescale change.

Proposition 8 *When* $T \to \infty$, *the debt capacities* $\hat{\kappa}^B \to \lambda$.

To understand this result, recall that the firm can only pledge the liquidation value with a single creditor. Although more creditors indeed enforce more repayment by proposition 2, the ultimate source of this extra repayment is from the *growth between two negotiation dates*. If the expected per period growth of the fundamental diminishes ($\hat{\mu} \to 1$), then the incremental pledgeability vanishes as well. Since renegotiation is closely related to the debt maturity in the model and a troubled firm typically negotiates the repayment at maturity in practice, the renegotiation frequency can be interpreted as the debt maturity. With very short maturity,[12] having multiple creditors provides no extra pledgeability.

[12] For example, an overnight repo agreement.

9 Possible Extensions

9.1 Uneven Concentration

Throughout the model, I have assumed that creditors are exante identical. However, this assumption is not crucial for the previous results. The key economic force here is that having more creditors means that each one of them can pose greater externalities on the others, causing coordination problems, which, on the other hand, improves their collective bargaining position with the entrepreneur. Allowing creditors to have different shares of the loan does not eliminate these channels.

9.2 Private Savings by the Entrepreneur

Suppose the entrepreneur can save; that is, instead of raising just enough money to roll over maturing debts, the firm can now borrow more and keep internal cash. The relevant question regarding the number of creditors the firm chooses centers on whether the firm wishes to borrow from more creditors and save for the future. A rigorous analysis of this problem is beyond the scope of this paper, but intuitively we know that the firm has no incentive to do so. First, having more creditors increases the chance of liquidation for the firm. Moreover, having internal cash does not benefit the firm, but instead gives each creditor a stronger bargaining position as the liquidation value of the firm increases and exacerbates the coordination problem.

9.3 Entrepreneur's Liquidation Incentive

The endogenous parameter assumption $\kappa^{\theta} \geq \lambda$ in proposition 1 rules out the entrepreneur's incentive to voluntarily liquidate the project. Without this assumption, the entrepreneur may wish to liquidate in equilibrium. For example, in a bad state, if the entrepreneur definitely foresees a liquidation tomorrow, he is better off voluntarily liquidating today, since the liquidation payoff λY_t is higher than the continuation value $(1-\pi)\lambda E_t(Y_t z_{t+1}) =$

$(1-\pi)\mu\lambda Y_t$. An interesting study would investigate how creditor dispersion interacts with the entrepreneur's liquidation decision. This topic is left for future research.

10 Conclusion

I build a dynamic model in which the firm must repeatedly roll over debt and can renegotiate repayment. Having more creditors brings the disadvantage of coordination problems, which in bad times make it harder for a firm to restructure its debt to avoid liquidation. In good times, however, these same coordination problems enhance pledgeability by making it harder for a firm to opportunistically hold up its creditors. In the model, the firm actively chooses the number of creditors over time by optimally trading off pledgeability with the liquidation probability.

Analysis of the model shows that firms increase the number of creditors when they perform badly. Doing so increases the liquidation probability and lowers the firm value. Allowing for coordination failure in equilibrium is valuable and policies that commit the creditors to ex post efficient coordination reduce the firm value and may raise the liquidation probability. If the firm can renegotiate the debt very frequently, the enhanced pledgeability from multiple creditors diminishes.

The model's implications highlight the potential for selection bias in empirical studies that investigate the effect of creditor dispersion. For example, an exogenous increase in the number of creditors lowers the required interest rate due to the firm's better repayment incentives. In equilibrium, however, this relationship is reversed because firms choose more creditors when they are in trouble, which in turn leads to higher interest rate.

Finally, having outstanding debt may provide the entrepreneur with the incentive to inefficiently continue the project, for example, risk shifting and gambling for survival. The received wisdom is that a higher level of debt exacerbates the problem and increases the inefficiency associated with such continuation bias. In this paper, I make parameter assumptions such that continuing the project is always efficient.[13] Therefore, there is no debt-equity conflict in continuing the project inefficiently. Instead, if abandoning the project is optimal in certain states, then having outstanding debt generates non-monotonic outcomes in my model, in contrast with the aforementioned intuition. When leverage is low, the entrepreneur implements the first best liquidation strategy. When leverage is high, the efficient liquidation can still be implemented. In this case, even though the entrepreneur is willing to gamble for survival, the creditors refuse to rollover and force an efficient termination. In addition, an intermediate case may exist, in which the debt level is high enough to distort the entrepreneur's liquidation incentive, but not too high to spur the creditors into action. Intuitively, having more creditors in this intermediate case may facilitate restoring the efficient liquidation strategy and correct the entrepreneur's continuation bias. A more rigorous analysis is required to further investigate this problem and I look forward to future research that can shed light on this issue.

References

[1] Albuquerque, Rui and Hugo A. Hopenhayn, Optimal Lending Contracts and Firm Dynamics, Review of Economic Studies (2004) 71, 285–315.

[2] Barrett, Rick, School Supply Distributor School Specialty Files for Bankruptcy Protection, http://www.jsonline.com/business/school-supply-distributor-school-specialty-files-for-bankruptcy-protection-f48hsvo-188661101.html, (2013, Jan. 28).

[3] Berglöf, Erik and Ernst-Ludwig von Thadden, Short-Term Versus Long-Term Interests: Capital Structure with Multiple Investors, The Quarterly Journal of Economics, Vol. 109, No. 4 (Nov., 1994), pp. 1055-1084.

[4] Bolton, Patrick and David S. Scharfstein, Optimal Debt Structure and the Number of Creditors, Journal of Political Economy, Vol. 104, No. 1 (Feb., 1996), pp. 1-25.

[13] To be specific, condition (4).

[5] Brunnermeier, Markus K. and Martin Oehmke ,The Maturity Rat Race, The Journal of Finance Volume 68, Issue 2, pages 483–521, April 2013.

[6] Degryse, Hans and Steven Ongena, Bank Relationships and Firm Profitability, Financial Management, Vol. 30, No. 1 (Spring, 2001), pp. 9-34.

[7] Detragiache, Enrica, Paolo Garella and Luigi GuisoMultiple versus Single Banking Relationships: Theory and Evidence, The Journal of Finance, Vol. 55, No. 3 (Jun., 2000), pp. 1133-1161.

[8] Diamond, Douglas W. and Philip H. Dybvig, Bank Runs, Deposit Insurance, and Liquidity, Journal of Political Economy, Vol. 91, No. 3 (Jun., 1983), pp. 401-419.

[9] Diamond, Douglas W., Financial Intermediation and Delegated Monitoring, The Review of Economic Studies, Vol. 51, No. 3 (Jul., 1984), pp. 393-414.

[10] Diamond, Douglas W., Presidential Address, Committing to Commit: Short-term Debt When Enforcement Is Costly, The Journal of Finance Vol. 59, No. 4 (Aug., 2004) (pp. 1447-1479).

[11] Dugan, Ianthe Jeanne, As Banks Retreat, Hedge Funds Smell Profit, http://online.wsj.com/news/articles/SB10001424127887324637504578567383459564510, (2013, Jul. 22).

[12] Farinha, Luísa A., João A. C. Santos, Switching from Single to Multiple Bank Lending Relationships: Determinants and Implications, Journal of Financial Intermediation 11, 124–151 (2002).

[13] Gale, Douglas and Martin Hellwig, Incentive-Compatible Debt Contracts: The One-Period Problem, Review of Economic Studies (1985) LII, 647-663.

[14] Gilson, Stuart C., Kose John and Larry H.P. Lang, Troubled debt restructurings An empirical study of private reorganization of firms in default, Journal of Financial Economics 27 (1990) 315-353.

[15] Goldstein, Itay and Ady Pauzner, Demand-Deposit Contracts and the Probability of Bank Runs, The Journal of Finance, Vol. 60, No. 3 (Jun., 2005), pp. 1293-1327.

[16] He, Zhiguo and Wei Xiong, Dynamic Debt Runs, Review of Financial Studies 25, pp. 1799-1843.

[17] He, Zhiguo and Wei Xiong, Rollover Risk and Credit Risk, Journal of Finance 67, 2012, 391-429.

[18] Hege, Ulrich and Pierre Mella-Barral, Repeated Dilution of Diffusely Held Debt, The Journal of Business, Vol. 78, No. 3 (May 2005), pp. 737-786.

[19] Kahn, Charles and Andrew Winton, Moral Hazard and Optimal Subsidiary Structure for Financial Institutions, The Journal of Finance, Vol. 59, No. 6 (Dec., 2004), pp. 2531-2575.

[20] Leland, Hayne E., Agency Costs, Risk Management, and Capital Structure, Volume 53, Issue 4, pages 1213–1243, August 1998.

