Asset Pricing in a Production Economy with Heterogeneous Investors

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This paper is a theoretical examination of the stochastic behavior of equilibrium asset prices in an economy consisting of a production process controlled by a state variable representing the state of technology. The investors with different degrees of risk aversion and time preferences trade and lend among themselves in order to maximize their individual utilities of life time consumption. The allocation of wealth fluctuates randomly among them and acts as a state variable against which each investor wants to hedge. This hedging motive complicates the investor's portfolio choice and the equilibrium in the production economy. A general method of constructing equilibrium asset prices is developed and the wealth effect in the general equilibrium is discussed.

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Abstract

This paper is a theoretical examination of the stochastic behavior of equilibrium asset prices in an economy consisting of a production process controlled by a state variable representing the state of technology. The investors with different degrees of risk aversion and time preferences trade and lend among themselves in order to maximize their individual utilities of life time consumption. The allocation of wealth fluctuates randomly among them and acts as a state variable against which each investor wants to hedge. This hedging motive complicates the investor's portfolio choice and the equilibrium in the production economy. A general method of constructing equilibrium asset prices is developed and the wealth effect in the general equilibrium is discussed.

1 Introduction

This paper is a theoretical examination of the stochastic behavior of equilibrium asset prices in an economy consisting of a production process controlled by a state variable representing the state of technology. The investors in the economy maximize their individual utilities of life time consumption. Heterogeneity in preferences introduces trading and lending among investors. Market clearing condition determines the general equilibrium in the production economy. The allocation of wealth fluctuates randomly among the investors and acts as a state variable against which each investor wants to hedge. This hedging motive complicates the investor's portfolio choice and the equilibrium in the production economy. As a result, the equilibrium market prices of risk and interest rate, and the investor's optimal investment and consumption strategies will fluctuate according to the wealth fluctuation among investors. Our objective will be to understand the relation between the dynamic equilibrium and the wealth distribution, and the interaction of the optimal indirect utilities between the heterogeneous investors.

In three related papers, Cox, Ingersoll and Ross (1985), Dumas (1989) and Vasicek (2005) consider equilibrium asset prices in *production economy*. Cox, Ingersoll and Ross develop an equilibrium model in a production economy with multiple production processes controlled by several state variables. They only deal with one representative agent[2], therefore their market clearing condition is superficial. Dumas uses a production economy with no technology change to model the capital market and investigates equilibrium conditions in the economy with two investors, say A and B. Investor A is myopic, i.e., has logarithmic utility function; investor B has non-logarithmic isoelastic utility function. He assumes that the myopic investor A's optimal strategy is not affected by that of investor B. He then studies the influence of investor A upon investor B. Vasicek derives equilibrium conditions

[2]Both terms *agent* and *investor* are used in the literature. They have the same meaning in this paper.

in a production economy with technology change and with several investors. Our paper is more general than Dumas (1989). It includes Dumas' result as one of the special cases. It improves Vasicek (2005) in the ways of finding joint optimal conditions between investors and handling the wealth effect of each investor. With a proper procedure of applying market clearing condition, we find that the wealth distribution among investors plays an important role in the equilibrium.

Other related work includes equilibrium models in pure *exchange economy* developed by Lucas (1978), Wang (1996) and Chan and Kogan (2002). Lucas considers equilibrium in a one-good pure exchange economy with one representative investor. Wang looks at an exchange economy with two heterogeneous investors. He observes that in a pure exchange economy, there is no intertemporal transformation of resources. The intertemporal resource constraint is simply the collection of resource constraints for each date and each state. Maximizing the expected intertemporal welfare function is equivalent to maximizing the welfare function period by period and state by state subject to the corresponding resource constraint. With this *clever* observation, Wang obtains a closed-form market equilibrium in the economy of two heterogeneous investors. Chan and Kogan analyze an exchange economy with heterogeneous investors, where each individual's utility is a function of consumption measured in units of an average aggregate endowment. They obtain the equilibrium for heterogeneous investors with a continuous weight distribution.

In a production economy, the wealth of each participant can be invested in a production process for a possible growth. The intertemporal transformation of resources plays a crucial role in the dynamic equilibrium. The joint optimization problem of investors' expected utility functions is much more difficult to solve. Market clearing between the supply and demand created by the investors plays a critical role in determining the general equilibrium. In general, one is unable to find a closed-form market equilibrium. Our target in this paper is to establish a proper economic model for the equilibrium conditions, so that the quantitative stochastic behavior of equilibrium market prices of risks and interest rate can be studied numerically if necessary.

In the next section, we describe the model of production economy and heterogeneous investors. Our main results are presented in Section 3. For completeness, we also include the results of one representative investor discussed by Cox, Ingersoll and Ross (1985) in this section. The paper is concluded in Section 4.

2 The model

To model the production economy, we use Vasicek's (2005) setup for the economy with one production process controlled by one state variable, which is a simplified version of Cox, Ingersoll and Ross' (1985) economy with multiple production processes controlled by several state variables. We also inherit Vasicek's (2005) notations for the convenience of the reader.

Consider a *production process* whose rate of return dA/A on an investment in *production variable* A is

$$\frac{dA}{A} = \mu dt + \sigma dy, \tag{1}$$

where $y(t)$ is a Wiener process that models the *production risk*. The development of the production process is affected by a state variable, $X(t)$, representing the *state of technology*. Both the expected return function, $\mu = \mu(X(t), t)$, and the volatility function, $\sigma = \sigma(X(t), t)$, are exogenously given.

The dynamics of the state variable is also exogenously given

$$dX = \zeta dt + \psi dy + \phi dx, \tag{2}$$

where $x(t)$ is another Wiener process independent of $y(t)$. The new Wiener process is used to model the *state risk* that is independent of the production risk. The parameters ζ, ψ and ϕ are exogenously given functions of $X(t)$ and t. The function ψ here is designed to allow a freedom to model the correlation between the production process and the progress of the state of technology.

The economy allows unrestricted borrowing and lending at any maturity. The *risk-free rate* is denoted by r. Then the *money market account*, $M(t)$, follows

$$\frac{dM}{M} = rdt. \tag{3}$$

The *asset price*, $P(t)$ of any asset in the economy must satisfy the equation

$$\frac{dP}{P} = (r + \beta\lambda + \delta\eta)dt + \beta dy + \delta dx, \tag{4}$$

where β and δ are the *risk exposures* of the asset to the production risk y and the state risk x. And λ is the *market price of production risk*, η is the *market price of state risk.*

Since there are only two risk sources, we only need one more asset with $\delta \neq 0$ to complete[3] the economy. The asset with $\delta \neq 0$ can be a derivative contract, such as an option written on the production variable A with certain strike price and maturity date. The price of the derivative depends on both the production risk y and the state risk x.

The production variable itself can be understood as a traded asset with $\beta = \sigma$ and $\delta = 0$, then the expected return of an investment in the production process satisfies following relationship

$$\mu = r + \sigma\lambda, \qquad \Longrightarrow \qquad r = \mu - \sigma\lambda. \tag{5}$$

The equality will be used to derive interest rate r once we know the market price of production risk λ from the equilibrium.

In particular, there exists a numeraire asset $Z(t)$ (Long Jr 1990) with the dynamics

$$\frac{dZ}{Z} = (r + \lambda^2 + \eta^2)dt + \lambda dy + \eta dx, \tag{6}$$

[3]By *complete* we mean that any asset in the economy can be replicated dynamically by the production variable and this additional derivative contract.

such that the price P of any asset satisfies

$$\frac{P(t)}{Z(t)} = E_t\left[\frac{P(s)}{Z(s)}\right], \tag{7}$$

which means that $P(s)/Z(s)$ is a martingale. If we write $\pi(t) = 1/Z(t)$, then

$$P(t) = \frac{1}{\pi(t)} E_t\left[\pi(s)P(s)\right]. \tag{8}$$

The *pricing kernel* or *stochastic discount factor*, $\pi(t)$, follows

$$\frac{d\pi}{\pi} = -rdt - \lambda dy - \eta dx. \tag{9}$$

In integral form, the pricing kernel is written analytically as

$$\frac{\pi(s)}{\pi(t)} = \exp\left(-\int_t^s rd\tau - \frac{1}{2}\int_t^s (\lambda^2 + \eta^2)d\tau - \int_t^s \lambda dy - \int_t^s \eta dx\right). \tag{10}$$

We now describe the investors with heterogeneous preferences. Suppose that the economy has n participants, $k = 1, 2, , \cdots, n$ and let $W_k(0)$ be the initial wealth of kth investor. Each investor maximizes the expected utility of his life time consumption,

$$\max E_0 \int_0^T p_k(t)U_k(c_k(t))dt, \tag{11}$$

where $c_k(t)$ is the rate of consumption at time t, $U_k(c)$ is a utility function with $U_k' > 0$, $U_k'' < 0$, and $p_k(t) \geq 0$, $0 \leq t \leq T$ is a time preference function. The time preference function can be very general. If it is concentrated at time T, then the investor maximizes the expected utility of terminal wealth, i.e, $\max E_0 U_k(W_k(T))$. The problem becomes a pure investment problem without consumption.