[21] Mella-Barral, Pierre and William Perraudin, Strategic Debt Service, The Journal of Finance Vol. 52, No. 2 (Jun., 1997) (pp. 531-556).

[22] Mella-Barral, Pierre, The dynamics of default and debt reorganization, Rev. Financ. Stud. (1999) 12 (3): 535-578.

[23] Ongena, Steven and David C. Smith, What Determines the Number of Bank Relationships? Cross-Country Evidence, Journal of Financial Intermediation 9, 26–56 (2000).

[24] Petersen, Mitchell and Raghuram Rajan, The Benefits of Firm-Creditor Relationships: Evidence from small business data, Journal of Finance, March 1994, 49(1), pp. 3-37.

[25] Petersen, Mitchell and Raghuram Rajan, The Effect of Credit Market Competition on Lending Relationships, Quarterly Journal of Economics, March 1995, 110(2), pp. 407-443.

[26] Williamson, Stephen D., Costly Monitoring, Financial Intermediation, and Equilibrium Credit Rationing, Journal of Monetary Economics 18 (1986) 159-179.

Appendix

Lemma A-1 *(Multi-dimensional Blackwell's Sufficient Condition) Let* $X \subseteq \mathbb{R}^K$ *and* $B^L(X)$ *be the space of bounded vector-valued functions:* $v = (v_1, v_2, ..., v_L) : X \to \mathbb{R}^L$*, where* $L < \infty$*. Equipe* $B^L(X)$ *with the sup norm over coordinates, i.e.* $||v|| = \max_{i \leq L}\{\sup_x v_i(x)\}$*. Suppose* $v, w \in B^L(X)$*, and define* $v \geq w$ *if and only if* $v_i \geq w_i$ *for all* $i \leq L$*. If the operator* $T : B^L(X) \to B^L(X)$ *satisfies that*

1. *(monotonicity) if* $v \geq w$*, then* $T(v) \geq T(w)$*, and*

2. *(discounting) there exists a constant* β *such that for any constant* a*,* $T(v+a) \leq T(v) + \beta a$*,*

then T *is a contraction mapping with coefficient* β*, namely* $||Tv - Tw|| \leq \beta||v - w||$ *for any* $v, w \in B^L(X)$*.*

Proof. Since $w \leq v + ||w - v||$, so monotonicity of T implies $T(w) \leq T(v + ||w - v||)$. The latter expession is in turn bounded by $T(v) + \beta||w - v||$ by discounting. Therefore,

$$T(w) - T(v) \leq \beta||w - v||.$$

Similarly, one can derive the opposite side $T(v) - T(w) \leq \beta||w - v||$. By the definition of the norm on $B^L(X)$, $||T(w) - T(v)|| \leq \beta||w - v||$. T is therefore a contraction mapping with coefficient β. ■

Proof of Lemma 1: In order to be consistent with the notations in the main text following the lemma, denote the values given in (2) by $V_{FB}^{\theta\star}(Y)$. They can be recursively formulated as following:

$$
\begin{aligned}
V_{FB}^{G\star}(Y) &= E\{\pi Y z + (1-\pi)[p^G V_{FB}^{G\star}(Yz) + (1-p^G)V_{FB}^{B\star}(Yz)]\} \\
V_{FB}^{B\star}(Y) &= (1-\pi)E[p^B V_{FB}^{B\star}(Yz) + (1-p^B)V_{FB}^{G\star}(Yz)].
\end{aligned} \tag{A-30}
$$

The first part $\pi Y z$ captures the final dividend, which is materialized only in the good state $\theta = G$. This case occurs with probability π. The second part captures the continuation payoff taking into account a potential switch in the state θ. Normalizing by Y and letting $v_{FB}^{\theta}(Y) = \frac{V_{FB}^{\theta\star}(Y)}{Y}$, (A-30) becomes

$$
\begin{aligned}
v_{FB}^{G}(Y) &= E\{\pi z + (1-\pi)[p^G v_{FB}^{G}(Yz)z + (1-p^G)v_{FB}^{B}(Yz)z]\} \\
v_{FB}^{B}(Y) &= (1-\pi)E[p^B v_{FB}^{B}(Yz)z + (1-p^B)v_{FB}^{G}(Yz)z].
\end{aligned} \tag{A-31}
$$

For any bounded continuous functions on $\mathbb{R}_+$: $v_{FB}^{\theta} \in B^1(\mathbb{R}_+)$, $(\theta = G, B)$, it is easy to check that the right hand side of (A-31) induces a natural operator $T : C_B^2(\mathbb{R}_+) \to C_B^2(\mathbb{R}_+)$ as following:

$$
T(v_{FB}^{G}, v_{FB}^{B}) = \begin{cases} E\{\pi z + (1-\pi)[p^G v_{FB}^{G}(Yz)z + (1-p^G)v_{FB}^{B}(Yz)z]\} \\ (1-\pi)E[p^B v_{FB}^{B}(Yz)z + (1-p^B)v_{FB}^{G}(Yz)z]. \end{cases}
$$

Clearly T satisfies the monotonicity condition in lemma A-1. To verify the discounting condition, notice

$$
T(v_{FB} + a) = T(v_{FB}) + (1-\pi)E(az) = T(v_{FB}) + (1-\pi)\mu a.
$$

By assumption (1) and lamma A-1, T is a contraction. Therefore, Banach fixed point theorem states that T has a unique fixed point, which implies (A-31) and thereby (A-30) have a unique solution. Finally, to find this solution, observe that (A-31) has a constant solution $(v_{FB}^{G\star}, v_{FB}^{B\star})$ that satisfies:

$$\begin{aligned} v_{FB}^{G} &= \pi\mu + (1-\pi)\mu[p^G v_{FB}^G + (1-p^G)v_{FB}^B]\} \\ v_{FB}^{B} &= (1-\pi)\mu[p^B v_{FB}^B + (1-p^B)v_{FB}^G]. \end{aligned}$$

Solving the above system for (v_{FB}^G, v_{FB}^B) gives (3).

Proof of Proposition 1: The proof contains three parts to verify the proposed equilibrium. First, given the conjectured properties stated in the proposition, I show that the conjectured strategy profile indeed constitutes a subgame perfect equilibrium. Part II (III) proves that the conjectured properties for the value of debt (firm) indeed hold in this equilibrium. In the following proof, the time indices and the arguements in the strategies are sometimes omitted when there is no confusion.

Part I: Given the stated properties of D_N^θ and V_N^θ, I check that the proposed strategy profile is subgame perfect. If the firm survives the period t stage game, then following the equilibrium strategies, the expected payoff to the entrepreneur is

$$V_{N_+^\star}^{\theta_{t+1}}(F_+^\star, Y_t) - \min(F_t, N\lambda Y_t) = \max_{N_+} V_{N_+}^{\theta_{t+1}}(F_+(X^*), Y_t) - X^*,$$

where $X^* = \min(F_t, N\lambda Y_t)$ and $F_+(X)$ is the smallest solution to $D_{N_+}^\theta(F_+, Y) = X$. By the definition of κ^θ, the conjectured property that $V_{N_+}^\theta(F_+, Y) \geq \kappa_{N_+}^\theta Y$, and the endogenous assumption $\kappa^\theta \geq \lambda$, the above equality implies:

$$V_{N_+^\star}^{\theta_{t+1}}(F_+^\star, Y_t) - \min(F_t, N\lambda Y_t) \geq \kappa^{\theta_{t+1}} Y - F_t \geq \lambda Y_t - F_t.$$

Therefore, the continuation payoff is weakly higher than the liquidation payoff. Thus the firm has no strict incentive to voluntarily liquidate nor to offer $S_i < \min(\frac{F}{N}, \lambda Y)$ and induce an immediate liquidation. Suppose the firm offers $S_i > \min(\frac{F}{N}, \lambda Y)$. Two possible cases can happen. If the offer is infeasible, i.e., $\sum_{j \leq i} S_j + (N-i)\min(\frac{F}{N}, \lambda Y) > DC^\theta(Y)$, then the creditor rejects the offer and the project is liquidated. This case is clearly dominated by the equilibrium outcome as discussed before. Alternatively if the offer is feasible. Let X be the total negotiated repayment following S_i. Clearly, it must be $X > X^*$, which implies $F_+(X) > F_+(X^*)$ for any given N_+. Because we have conjectured that $V_N^\theta(F, Y)$ is weakly decreasing in F, so

$$V_{N_+}^{\theta_{t+1}}(F_+(X^*), Y_t) - X^* \geq V_{N_+}^{\theta_{t+1}}(F_+(X), Y_t) - X^* > V_{N_+}^{\theta_{t+1}}(F_+(X), Y_t) - X,$$

for any N_+. Therefore, the entrepreneur is strictly worse off by offering any $S_i > \min(\frac{F}{N}, \lambda Y)$. In all, the offering strategy $S_i^\star = \min(\frac{F}{N}, \lambda Y)$ is optimal.