We consider the class of isoelastic utility functions

$$U_k(c) = \begin{cases} \dfrac{c^{(\gamma_k-1)/\gamma_k}}{\gamma_k - 1} & \gamma_k > 0,\ \gamma_k \neq 1, \\ \ln c & \gamma_k = 1. \end{cases} \tag{12}$$

The wealth process of kth investor is written as

$$dW_k = [W_k(r + \beta_k\lambda + \delta_k\eta) - c_k]dt + W_k\beta_k dy + W_k\delta_k dx, \tag{13}$$

where the risk exposures β_k and δ_k can be achieved by a portfolio of investing in the production A, a basic derivative asset P and the money market account M. These risk exposures fully describe the particular investment strategy of investor k.

By summing up equation (13) according to the index k for all investors, we obtain that the total wealth of the n investors in the economy,

$$W(t) = \sum_{k=1}^{n} W_k(t), \tag{14}$$

follows the process

$$dW = \left[W\left(r + \lambda\sum_{k=1}^{n}\omega_k\beta_k + \eta\sum_{k=1}^{n}\omega_k\delta_k\right) - \sum_{k=1}^{n}c_k\right]dt + W\sum_{k=1}^{n}\omega_k\beta_k dy + W\sum_{k=1}^{n}\omega_k\delta_k dx, \tag{15}$$

where $\omega_k = W_k/W$ is the *wealth ratio* of investor k's wealth W_k to the total wealth W. By definition, the wealth ratios satisfy

$$\sum_{k=1}^{n}\omega_k = 1. \tag{16}$$

The *market clearing condition* is that the total wealth must be invested in the production process, i.e.,

$$dW = \left(\mu W - \sum_{k=1}^{n}c_k\right)dt + \sigma W dy. \tag{17}$$

Comparing (15) and (17) gives two restrictions on the investment strategies

$$\sum_{k=1}^{n}\omega_k\beta_k = \sigma, \qquad \sum_{k=1}^{n}\omega_k\delta_k = 0. \tag{18}$$

The problem is to determine the equilibrium risk-free rate r, the market prices of production risk λ, and the market price of state risk η. Once we know their dynamics, the stochastic discount factor can be determined by equation (10). Any asset with a known payoff, $P(s)$, at future time s, can be priced by equation (8) accordingly.

3 Main Results

The new results in this paper are the equilibrium conditions for the production economy with heterogeneous investors. For completeness, we also present the results of one representative investor, which has been studied by Cox, Ingersoll and Ross (1985).

3.1 One representative investor

Proposition 1 *In a production economy of one representative investor with a general utility function $U(c)$, the equilibrium market prices of risk and risk-free rate are*

$$\lambda = -\frac{V_{WW}}{V_W} W\sigma - \frac{V_{WX}}{V_W}\psi, \qquad \eta = -\frac{V_{WX}}{V_W}\phi, \qquad r = \mu - \sigma\lambda. \tag{19}$$

The indirect utility function, $V = V(W, X, t)$, is determined by a nonlinear partial differential equation (PDE)[4]

$$\begin{aligned} V_t + V_W(\mu W - U'^{-1}(V_W/p)) + V_X\zeta + \frac{1}{2}V_{WW}W^2\sigma^2 \\ + V_{WX}W\sigma\psi + \frac{1}{2}V_{XX}(\psi^2 + \phi^2) + pU(U'^{-1}(V_W/p)) = 0, \end{aligned} \tag{20}$$

subject to a final condition $V(W, X, T) = 0$. The notation U'^{-1} stands for the inverse function of the marginal utility function of consumption, $U'(c)$.

The investor's optimal investment strategy is $\beta = \sigma$, $\delta = 0$. His optimal consumption strategy is $c = U'^{-1}(V_W/p)$. His wealth process is described by

$$dW = [\mu W - U'^{-1}(V_W/p)]dt + \sigma W dy. \tag{21}$$

Proof. See appendix A.