The entrepreneur's financing strategy $(N_+^{\theta\star}, F_+^{\theta\star})$ is just a repetition of the equilibrium definition. The ith incumbent creditor clearly has no incentive to accept any offer lower than the liquidation payoff. On the other hand, if the payoff is not feasible such that (12) fails, the project will be liquidated following the equilibrium strategies by other creditors. In this case, creditor i either gets $\min(\frac{F_t}{N_t}, \frac{1}{N_t - 1}\max(0, \lambda Y_t - \frac{F_t}{N_t}))$ or $\frac{1}{N_t}\min(F_t, \lambda Y_t)$, both are weakly dominated by $\min(\frac{F_t}{N_t}, \lambda Y_t)$. Finally, the optimality of the new creditors' strategies $r_i^{\theta\star}$ is trivial to verify.

Part II: Given the above strategies, I now show that there exists a consistent linear conjecture of the debt capacities, i.e. $DC^\theta(Y) = \kappa^\theta Y$ for some constants κ^θ. In addition, the value of debt $D_N^\theta(F, Y)$ is continuous and HD1 in (F, Y).

Under the conjecture $DC^\theta(Y) = \kappa^\theta Y$, the equilibrium strategies (condition (12) in particular) imply that rollover is possible if and only if (14) holds. Under this condition, the value of debt can be rewritten as (15). The value of debt $D_N^\theta(F, Y)$ is clearly HD1, because one can verify that

$$D_N^\theta(F, Y) = Y D_N^\theta(\frac{F}{Y}, 1) \equiv Y D_N^\theta(f, 1).$$

where $f \equiv \frac{F}{Y}$. The ratio $\frac{\overline{F}_N^\theta(Y)}{Y}$ being a constant independ of Y is a simple corollary of HD1. In fact, one can readily see $\overline{f}_N^\theta(Y) = \arg\max_f D_N^\theta(f, 1)$. In addition, the debt capacity with

N creditors is linear as given by (16). Finally, $D_N^\theta(f,1)$ is continuous in f, since it can be expressed as the sum of integrals in the form of $\int_{B(f)}^{A(f)} C(f,z)dz$, where A, B, C are continuous functions in their arguments. For example, when $N > \frac{\max_\theta \kappa^\theta}{\lambda}$,

$$\begin{aligned} D_N^\theta(f,1) \quad \equiv \quad & \pi \int_{\underline{z}}^{\overline{z}} \min(f,z)\mathbf{1}_{\theta=G} g(z)dz + (1-\pi)\sum\nolimits_{\theta'=G,B} \\ & P(\theta'|\theta)[\int_{\max(\underline{z},\frac{f}{\kappa^{\theta'}})}^{\max(\overline{z},\frac{f}{\kappa^{\theta'}})} g(z)fdz + \int_{\min(\underline{z},\frac{f}{\kappa^{\theta'}})}^{\min(\overline{z},\frac{f}{\kappa^{\theta'}})} \lambda z g(z)dz] \end{aligned} \tag{A-32}$$

which is clearly continuous in f. The remaining cases are similar. Finally, I show that there exists a consistent conjecture of $\{\kappa_N^\theta, \kappa^\theta\}_{N\in\mathbb{N}}^{\theta=G,B}$. Notice that (16) is a function of (κ^G, κ^B). Denote $\hat{\kappa}_N^\theta(\kappa^G,\kappa^B)$ to be this function and let $L^\theta(\kappa^G,\kappa^B) \equiv \max\{\hat{\kappa}_1^\theta, ..., \hat{\kappa}^\theta_{[\frac{\max_\theta k^\theta}{\lambda}]+1}\}$. So a consistent conjecture of $\{\kappa_N^\theta, \kappa^\theta\}_{N\in\mathbb{N}}^{\theta=G,B}$ is a solution to (13) which is in turn a fixed point of $L \equiv (L^G, L^B) : \mathbb{R}_+^2 \to \mathbb{R}_+^2$. Equipe $\mathbb{R}_+^2$ with the usual partial order $\leq$ such that $x \leq y$ if and only if $x_1 \leq y_1$ and $x_2 \leq y_2$. Apparently L is order-preserving, since D_N^θ is weakly increasing in κ^θ. I shall then construct a complete lattice $\Omega \subseteq \mathbb{R}_+^2$ such that $L(\Omega) \subseteq \Omega$. By Tarski's fixed point theorem, L has a fixed point and therefore a solution to (13) exists. The remainder of the proof is to construct such an Ω.

By (1), it is possible to choose M big enough such that

$$(\pi + (1-\pi)M)\mu < M. \tag{A-33}$$

Let $\Omega \equiv [0,M] \times [0,M]$ be a complete lattice. Suppose $(\kappa^G,\kappa^B) \in \Omega$, (16) and (A-33) imply that for all $N \leq [\frac{M}{\lambda}] + 1$,

$$\begin{aligned} \hat{\kappa}_N^\theta \quad = \quad & \max\nolimits_f E\{\pi \min(f,z)\mathbf{1}_{\theta=G} + (1-\pi) \\ & [\mathbf{1}_{\min(f,N\lambda z)\leq\kappa^{\theta'} z} \min(f, N\lambda z) + \mathbf{1}_{\min(f,N\lambda z)>\kappa^{\theta'} z} \min(f,\lambda z)]\} \\ \leq \quad & \pi E(z) + (1-\pi)E[\mathbf{1}_{\min(f,N\lambda z)\leq\kappa^{\theta'} z}\kappa^{\theta'} z + \mathbf{1}_{\min(f,N\lambda z)>\kappa^{\theta'} z}\lambda z] \\ \leq \quad & \pi\mu + (1-\pi)M\mu \\ < \quad & M. \end{aligned}$$

Therefore, $L^{\theta}(\kappa_N^{\theta}) < M$, which implies that Ω is invariant under L . This completes the proof.

Part III: Finally, for any pair of $\kappa^{\theta} \geq \lambda$, I will show there exists a unique continuous HD1 function $V_N^{\theta}(F, Y)$ which is increasing in Y, weakly decreasing in F and $V_N^{\theta}(F, Y) \geq \kappa_N^{\theta} Y$ for any $F \leq \overline{F}_N^{\theta}$. By the discussion following proposition 1, in the conjectured equilibrium, the firm's problem can be rewritten as a dynamic programming problem (17) and (18). By the definition of $\overline{N}$ in (19) and the discussion following it, we can confine the choice of N_+^{θ} to $\{1, 2, ..., \overline{N}\}$ without loss of generality.

Define an auxiliary problem:

$$v_N^{\theta}(f) \quad = \quad E\{\pi z \mathbf{1}_{\theta=G} + (1-\pi)[\mathbf{1}_{\min(f, N\lambda z) \leq \kappa^{\theta'} z} \max\nolimits_{N_+} v_{N_+}^{\theta'}(f_+) z + \mathbf{1}_{\min(f, N\lambda z) > \kappa^{\theta'} z} \lambda z]\} \tag{A-34}$$

where $f_+(\frac{f}{z}, N)$ is the minimum solution to

$$D_{N_+}^{\theta'}(f_+, 1) = \min(\frac{f}{z}, N\lambda). \tag{A-35}$$

By the definition of $\overline{f}_N^{\theta}$ in proposition 1, it must be $f_+ \leq \overline{f}_{N_+}^{\theta'}$. Denote $T_N^{\theta} : B^{2\overline{N}} \to B$ to be the operator on $(v_i^{\theta})_{i \leq \overline{N}}^{\theta=G,B}$ induced by the right-hand side of (A-34) and let $T \equiv (T_N^{\theta}) : B^{2\overline{N}} \to B^{2\overline{N}}$.

First, notice that if $v \in B^{2\overline{N}}$ is bounded by some $M > 1$, then $||Tf(v)|| \leq \pi(1+\mu) + (1-\pi)M(1+\mu)$ is also bounded. So T is indeed well-defined. Then I prove that T is a contraction mapping by verifying monotonicity and discounting conditions in lemma A-1. Monotonicity is trivial. For any constant a, $T(v+a) \leq T(v) + (1-\pi)(1+\mu)a$. So the discounting condition holds by (1).

Denote $C_{a,l} = \{v \ : \ v$ is bounded, continuous, decreasing, and $v|_{[0,a]} \geq l\} \subseteq B^1$ to be the subset of all bounded continuous decreasing functions taking values in $[l, \infty)$ when restricted to $[0, a]$. Consider $C \equiv \times_{N \leq \overline{N}, \theta=G,B} C_{\overline{f}_N^{\theta}, \kappa_N^{\theta}}$. Clearly C is a closed subset of $B^{2\overline{N}}$. Next I show $T(C) \subseteq C$. Suppose $v \in C$ and $f_1 \leq f_2$. By the definition of f_+, we have $f_+(\frac{f_1}{z}, N) \leq$

$f_+(\frac{f_2}{z}, N)$. To simplify notation, let $f_{1+} \equiv f_+(\frac{f_1}{z}, N)$. The following inequalities must hold:

$$\begin{aligned}
T_N^\theta(v)(f_1) &= E\{\pi z \mathbf{1}_{\theta=G} + (1-\pi)[\mathbf{1}_{\min(f_1,N\lambda z)\le\kappa^{\theta'}z} \max_{N_+} v_{N_+}^{\theta'}(f_{1+})z + \mathbf{1}_{\min(f_1,N\lambda z)>\kappa^{\theta'}z}\lambda z]\} \\
&\ge E\{\pi z \mathbf{1}_{\theta=G} + (1-\pi)[\mathbf{1}_{\min(f_1,N\lambda z)\le\kappa^{\theta'}z} \max_{N_+} v_{N_+}^{\theta'}(f_{2+})z + \mathbf{1}_{\min(f_1,N\lambda z)>\kappa^{\theta'}z}\lambda z]\} \\
&\ge E\{\pi z \mathbf{1}_{\theta=G} + (1-\pi)[\mathbf{1}_{\min(f_2,N\lambda z)\le\kappa^{\theta'}z} \max_{N_+} v_{N_+}^{\theta'}(f_{2+})z + \mathbf{1}_{\min(f_2,N\lambda z)>\kappa^{\theta'}z}\lambda z]\} \\
&= T_N^\theta(v)(f_2).
\end{aligned}$$

The last inequality is because that $v_{N_+}^\theta(f_{2+}) \ge \kappa^\theta \ge \lambda$ and $\{z|\min(f_1, N\lambda z) \le \kappa^\theta z\} \supseteq \{z|\min(f_2, N\lambda z) \le \kappa^\theta z\}$ for $\theta = G, B$. So each coordinate in $T(v)$ is also a decreasing function. In addition,

$$\begin{aligned}
T_N^\theta(v)(\overline{f}_N^\theta) &= E\{\pi z \mathbf{1}_{\theta=G} + (1-\pi)[\mathbf{1}_{\min(\overline{f}_N^\theta,N\lambda z)\le\kappa^{\theta'}z} \max_{N_+} v_{N_+}^{\theta'}(f_+)z + \mathbf{1}_{\min(\overline{f}_N^\theta,N\lambda z)>\kappa^{\theta'}z}\lambda z]\} \\
&\ge E\{\pi z \mathbf{1}_{\theta=G} + (1-\pi)[\mathbf{1}_{\min(\overline{f}_N^\theta,N\lambda z)\le\kappa^{\theta'}z} \kappa^{\theta'} z + \mathbf{1}_{\min(\overline{f}_N^\theta,N\lambda z)>\kappa^{\theta'}z}\lambda z]\} \\
&\ge E\{\pi z \mathbf{1}_{\theta=G} + (1-\pi)[\mathbf{1}_{\min(\overline{f}_N^\theta,N\lambda z)\le\kappa^{\theta'}z} \min(\overline{f}_N^\theta, N\lambda z) + \mathbf{1}_{\min(\overline{f}_N^\theta,N\lambda z)>\kappa^{\theta'}z} \min(\overline{f}_N^\theta, N\lambda z)]\} \\
&= \kappa_N^\theta
\end{aligned}$$

The first inequality uses the fact $\max_{N_+} v_{N_+}^\theta(f_+) \ge \max_{N_+} \kappa_{N_+}^\theta = \kappa^\theta$ for both $\theta = G, B$, since $v \in C$. The second inequality holds because $\min(\overline{f}_N^\theta, N\lambda z) \le \kappa^\theta z$ over the relevant region. The last equality is by the definition of $\overline{f}_N^\theta$ and (16). Because $T_N^\theta(v)$ is a decreasing function, so $T_N^\theta(v)|_{[0,\overline{f}_N^\theta]} \ge \kappa_N^\theta$. Finally, I show that $T_N^\theta(v)$ must be a continuous function. Consider $\frac{f_2}{f_1} = 1 + \delta$. By definition (A-34)

$$\begin{aligned}
T_N^\theta(v)(f_2) &= \pi\mu \mathbf{1}_{\theta=G} + (1-\pi)\textstyle\sum_{\theta'=G,B} P(\theta'|\theta) \\
&\quad [\textstyle\int_{\min(f_2,N\lambda z)\le\kappa^{\theta'}z} \max_{N_+} v_{N_+}^{\theta'}(f_{2+}) z g(z) dz + \int_{\min(f_2,N\lambda z)>\kappa^{\theta'}z} \lambda z g(z) dz] \\
&= \pi\mu \mathbf{1}_{\theta=G} + (1-\pi)\textstyle\sum_{\theta'=G,B} P(\theta'|\theta) \\
&\quad [\textstyle\int_{\min(f_1,N\lambda z')\le\kappa^{\theta'}z'} \max_{N_+} v_{N_+}^{\theta'}(f_{1+})(1+\delta)^2 z' g[z'(1+\delta)] dz' \\
&\quad + \textstyle\int_{\min(f_1,N\lambda z')>\kappa^{\theta'}z'} \lambda(1+\delta) z' g[z'(1+\delta)] dz']
\end{aligned}$$

where the change of variable $z = (1+\delta)z'$. Notice that, by assumption, v_N^θ are bounded by some constant M and g is a density function, so, as $\delta \to 0$, the functions under the integrals in the above expression are dominated by $2Mz'g(2z')$. Because the random variable z has a finite mean, so $\int 2Mz'g(2z')dz' < \infty$. The dominated convergence theorem then implies that as $\delta \to 0$, the last expression converges to $T_N^\theta(v)(f_1)$. Therefore, the function $T_N^\theta(v)$ is continuous. In all, I have established that the contraction mapping T maps C into itself.

By contraction mapping theorem, the operator T has a unique fixed point $v^\star \in B^{2\overline{N}}$. Furthermore, this fixed point must belong to C. Define

$$V_N^\theta(F, Y) = v_N^{\theta\star}(\frac{F}{Y})Y. \tag{A-36}$$

which is decreasing in F. It is very easy to verify that the constructed solution satisfies the original recursive problem (17) with (18). Because $v_N^{\theta\star}(\frac{F}{Y})$ is increasing in Y, so V_N^θ as defined above is also increasing in Y. This completes the full proof of this proposition.

Proof of Lemma 2: Suppose otherwise if $\kappa^G \le \kappa^B$, then

$$\begin{aligned} & \mathbf{1}_{\min(F,N\lambda z)\le\kappa^G z}\min(f, N\lambda z) + \mathbf{1}_{\min(F,N\lambda z)>\kappa^G z}\min(f, \lambda z) \\ \le\ & \mathbf{1}_{\min(F,N\lambda z)\le\kappa^B z}\min(f, N\lambda z) + \mathbf{1}_{\min(F,N\lambda z)>\kappa^B z}\min(f, \lambda z). \end{aligned}$$

So (16) implies:

$$\begin{aligned} \kappa^B &\le \max\nolimits_{N,f}(1-\pi)E[\mathbf{1}_{\min(F,N\lambda z)\le\kappa^B z}\min(f, N\lambda z) + \mathbf{1}_{\min(F,N\lambda z)>\kappa^B z}\min(f, \lambda z)] \\ &\le (1-\pi)E\kappa^B z = (1-\pi)\mu\kappa^B \\ &< \kappa^B. \end{aligned}$$

Contradiction! So it must be $\kappa^G > \kappa^B$.