Since there is only one representative investor in the economy, there is no counter-party to trade or borrow. The market clearing condition requires that the investor must invest all

[4]By *nonlienar PDE* we mean that the dependent variable V appears in the PDE in a nonlinear way. For example, $U'^{-1}(V_W/p)V_W$ is a nonlinear term in equation (20).

his wealth into the production process, i.e., $\beta = \sigma$, $\delta = 0$. The strategy is the only choice to the investor, but in order to make it optimal, the market prices of risk have to satisfy certain relations given by equation (19), which are associated with the *marginal utility of wealth*, V_W.

With Ito's Lemma, the process of marginal utility of wealth can be written as

$$\frac{dV_W}{V_W} = \mu_{V_W} dt + \left(\frac{V_{WW}}{V_W} W\sigma + \frac{V_{WX}}{V_W}\psi \right) dy + \frac{V_{WX}}{V_W}\phi dx = \mu_{V_W} dt - \lambda dy - \eta dx,$$

where μ_{V_W} represents the drift of the process. It depends on the partial derivatives of V_W w.r.t. to W, X and t. One may observe that the market price of risk for each risk source is the negative of the volatility on the corresponding risk source of the marginal utility of wealth. The interest rate is written as

$$r = \mu + \frac{V_{WW}}{V_W} W\sigma^2 + \frac{V_{WX}}{V_W}\sigma\psi = \mu + \frac{1}{dt}\text{cov}\left(\frac{dW}{W}, \frac{dV_W}{V_W} \right).$$

It says that the equilibrium interest rate r is the sum of expected rate of return on wealth μ and covariance of the rate of return on wealth with the rate of change in the marginal utility of wealth. The result is presented by Cox, Ingersoll and Ross (1985) in a more general setting of multiple production processes controlled by several state variables.

For a given utility function $U(c)$, both market prices of risk and risk-free rate are functions of wealth level W, state of technology X and time t. A full description on them relies on the detailed structure of the indirect utility function of wealth $V(W, X, t)$, which has to be determined by solving the nonlinear PDE (20). In general, the problem cannot be solved analytically, but it can be simplified for some specified utility functions such as a logarithmic utility function and isoelastic utility function. The equilibrium conditions for the case of these two kinds of utility functions will be discussed in the next two propositions.

Proposition 2 *In a production economy of one representative investor with a logarithmic* *utility function,* $U(c) = \ln c$*, the equilibrium market prices of risk and risk-free rate are*

$$\lambda = \sigma, \qquad \eta = 0, \qquad r = \mu - \sigma^2. \tag{22}$$

The investor's optimal investment strategy is $\beta = \sigma$, $\delta = 0$*. His optimal consumption strategy is given by*

$$c = \frac{p}{\int_t^T p(\tau)d\tau} W. \tag{23}$$

His wealth process is described by

$$dW = \left[\mu - \frac{p}{\int_t^T p(\tau)d\tau} \right] W dt + \sigma W dy, \tag{24}$$

and his indirect utility function is given by

$$V(W, X, t) = \int_t^T p(\tau)d\tau \ln W + G(X, t), \tag{25}$$

where $G(X, t)$ *is determined by a linear PDE*

$$G_t + G_X \zeta + \frac{1}{2} G_{XX}(\psi^2 + \phi^2) + \left(\mu - \frac{1}{2}\sigma^2 \right) \int_t^T p(\tau)d\tau - p + p \ln \frac{p}{\int_t^T p(\tau)d\tau} = 0, \tag{26}$$

subject to a final condition $G(X, T) = 0$*.*

Proof. See appendix B.

For the case of logarithmic utility function, the market price of production risk is equal to the volatility of production process, while the market price of state risk is zero. The result is true for any investor's time preference function $p(t)$ and for any process that drives the state variable X. The investor's investment and consumption strategies and wealth process do not depend explicitly on the factors that drive the state variable. The state variable affects the indirect utility function $V(W, X, t)$ only through the level function $G(X, t)$. This result somehow reflects the myopic (nearsighted) property of the logarithmic utility investors.