Proof of Proposition 2: First, if $N_1 = 1$, then by (15),

$$\begin{aligned} D^{\theta}_{N_2}(F,Y) &= E\{\pi \min(F,Yz)\mathbf{1}_{\theta=G} \\ &+ (1-\pi)[\mathbf{1}_{\min(F,N_2\lambda Yz)\leq\kappa^{\theta'}Yz}\min(F,N_2\lambda Yz) + \mathbf{1}_{\min(F,N_2\lambda Yz)>\kappa^{\theta'}Yz}\min(F,\lambda Yz)]\} \\ &\geq E[\pi\min(F,Yz)\mathbf{1}_{\theta=N} + (1-\pi)\min(F,\lambda Yz)] \\ &= D^{\theta}_{1}(F,Y). \end{aligned}$$

If $\underline{N} \geq N_2 > N_1$, by the definition of $\underline{N}$ in (20), then the liquidation region $\{z|\min(F,N_i\lambda Yz) > \kappa^{\theta}Yz\} = \emptyset$ for $\theta = G, B$. Therefore,

$$\begin{aligned} D^{\theta}_{N_2}(F,Y) &= E[\pi\min(F,Yz)\mathbf{1}_{\theta=G} + (1-\pi)\min(F,N_2\lambda Yz)] \\ &\geq E[\pi\min(F,Yz)\mathbf{1}_{\theta=G} + (1-\pi)\min(F,N_1\lambda Yz)] \\ &= D^{\theta}_{N_1}(F,Y). \end{aligned}$$

Finally, if $\overline{N} > N_2 > N_1 \geq \underline{N}$, then $\{z|\min(F,N_i\lambda Yz) > \kappa^{G}Yz\} = \emptyset$ and $\{z|\min(F,N_i\lambda Yz) > \kappa^{B}Yz\} = \{z|F > \kappa^{B}Yz\}$. Therefore,

$$\begin{aligned} D^{\theta}_{N_2}(F,Y) &= E\{\pi\min(F,Yz)\mathbf{1}_{\theta=G} + (1-\pi)\{P(G|\theta)\min(F,N_2\lambda Yz) \\ &+ P(B|\theta)[\mathbf{1}_{F\leq\kappa^{B}Yz}\min(F,N_2\lambda Yz) + \mathbf{1}_{F>\kappa^{B}Yz}\min(F,\lambda Yz)]\}\} \\ &\geq E\{\pi\min(F,Yz)\mathbf{1}_{\theta=G} + (1-\pi)\{P(G|\theta)\min(F,N_1\lambda Yz) \\ &+ P(B|\theta)[\mathbf{1}_{F\leq\kappa^{B}Yz}\min(F,N_1\lambda Yz) + \mathbf{1}_{F>\kappa^{B}Yz}\min(F,\lambda Yz)]\}\} \\ &= D^{\theta}_{N_1}(F,Y). \end{aligned}$$

So statement 1 holds. Higher debt capacity with N_2 in each category ($\kappa^{\theta}_{N_2} \geq \kappa^{\theta}_{N_1}$) is a direct implication of the previous statement.

Finally, to show the last statement, by definition (15), $D^{\theta}_{N}(F,Y)$ is continuous in F with $D^{\theta}_{N}(0,Y) = 0$. Intermediate value theorem guarantees the existence of the solutions $F^{\theta}_{N_i}$.

Utilizing statement 1,

$$D_{N_2}^{\theta}(F_{N_2}^{\theta}, Y) = S = D_{N_1}^{\theta}(F_{N_1}^{\theta}, Y) \leq D_{N_2}^{\theta}(F_{N_1}^{\theta}, Y).$$

Again by intermediate value theorem, the minimum solution to $D_{N_2}^{\theta}(F_{N_2}^{\theta}, Y) = S$ must be within $(0, F_{N_1}^{\theta}]$, completing the proof of the proposition.

Proof of Proposition 4: For any continuation number of creditors N_+, define F_{+,N_i} $(i = 1, 2)$ to be the minimum solution such that $D_{N_+}^{\theta}(F_{+,N_i}, Yz) = \min(F, N_i \lambda Yz)$. For a given N_+

$$\begin{aligned} D_{N_+}^{\theta}(F_{+,N_2}, Yz) &= \min(F, N_2 \lambda Yz) \\ &\geq \min(F, N_1 \lambda Yz) \\ &= D_{N_+}^{\theta}(F_{+,N_1}, Yz). \end{aligned}$$

Thus $F_{+,N_2} \geq F_{+,N_1}$, so for any given continuation number of creditors N_+, having more incumbent creditors $N_2 > N_1$ implies higher continuation face value $F_{+,N_2} \geq F_{+,N_1}$. By the recursive formulation (17) and proposition 1 we have:

$$\begin{aligned} V_{N_2}^{\theta}(F, Y) &= E\{\pi Yz \mathbf{1}_{\theta=G} \\ &+ (1-\pi)[\mathbf{1}_{\min(F, N_2\lambda Yz) \leq \kappa^{\theta'} Yz} \max\nolimits_{N_+} V_{N_+}^{\theta'}(F_{+,N_2}, Yz) + \mathbf{1}_{\min(F, N_2\lambda Yz) > \kappa^{\theta'} Yz} \lambda Yz]\} \\ &\leq E\{\pi Yz \mathbf{1}_{\theta=G} \\ &+ (1-\pi)[\mathbf{1}_{\min(F, N_1\lambda Yz) \leq \kappa^{\theta'} Yz} \max\nolimits_{N_+} V_{N_+}^{\theta'}(F_{+,N_2}, Yz) + \mathbf{1}_{\min(F, N_1\lambda Yz) > \kappa^{\theta'} Yz} \lambda Yz]\} \\ &\leq E\{\pi Yz \mathbf{1}_{\theta=G} \\ &+ (1-\pi)[\mathbf{1}_{\min(F, N_1\lambda Yz) \leq \kappa^{\theta'} Yz} \max\nolimits_{N_+} V_{N_+}^{\theta'}(F_{+,N_1}, Yz) + \mathbf{1}_{\min(F, N_1\lambda Yz) > \kappa^{\theta'} Yz} \lambda Yz]\} \\ &= V_{N_1}^{\theta}(F, Y). \end{aligned}$$

The first equality is by definition. The second inequality is because $\{z | \min(F, N_2 \lambda Yz) \leq \kappa^{\theta} Yz\} \subseteq \{z | \min(F, N_1 \lambda Yz) \leq \kappa^{\theta} Yz\}$ and $V_{N_+}^{\theta}(F_+, Yz) \geq \lambda Yz$ by proposition 1. The third inequality is because $F_{+,N_2} \geq F_{+,N_1}$ and the fact that $V_{N_+}^{\theta}(F_+, Yz)$ is decreasing in F_+ by proposition 1. Thus $V_{N_2}^{\theta}(F, Y) \leq V_{N_2}^{\theta}(F, Y)$.

Proof of Proposition 6: The proof shares the same spirit as the existence proof of κ^θ in proposition 1 part II. Define the same order-preserving function $L : \mathbb{R}_+^2 \to \mathbb{R}_+^2$ as in the proof of proposition 1 with the expectations taken under the distribution g_a. Pick any pair of $\kappa^{\theta,b}$. I shall prove that there exists a fixed point $\kappa^{\theta,a} \in \Omega$ of L, where $\Omega = [\kappa^{G,b}, M] \times [\kappa^{B,b}, M]$ is a complete lattice and M is given by (A-33). For any N and $\kappa^{\theta,a} \in \Omega$,

$$\begin{aligned}
\hat{\kappa}_N^{\theta,a} &= \max\nolimits_f E_{g_a}\{\pi \min(f,z)\mathbf{1}_{\theta=G} \\
&+ (1-\pi)[\mathbf{1}_{\min(f,N\lambda z)\leq \kappa^{\theta',a}z}\min(f,N\lambda z) + \mathbf{1}_{\min(f,N\lambda z)>\kappa^{\theta',a}z}\min(f,\lambda z)]\} \\
&\geq \max\nolimits_f E_{g_a}\{\pi \min(f,z)\mathbf{1}_{\theta=G} \\
&+ (1-\pi)[\mathbf{1}_{\min(f,N\lambda z)\leq \kappa^{\theta',b}z}\min(f,N\lambda z) + \mathbf{1}_{\min(f,N\lambda z)>\kappa^{\theta',b}z}\min(f,\lambda z)]\} \\
&\geq \max\nolimits_f E_{g_b}\{\pi \min(f,z)\mathbf{1}_{\theta=G} \\
&+ (1-\pi)[\mathbf{1}_{\min(f,N\lambda z)\leq \kappa^{\theta',b}z}\min(f,N\lambda z) + \mathbf{1}_{\min(f,N\lambda z)>\kappa^{\theta',b}z}\min(f,\lambda z)]\} \\
&= \kappa_N^{\theta,b}.
\end{aligned}$$

The first inequality is because $\min(f, N\lambda z) \geq \min(f, \lambda z)$ and $\{z|\min(f, N\lambda z) \leq \kappa^{\theta,b}z\} \subseteq \{z|\min(f, N\lambda z) \leq \kappa^{\theta,a}z\}$. The second inequality uses first order stochastic dominance and the fact that the function under the expectation is weakly increasing in z. Therefore, for any $\kappa^{\theta,a} \in \Omega$, $L^\theta(\kappa^{\theta,a}) = \max_N \hat{\kappa}_N^{\theta,a} \geq \max_N \kappa_N^{\theta,b} \geq \kappa^{\theta,b}$. So $L(\Omega) \subseteq \Omega$ and Tarski's fixed point theorem completes the argument. The omitted proof for the other direction is very similar, with the auxiliary set $\Omega = [0, \kappa_N^{\theta,a}]$.