Proposition 3 *In a production economy of one representative investor with a non-logarithmic isoelastic utility function,* $U(c) = \frac{c^{(\gamma-1)/\gamma}}{\gamma-1}$, $\gamma > 0$, $\gamma \neq 1$, *the equilibrium market prices of risk and risk-free rate are*

$$\lambda = \frac{1}{\gamma}\sigma - \frac{Q_X}{Q}\psi, \qquad \eta = -\frac{Q_X}{Q}\phi, \qquad r = \mu - \sigma\lambda, \tag{27}$$

where $Q = Q(X,t)$ *is determined by a nonlinear PDE*

$$Q_t + \frac{\gamma-1}{\gamma}\left(\mu - \frac{1}{2\gamma}\sigma^2\right)Q + \left(\zeta + \frac{\gamma-1}{\gamma}\sigma\psi\right)Q_X + \frac{1}{2}(\psi^2+\phi^2)Q_{XX} + \frac{1}{\gamma}\frac{p^\gamma}{Q^{\gamma-1}} = 0. \tag{28}$$

subject to a final condition $Q(X,T) = 0$.

The investor's optimal investment strategy is $\beta = \sigma$, $\delta = 0$. *His optimal consumption strategy is given by*

$$c = \frac{p^\gamma}{Q^\gamma}W. \tag{29}$$

His wealth process is described by

$$dW = \left[\mu - \frac{p^\gamma}{Q^\gamma}\right]Wdt + \sigma W dy, \tag{30}$$

and his indirect utility function is given by

$$V(W,X,t) = Q(X,t)\frac{W^{(\gamma-1)/\gamma}}{\gamma-1}. \tag{31}$$

If $\mu - \frac{1}{2\gamma}\sigma^2$ *is a function of time only, then* Q *can be solved to be a function of time*

$$Q(t) = \left[\int_t^T p^\gamma(\tau)e^{(\gamma-1)\int_t^\tau(\mu-\frac{1}{2\gamma}\sigma^2)ds}d\tau\right]^{1/\gamma}, \tag{32}$$

then the equilibrium market prices of risk and risk-free rate will be

$$\lambda = \frac{1}{\gamma}\sigma, \qquad \eta = 0, \qquad r = \mu - \frac{1}{\gamma}\sigma^2. \tag{33}$$

Proof. See appendix C.

For the case of non-logarithmic isoelastic utility function, the market price of state risk η is no longer zero in general. Both market prices of production risk λ and state risk η depends on the evolution of the state variable through the coefficient function $Q(X,t)$ of the utility function of wealth, $V(W,X,t) = Q(X,t)\frac{W^{(\gamma-1)/\gamma}}{\gamma-1}$. In order to obtain the $Q(X,t)$ function, one needs to solve the nonlinear PDE (28) for some given μ, σ, ζ, ψ and ϕ functions of X and t, and $p(t)$. In general, very little is known about nonlinear PDEs, and the equation needs to be investigated case by case. The investor's optimal consumption rate is also proportional to his wealth level, $c = \frac{p^\gamma}{Q^\gamma}W$, but the coefficient depends on the state variable X again through the Q function.

Only under a special circumstance that $\mu - \frac{1}{2\gamma}\sigma^2$ is a function of time only, one can show that $Q_X = 0$. Equation (28) can be linearized. Its solution is given by (32). In this case, the non-myopic investor does not price state risk just like myopic one.

3.2 Two investors

In a production economy of two heterogeneous investors with utility functions, $U_1(c)$ and $U_2(c)$, the trading of the basic derivative asset P and borrowing/lending becomes possible. Since each investor controls his own consumption rate that affects the rate of change in his wealth process, his wealth level will then become a new state variable in the economy. One investor's wealth level enters into another investor's indirect utility function as a new state variable. As Dumas (1989) puts it

> The allocation of wealth fluctuates randomly among them (the investors) and acts as a state variable against which each market participant will want to hedge. This hedging motive complicates the investors' portfolio choice and the equilibrium in the capital market (production economy).

Therefore, we have following observation.

Property. *In a production economy of two heterogeneous investors, the optimal indirect utility functions of the two investors, V_1 and V_2, are functions of their wealth levels, W_1 and W_2, state variable X and time t, i.e.,*

$$V_1 = V_1(W_1, W_2, X, t), \qquad V_2 = V_2(W_1, W_2, X, t). \tag{34}$$

It is quite intuitive that each investor's indirect utility function depends on his own wealth level. The dependency of one investor's indirect utility function on the other one's wealth level requires a little bit more reasoning. There are only two investors in the economy, one investor's demand must meet the other one's supply. Each investor has to take account of the optimizing decision of the other one. He knows what the decision will be, even though he cannot affect it. Therefore the indirect utility function of one investor relies on the other one's indirect utility function, hence the other one's wealth level.

The equilibrium conditions of the two-party dynamic game are presented in the next proposition.

Proposition 4 *In a production economy of two heterogeneous investors with general utility functions, $U_1(c)$ and $U_2(c)$, the equilibrium market prices of risk and risk-free rate are given by*

$$\lambda = \frac{\Delta_\lambda}{\Delta}, \qquad \eta = \frac{\Delta_\eta}{\Delta}, \qquad r = \mu - \sigma\lambda, \tag{35}$$

where

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ W_1 & W_2 & 0 \end{vmatrix}, \quad \Delta_\lambda = \begin{vmatrix} a_{11} & a_{12} & b_1\psi \\ a_{21} & a_{22} & b_2\psi \\ W_1 & W_2 & \sigma(W_1 + W_2) \end{vmatrix}, \quad \Delta_\eta = \begin{vmatrix} a_{11} & a_{12} & b_1\phi \\ a_{21} & a_{22} & b_2\phi \\ W_1 & W_2 & 0 \end{vmatrix}.$$

$$a_{11} = \frac{\partial^2 V_1}{\partial W_1^2} W_1^2, \quad a_{12} = \frac{\partial^2 V_1}{\partial W_1 \partial W_2} W_1 W_2, \quad a_{13} = \frac{\partial V_1}{\partial W_1} W_1, \quad b_1 = -\frac{\partial^2 V_1}{\partial W_1 \partial X} W_1,$$

$$a_{21} = \frac{\partial^2 V_2}{\partial W_2 \partial W_1} W_2 W_1, \quad a_{22} = \frac{\partial^2 V_2}{\partial W_2^2} W_2^2, \quad a_{23} = \frac{\partial V_2}{\partial W_2} W_2, \quad b_2 = -\frac{\partial^2 V_2}{\partial W_2 \partial X} W_2.$$

The indirect utility functions of two investors, $V_1(W_1, W_2, X, t)$ and $V_2(W_1, W_2, X, t)$, are determined by the two PDEs jointly

$$\begin{aligned}
&\frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial W_1}[W_1(r + \beta_1\lambda + \delta_1\eta) - c_1] + \frac{\partial V_1}{\partial W_2}[W_2(r + \beta_2\lambda + \delta_2\eta) - c_2] + \frac{\partial V_1}{\partial X}\zeta \\
&\quad + \frac{1}{2}\frac{\partial^2 V_1}{\partial W_1^2}W_1^2(\beta_1^2 + \delta_1^2) + \frac{1}{2}\frac{\partial^2 V_1}{\partial W_2^2}W_2^2(\beta_2^2 + \delta_2^2) + \frac{1}{2}\frac{\partial^2 V_1}{\partial X^2}(\psi^2 + \phi^2) \\
&\quad + \frac{\partial^2 V_1}{\partial W_1 \partial W_2}W_1W_2(\beta_1\beta_2 + \delta_1\delta_2) + \frac{\partial^2 V_1}{\partial W_1 \partial X}W_1(\beta_1\psi + \delta_1\phi) \\
&\quad + \frac{\partial^2 V_1}{\partial W_2 \partial X}W_2(\beta_2\psi + \delta_2\phi) + p_1U_1(c_1) = 0,
\end{aligned} \tag{36}$$

$$\begin{aligned}
&\frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial W_1}[W_1(r + \beta_1\lambda + \delta_1\eta) - c_1] + \frac{\partial V_2}{\partial W_2}[W_2(r + \beta_2\lambda + \delta_2\eta) - c_2] + \frac{\partial V_2}{\partial X}\zeta \\
&\quad + \frac{1}{2}\frac{\partial^2 V_2}{\partial W_1^2}W_1^2(\beta_1^2 + \delta_1^2) + \frac{1}{2}\frac{\partial^2 V_2}{\partial W_2^2}W_2^2(\beta_2^2 + \delta_2^2) + \frac{1}{2}\frac{\partial^2 V_2}{\partial X^2}(\psi^2 + \phi^2) \\
&\quad + \frac{\partial^2 V_2}{\partial W_1 \partial W_2}W_1W_2(\beta_1\beta_2 + \delta_1\delta_2) + \frac{\partial^2 V_2}{\partial W_1 \partial X}W_1(\beta_1\psi + \delta_1\phi) \\
&\quad + \frac{\partial^2 V_2}{\partial W_2 \partial X}W_2(\beta_2\psi + \delta_2\phi) + p_2U_2(c_2) = 0
\end{aligned} \tag{37}$$

subject to final conditions, $V_1(W_1, W_2, X, T) = 0$ and $V_2(W_1, W_2, X, T) = 0$.