Proof of Proposition 7: First I show $V_{CF}^\theta(F,Y) < V_1^\theta(F,Y)$. Recall the function space C and the mapping T defined in proposition 1 part III. Define a new closed subset of functions in $B^{2(\overline{N}+1)}$: $C_A = \{(v_{CF}^\theta, v_N^\theta)_{N\leq\overline{N}}^{\theta=G,B} | (v_N^\theta)_{N\leq\overline{N}}^{\theta=G,B} \in C \text{ and } v_{CF}^\theta \leq v_1^\theta\} \subset B^{2(\overline{N}+1)}$. Let $C_B = \{(v_{CF}^\theta, v_N^\theta)_{N\leq\overline{N}}^{\theta=G,B} \in C_A | v_{CF}^\theta < v_1^\theta \text{ for all } f > 0\} \subset C_A$. Finally let $C_\beta = \{(v_{CF}^\theta, v_N^\theta)_{N\leq\overline{N}}^{\theta=G,B} \in C_A | v_{CF}^\theta(f) < v_1^\theta(f) \text{ for all } f > \beta\}$. Clearly $C_B = C_0 \subset C_{\beta_2} \subset C_{\beta_1} \subset C_\infty = C_A$ for any $\beta_2 < \beta_1$. Define a new mapping $TC(v_{CF}^\theta, v_N^\theta)_{N\leq\overline{N}}^{\theta=G,B} = \left(T_{CF}[(v_{CF}^\theta)^{\theta=G,B}], T[(v_N^\theta)_{N\leq\overline{N}}^{\theta=G,B}]\right)$ on

$B^{2(\overline{N}+1)}$, where $T_{CF} = (T^G_{CF}, T^B_{CF})$ is given by

$$T^{\theta}_{CF}(v_{CF}) \quad = \quad \pi\mu\mathbf{1}_{\theta=G} + (1-\pi)E[\mathbf{1}_{\min(f,\lambda z)\leq\kappa_1^{\theta'}z} v^{\theta'}_{CF}(f_{+,1})z + \mathbf{1}_{\min(f,\lambda z)>\kappa_1^{\theta'}z}\lambda z]$$

where $f_{+,N}$ is an abbreviation for $f_+(\frac{f}{z}, N)$, the minimum solution to $D^{\theta}_N(f_+, 1) = \min(\frac{f}{z}, \lambda N)$ as before. Similar to the proof in proposition 1 part III, it is straight forward to check that TC defined above satisfies the monotonicity and discounting conditions stated in lemma A-1. So TC must have a unique fixed point $v^\star$ in $B^{2(\overline{N}+1)}$. Our goal is to show this $v^\star \in C_B$.

Claim: there exists a decreasing sequence of $\beta_n \to 0$ such that $\beta_0 = \infty$ and $TC(C_{\beta_n}) \subseteq C_{\beta_{n+1}}$.

Given this claim, we have

$$TC(C_A) = TC(C_\infty) \subseteq C_{\beta_1} \subseteq C_A. \tag{A-37}$$

The contraction mapping theorem states that the unique fixed point can be derived from repeated iterations starting from any point v, i.e., $v^\star = \lim_{n\to\infty} TC^{(n)}(v)$. Because the set C_A is closed, one can start the iteration from any point $v \in C_A$ and the limiting point $v^\star$ will stay in C_A by (A-37). Furthermore, for any n, one can argue $v^\star \in TC^{(n)}(C_A) \subseteq C_{\beta_n}$. As $n \to \infty$, $v^\star \in \lim_{n\to\infty} TC^{(n)}(C_A) \subseteq \lim_{n\to\infty} C_{\beta_n} = C_0 = C_B$. Therefore, $v^\star \in C_B$. Let $V^{\theta}_{CF}(F, Y) = v^{\theta\star}_{CF}(\frac{F}{Y})Y$. Following the same procedures in proposition 1 part III, one can check that it is indeed the firm's value function in the counterfactual case. The fact $v^\star \in C_B$ implies $V^{\theta}_{CF}(F, Y) < V^{\theta}_1(F, Y)$ for all $F > 0$, completing the first half of the statement in the proposition.

Finally, when $F < \lambda\underline{z}Y < \kappa^{\theta}Y$, the actual repayment in the true model must be $F = \min(F, \lambda NzY)$ regardless of the number of incumbent creditors N. The firm always survives the next period. Therefore, it is easy to see from (17) and (18) that the firm values do not depend on N when $F < \lambda\underline{z}Y$. Combining with the result we just proved, it is immediate that $V^{\theta}_N(F, Y) = V^{\theta}_1(F, Y) > V^{\theta}_{CF}(F, Y)$ for all N, establishing the proposition.

Proof of the claim: Suppose $(v^\theta_{CF}, v^\theta_N)^{\theta=G,B}_{N\le\overline{N}} \in C_A$. By the construction of the operator T^θ_{CF}, we have

$$\begin{aligned} T^\theta_{CF}(v_{CF}) &= \pi\mu\mathbf{1}_{\theta=G} + (1-\pi)E[\mathbf{1}_{\min(f,\lambda z)\le\kappa^{\theta'}_1 z} v^{\theta'}_{CF}(f_{+,1})z + \mathbf{1}_{\min(f,\lambda z)>\kappa^{\theta'}_1 z}\lambda z] \\ &\le \pi\mu\mathbf{1}_{\theta=G} + (1-\pi)E[\mathbf{1}_{\min(f,\lambda z)\le\kappa^{\theta'}_1 z} v^{\theta'}_1(f_{+,1})z + \mathbf{1}_{\min(f,\lambda z)>\kappa^{\theta'}_1 z}\lambda z] \end{aligned} \tag{A-38}$$

Because $\kappa^\theta_1 \le \kappa^\theta$ and $\kappa^B_1 = (1-\pi)\mu\lambda < \lambda < \kappa^B$, so whenever $f > (1-\pi)\mu\lambda\underline{z}$ there is a positive probability that $f > \kappa^B_1 z$. In addition, because $\max_{N_+} v^\theta_{N_+}(f_{+,N_+}) \ge \max_{N_+}\kappa^\theta_{N_+} = \kappa^\theta > \lambda$, the last expression in (A-38) is strictly dominated by

$$\begin{aligned} \text{(A-38)} &< \pi\mu\mathbf{1}_{\theta=G} + (1-\pi)E[\mathbf{1}_{\min(f,\lambda z)\le\kappa^{\theta'} z} \max\nolimits_{N_+} v^{\theta'}_{N_+}(f_{+,N_+})z + \mathbf{1}_{\min(f,\lambda z)>\kappa^{\theta'} z}\lambda z] \\ &= \pi\mu\mathbf{1}_{\theta=G} + (1-\pi)E[\max\nolimits_{N_+} v^{\theta'}_{N_+}(f_{+,N_+})z] \\ &= T^\theta_1((v^\theta_N)^{\theta=G,B}_{N\le\overline{N}}). \end{aligned} \tag{A-39}$$

Therefore, $TC((v^\theta_{CF}, v^\theta_N)^{\theta=G,B}_{N\le\overline{N}}) \in C_{(1-\pi)\mu\lambda\underline{z}}$ and we can pick $\beta_1 = (1-\pi)\mu\lambda\underline{z}$. Let $\beta_{n+1} = (1-\pi)\beta_n$. I shall prove that $TC(v) \in C_{(1-\pi)\beta_n}$ for all $v \in C_{\beta_n}$. Suppose $(v^\theta_{CF}, v^\theta_N)^{\theta=G,B}_{N\le\overline{N}} \in C_{\beta_n}$ and consider any $f \in (\beta_n(1-\pi), \beta_n]$. On one hand, from the rollover condition (A-35) and the fact that $\frac{f}{z} \le \frac{f}{\underline{z}} \le \frac{\beta_n}{\underline{z}} < \frac{\beta_1}{\underline{z}} = (1-\pi)\mu\lambda < \lambda$, we have

$$D^B_1(f^B_+, 1) = \min(\frac{f}{z}, \lambda) = \frac{f}{z}.$$

On the other hand, from expression (15) and the fact $f \le \beta_n \le \lambda\underline{z}$, we have

$$D^B_1(f^B_+, 1) = (1-\pi)f^B_+.$$

The above two equalities together imply that $f^B_{+,1} = \frac{f}{z(1-\pi)} > \frac{\beta_n}{z}$, which in turn implies that there is positive possibility that $f^B_{+,1} > \beta_n$. By the construction of the set C_{β_n}, $v^B_{CF}(f_{+,1}) < v^B_1(f_{+,1})$ holds strictly when $f_{+,1} > \beta_n$. Therefore, the inequality (A-38) holds strictly in this case. On the other hand, the weak inequality between (A-38) and (A-39) is trivial, so we again have $T^\theta_{CF}(v_{CF})(f) < T^\theta((v^\theta_N)^{\theta=G,B}_{N\le\overline{N}})(f)$ for all $f > (1-\pi)\beta_n$. Therefore, we have established $TC(C_{\beta_n}) \subseteq C_{\beta_{n+1}}$ for the constructed sequence of β_n that converges to zero, completing the proof of the claim and the whole proposition.