For investor 1, his optimal consumption strategy is given by

$$c_1 = U_1'^{-1}\left(\frac{1}{p_1}\frac{\partial V_1}{\partial W_1}\right). \tag{38}$$

His optimal investment strategy is given by

$$\beta_1 = \frac{\Delta_{\beta_1}}{\Delta}, \qquad \delta_1 = \frac{\Delta_{\delta_1}}{\Delta}, \tag{39}$$

where

$$\Delta_{\beta_1} = \begin{vmatrix} b_1\psi & a_{12} & a_{13} \\ b_2\psi & a_{22} & a_{23} \\ \sigma(W_1 + W_2) & W_2 & 0 \end{vmatrix}, \qquad \Delta_{\delta_1} = \begin{vmatrix} b_1\phi & a_{12} & a_{13} \\ b_2\phi & a_{22} & a_{23} \\ 0 & W_2 & 0 \end{vmatrix}.$$

For investor 2, his optimal consumption strategy is given by

$$c_2 = U_2'^{-1}\left(\frac{1}{p_2}\frac{\partial V_2}{\partial W_2}\right). \tag{40}$$

His optimal investment strategy is given by

$$\beta_2 = \frac{\Delta_{\beta_2}}{\Delta}, \qquad \delta_2 = \frac{\Delta_{\delta_2}}{\Delta}, \tag{41}$$

where

$$\Delta_{\beta_2} = \begin{vmatrix} a_{11} & b_1\psi & a_{13} \\ a_{21} & b_2\psi & a_{23} \\ W_1 & \sigma(W_1+W_2) & 0 \end{vmatrix}, \qquad \Delta_{\delta_2} = \begin{vmatrix} a_{11} & b_1\phi & a_{13} \\ a_{21} & b_2\phi & a_{23} \\ W_1 & 0 & 0 \end{vmatrix}.$$

Proof. See appendix D.

The equilibrium market prices of risk λ and η, and risk-free rate r depends on the two investors' indirect utility functions, V_1 and V_2, in particular their partial derivatives w.r.t. W_1, W_2 and X. The two indirect utility functions are then determined by solving the two PDEs (36) and (37) jointly, where λ, η, r, β_1, δ_1, c_1, β_2, δ_2 and c_2 need to be replaced by the relations in equations (35), (39), (38), (41) and (40). The two resulted PDEs are highly nonlinear and highly entangled. It is unlikely that one can solve them analytically.

For some specified utility functions of the two investors, the equilibrium conditions can be simplified. The dimension of independent variables can be reduced from four (W_1, W_2, X, t) to three (ω, X, t) at least, where ω is the wealth ratio. We present three simplified cases in the next three propositions.

Proposition 5 *In a production economy of two investors with logarithmic utility function, $U_1(c_1) = \ln c_1$ and $U_2(c) = \ln c_2$, and heterogeneous time preferences, the equilibrium market prices of risk and risk-free rate are given by*

$$\lambda = \sigma, \qquad \eta = 0, \qquad r = \mu - \sigma^2, \tag{42}$$

For investor k, $k = 1, 2$, his optimal consumption strategy is given by

$$c_k = \frac{p_k}{\int_t^T p_k(\tau)d\tau} W_k. \tag{43}$$

His optimal investment strategy is

$$\beta_k = \sigma, \qquad \delta_k = 0. \tag{44}$$

His indirect utility function is given by

$$V_k(W_k, X, t) = \int_t^T p_k(\tau) d\tau \ln W_k + G_k(X, t), \tag{45}$$

where the function $G_k(X,t)$ is determined by a PDE

$$\frac{\partial G_k}{\partial t} + \frac{\partial G_k}{\partial X}\zeta + \frac{1}{2}\frac{\partial^2 G_k}{\partial X^2}(\psi^2 + \phi^2) + \left(\mu - \frac{1}{2}\sigma^2\right)\int_t^T p_k(\tau)d\tau - p_k + p_k \ln \frac{p_k}{\int_t^T p_k(\tau)d\tau} = 0,$$

subject to a final condition $G_k(X,T) = 0$.

Proof. Substituting the assumed form of the optimal indirect utility function (45) into Proposition 4 gives us the required results.