Proof of Lemma 4: By definition (28), $\lim_{T\to\infty}\hat{\pi} = \lim_{T\to\infty}\frac{\pi}{T} = 0$ is obvious. Rewrite (29) using (3):

$$\frac{\hat{\pi}[1-(1-\hat{\pi})\hat{\mu}\hat{p}]\hat{\mu}}{[1-(1-\hat{\pi})\hat{\mu}][1-(1-\hat{\pi})\hat{\mu}(2\hat{p}-1)]} = \frac{\pi[1-(1-\pi)\mu p]\mu}{[1-(1-\pi)\mu][1-(1-\pi)\mu(2p-1)]}, \tag{A-40}$$

$$\frac{\hat{\pi}(1-\hat{p})(1-\hat{\pi})\hat{\mu}^2}{[1-(1-\hat{\pi})\hat{\mu}][1-(1-\hat{\pi})\hat{\mu}(2\hat{p}-1)]} = \frac{\pi(1-p)(1-\pi)\mu^2}{[1-(1-\pi)\mu][1-(1-\pi)\mu(2p-1)]}. \tag{A-41}$$

Adding the above two equations, we have

$$\frac{\hat{\pi}\hat{\mu}}{1-(1-\hat{\pi})\hat{\mu}} = \frac{\pi\mu}{1-(1-\pi)\mu}. \tag{A-42}$$

Plugging in $\hat{\pi} = \frac{\pi}{T}$ from (28), one can solve for $\hat{\mu} = \frac{T\mu}{T\mu-\mu+1} \to 1$ as $T\to\infty$. Finally, in order to calculate $\hat{p}$, divide (A-40) by (A-41) and then we have

$$\frac{1-(1-\hat{\pi})\hat{\mu}\hat{p}}{(1-\hat{p})(1-\hat{\pi})\hat{\mu}} = \frac{1-(1-\pi)\mu p}{(1-p)(1-\pi)\mu}.$$

Subtract 1 from both sides and multiply it by (A-42),

$$\frac{\hat{\pi}}{(1-\hat{p})(1-\hat{\pi})} = \frac{\pi}{(1-p)(1-\pi)}.$$

Plug in $\hat{\pi} = \frac{\pi}{T}$ and we can solve for $\hat{p} = \frac{T-1+p(1-\pi)}{T-\pi}$. Clearly, when $T \ge 1$, $\hat{\pi}, \hat{p} \in (0,1)$. In addition, $\lim_{T\to\infty}\hat{\pi} = 0$ and $\lim_{T\to\infty}\hat{p} = 1$. Finally,

$$(1-\hat{\pi})\hat{\mu} = \frac{\mu(T-\pi)}{T\mu-\mu+1} = 1 - \frac{1-\mu(1-\pi)}{T\mu-\mu+1} < 1,$$

by assumption (1). Therefore, the new parameters are well defined.

Proof of Proposition 8: First, notice $\hat{\kappa}^G$ must be bounded as $T \to \infty$. This is because

$$\hat{\kappa}_N^G Y = \max_F D_N^G(F, Y) \leq V_{FB}^{G\star}(Y).$$

So $\hat{\kappa}^G = \max_N \hat{\kappa}_N^G$ must be bounded by some upper bound M ($\frac{V_{FB}^{G\star}(Y)}{Y}$ for example) that is independent of T. Let $\hat{N}$ be the number of creditors such that $\hat{\kappa}_{\hat{N}}^B$ attains the total debt capacity $\hat{\kappa}^B$, then

$$\begin{aligned}
\hat{\kappa}^B &= \max_f (1-\hat{\pi})\{\hat{p}[\mathbf{1}_{\min(f,\hat{N}\lambda\mu)\leq\hat{\kappa}^B\hat{\mu}}\min(f,\hat{N}\lambda\hat{\mu}) + \mathbf{1}_{\min(f,\hat{N}\lambda\hat{\mu})>\hat{\kappa}_2^B\hat{\mu}}\min(f,\lambda\hat{\mu})] \\
&+ (1-\hat{p})[\mathbf{1}_{\min(f,\hat{N}\lambda\hat{\mu})\leq\hat{\kappa}_2^G\hat{\mu}}\min(f,\hat{N}\lambda\hat{\mu}) + \mathbf{1}_{\min(f,\hat{N}\lambda\mu)>\hat{\kappa}_2^G\hat{\mu}}\min(f,\lambda\hat{\mu})]\}. \qquad \text{(A-43)} \\
&\leq \max_f (1-\hat{\pi})\{\hat{p}[\mathbf{1}_{\min(f,\hat{N}\lambda\mu)\leq\hat{\kappa}^B\hat{\mu}}\min(f,\hat{N}\lambda\hat{\mu}) + \mathbf{1}_{\min(f,\hat{N}\lambda\hat{\mu})>\hat{\kappa}_2^B\hat{\mu}}\min(f,\lambda\hat{\mu})] \\
&+ (1-\hat{p})\min(f,\hat{N}\lambda\hat{\mu})\}. \qquad \text{(A-44)}
\end{aligned}$$

Let $f^\star$ be the optimal f such that (A-43) attains $\hat{\kappa}^B$. Suppose $f^\star \leq \hat{\kappa}^B\hat{\mu}$. Notice that the expression in (A-44) is increasing in $f \in [0, \hat{\kappa}^B\hat{\mu}]$ and $(1-\hat{\pi})\hat{\mu} < 1$ by lemma 4, so

$$\hat{\kappa}^B \leq (1-\hat{\pi})\min(\hat{\kappa}^B\hat{\mu}, \hat{N}\lambda\hat{\mu}) < \hat{\kappa}^B.$$

Contradiction! On the other hand, if $\hat{N}\lambda \leq \hat{\kappa}^B$, then it is optimal to set $f^\star$ arbitrarily large in (A-44) and $\hat{\kappa}^B = (1-\hat{\pi})\hat{N}\lambda\hat{\mu} < \hat{N}\lambda$. Again a contradiction! Therefore, at $f = f^\star$, it must be $\min(f^\star, \hat{N}\lambda\hat{\mu}) > \hat{\kappa}^B\hat{\mu}$ and (A-43) becomes:

$$\begin{aligned}
\hat{\kappa}^B &= (1-\hat{\pi})\{\hat{p}\min(f^\star,\lambda\hat{\mu}) \\
&+ (1-\hat{p})[\mathbf{1}_{\min(f^\star,\hat{N}\lambda\hat{\mu})\leq\hat{\kappa}^G\hat{\mu}}\min(f^\star,\hat{N}\lambda\hat{\mu}) + \mathbf{1}_{\min(f^\star,\hat{N}\lambda\hat{\mu})>\hat{\kappa}^G\hat{\mu}}\min(f^\star,\lambda\hat{\mu})]\} \\
&\leq (1-\hat{\pi})\{\hat{p}\min(f^\star,\lambda\hat{\mu}) + (1-p)\hat{\kappa}^G\hat{\mu}\} \\
&\leq (1-\hat{\pi})\{\hat{p}\lambda\hat{\mu} + (1-\hat{p})\hat{\mu}M\}, \qquad \text{(A-45)}
\end{aligned}$$

where the last inequality uses the fact that $\hat{\kappa}^G \leq M$, which is independent of T. As $T \to \infty$, lemma 4 states $\hat{\pi} \to 0$, $\hat{\mu} \to 1$, and $\hat{p} \to 1$, so the upper bound given by (A-45) approaches λ. Finally, because $\hat{\kappa}^B \geq \hat{\kappa}_1^B \to \lambda$ as $T \to \infty$, so we conclude $\lim_{T\to\infty} \hat{\kappa}^B = \lambda$.

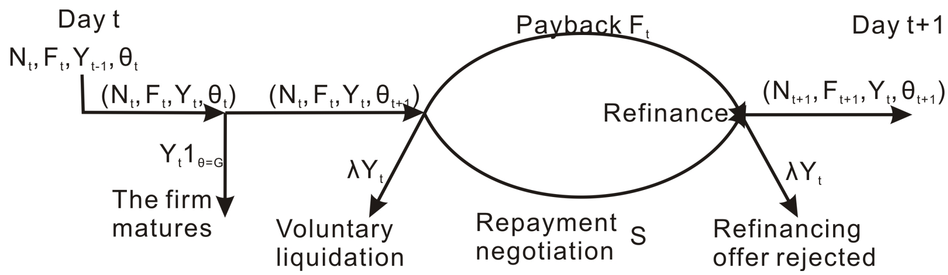

Figure 1 *The timeline and the evolution of the state variables.*

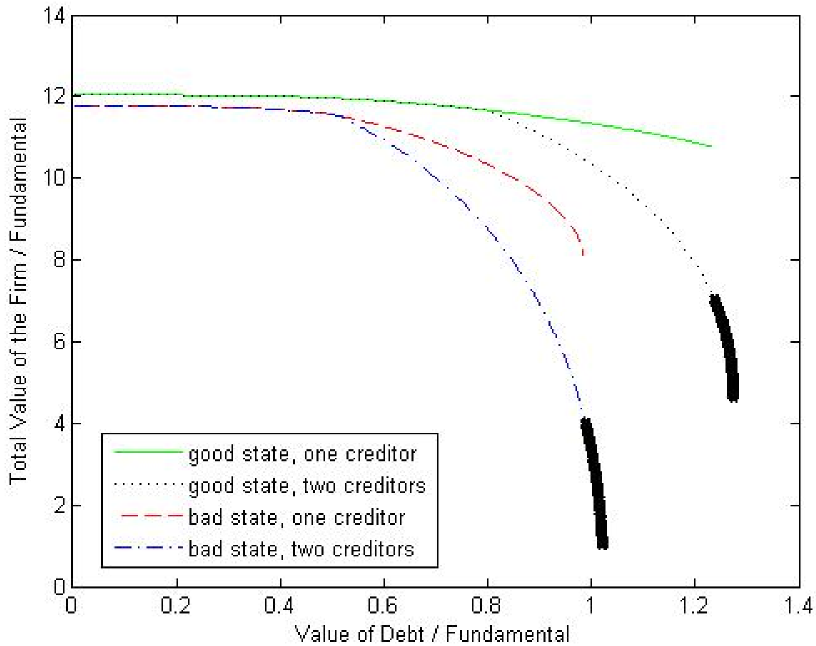

Figure 2 *The figure plots the expected total firm value against the expected value of the debt. The solid (dashed) line is the firm value with a single creditor when the fundamental* $\theta = G$ $(\theta = B)$. *The dotted (dash-dotted) line is the firm value with two creditors when the fundamental* $\theta = G$ $(\theta = B)$. *The thick solid black segments can be supported only by two creditors. Although the firm values are comparatively much lower along the thick lines, the firm cannot even reach that portion with just one creditor. When the value of debt is very low the choice between one and two creditors is irrelevant. As the fundamental worsens, the two groups of lines diverge and, when both are feasible, the single creditor structure always delivers a higher firm value.*

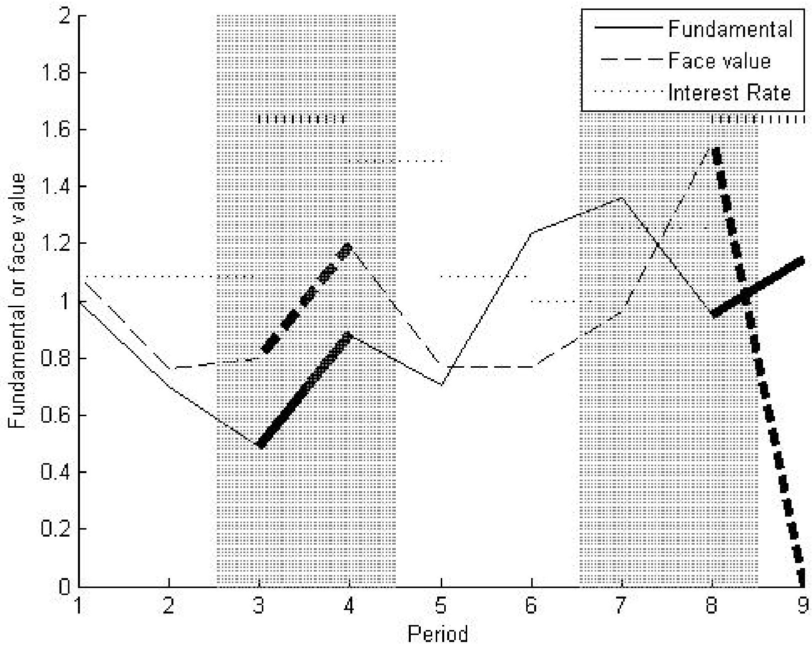

Figure 3 *The figure plots a typical sample path of the firm. Areas are shaded when the state is bad. The solid (dashed) line denotes the exogenous fundamental process* Y_{t-1} *(the face value process* F_t *determined in equilibrium). I use bold segments when the firm chooses two creditors. The values plotted at each period* t *are the state variables entering period* t*: number of creditors* N_t*, the promised face value* F_t*, state* θ_t*, and fundamental process* Y_{t-1}*. Finally, the dotted bars plot the interest rates* $\frac{F_t}{D_{N_t}^{\theta_t}(F_t,Y_{t-1})}$ *during each period.*

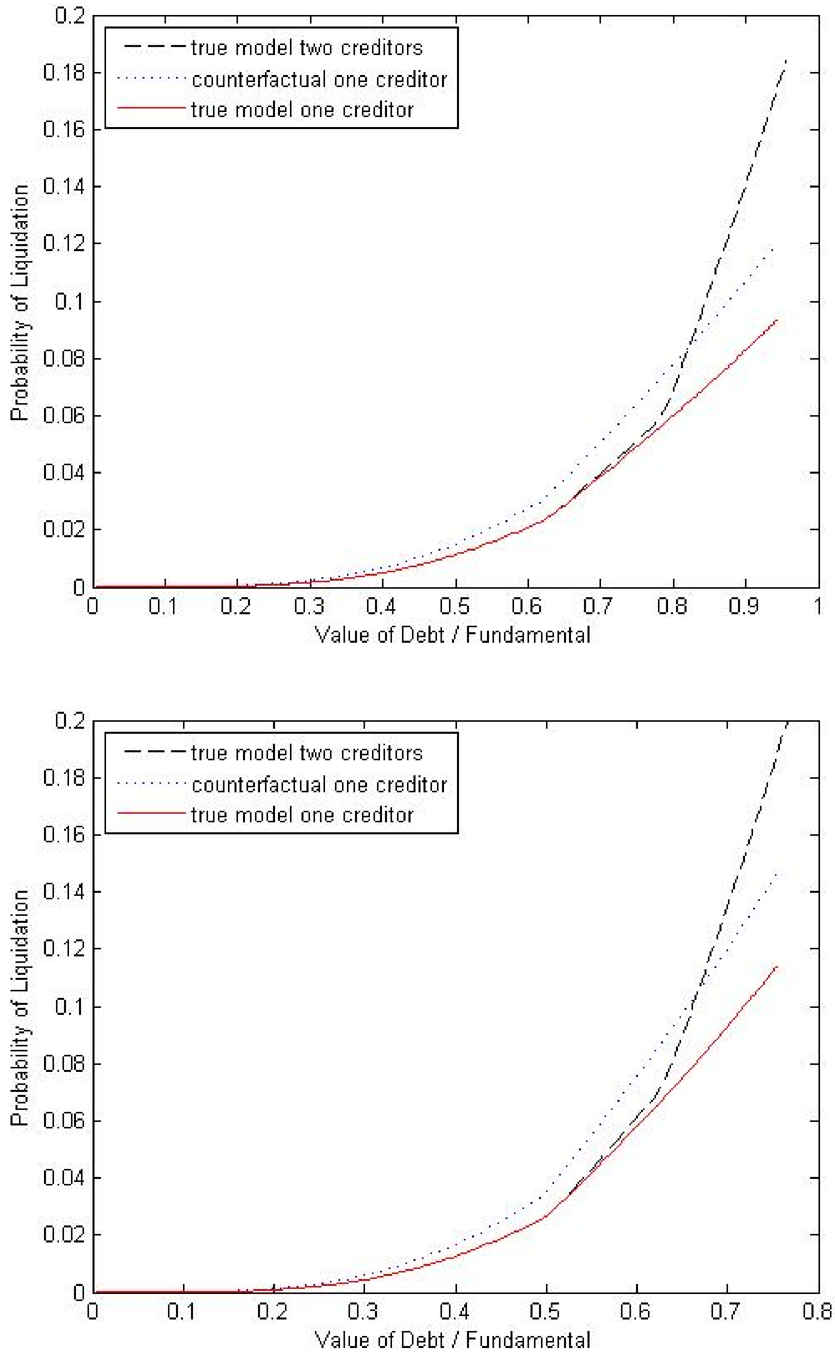

Figure 4 *The figure plots the expected probability of liquidation* $L_1^{\theta,\infty}(F,Y)$ *against the expected value of the debt conditional on the current state* $\theta = G$ *(top panel) and* $\theta = B$ *(bottom panel). The solid (dashed) line is the liquidation probability with a single creditor (two creditors) in the full model. The dotted line is for the counterfactual model in which the number*

of creditors is exogenously fixed at one. It is easy to see that having a single creditor in the true model means a lower liquidation probability compared to having two creditors as well as the counterfactual one creditor model. For a substantial range of fundamental values, the liquidation probability with two creditors *in the true model is strictly lower compared with the* single *creditor counterfactual.*