The result in this proposition is quite intuitive. From Proposition 2, we know that myopic investor prices the production and state risks in the same way no matter what kind of time preference function he has. Therefore the two myopic investors with different time preferences do not interact in the production economy. Each one ignores the existence of the other one and makes optimal consumption and investment decisions only based on his own expected utility function.

If one of them is non-myopic, then the two investors will interact. This case will be discussed in the next proposition.

Proposition 6 *In a production economy of two investors with utility functions,*

$$U_1(c_1) = \ln c_1, \qquad U_2(c_2) = \frac{c_2^{(\gamma-1)/\gamma}}{\gamma - 1}, \tag{46}$$

where γ is positive but not equal to 1, the equilibrium market prices of risk and risk-free rate are given by

$$\lambda = \frac{\Delta_\lambda}{\Delta}, \qquad \eta = \frac{\Delta_\eta}{\Delta}, \qquad r = \mu - \sigma\lambda, \tag{47}$$

where

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \omega_1 & \omega_2 & 0 \end{vmatrix}, \quad \Delta_\lambda = \begin{vmatrix} a_{11} & a_{12} & b_1\psi \\ a_{21} & a_{22} & b_2\psi \\ \omega_1 & \omega_2 & \sigma \end{vmatrix}, \quad \Delta_\eta = \begin{vmatrix} a_{11} & a_{12} & b_1\phi \\ a_{21} & a_{22} & b_2\phi \\ \omega_1 & \omega_2 & 0 \end{vmatrix},$$

and

$$\begin{aligned} a_{11} &= -\int_t^T p_1(\tau)d\tau + 2\omega_1^2\omega_2\frac{\partial G}{\partial \omega_2} + \omega_1^2\omega_2^2\frac{\partial^2 G}{\partial \omega_2^2}, \\ a_{12} &= (2\omega_2 - 1)\omega_1\omega_2\frac{\partial G}{\partial \omega_2} - \omega_1^2\omega_2^2\frac{\partial^2 G}{\partial \omega_2^2}, \\ a_{13} &= \int_t^T p_1(\tau)d\tau - \omega_1\omega_2\frac{\partial G}{\partial \omega_2}, \qquad b_1 = \omega_1\omega_2\frac{\partial^2 G}{\partial \omega_2 \partial X}, \end{aligned}$$

$$\begin{aligned} a_{21} &= \left(2\omega_1 - 1 + \frac{\gamma-1}{\gamma}\right)\omega_1\omega_2\frac{\partial Q}{\partial \omega_1} - \omega_1^2\omega_2^2\frac{\partial^2 Q}{\partial \omega_1^2}, \\ a_{22} &= -\frac{\gamma-1}{\gamma^2}Q + 2\left(\omega_2 - \frac{\gamma-1}{\gamma}\right)\omega_1\omega_2\frac{\partial Q}{\partial \omega_1} + \omega_1^2\omega_2^2\frac{\partial^2 Q}{\partial \omega_1^2}, \\ a_{23} &= \frac{\gamma-1}{\gamma}Q - \omega_1\omega_2\frac{\partial Q}{\partial \omega_1}, \qquad b_2 = -\frac{\gamma-1}{\gamma}\frac{\partial Q}{\partial X} + \omega_1\omega_2\frac{\partial^2 Q}{\partial \omega_1 \partial X}. \end{aligned}$$

The two functions, $G(\omega_2, X, t)$ and $Q(\omega_1, X, t)$, are determined by the two PDEs jointly

$$\begin{aligned} &\frac{\partial G}{\partial t} + \int_t^T p_1(\tau)d\tau \left\{\mu + (\beta_1 - \sigma)\lambda + \delta_1\eta - p_1\left[\int_t^T p_1(\tau)d\tau - \omega_1\omega_2\frac{\partial G}{\partial \omega_2}\right]^{-1} - \frac{1}{2}(\beta_1^2 + \delta_1^2)\right\} \\ &+ \frac{\partial G}{\partial \omega_2}\omega_1\omega_2\left\{ (\beta_2 - \beta_1)(\lambda - \sigma) + (\delta_2 - \delta_1)\eta - p_2^\gamma\left[Q - \frac{\gamma}{\gamma-1}\omega_1\omega_2\frac{\partial Q}{\partial \omega_1}\right]^{-\gamma} \right. \\ &\qquad \left. + p_1\left[\int_t^T p_1(\tau)d\tau - \omega_1\omega_2\frac{\partial G}{\partial \omega_2}\right]^{-1}\right\} \\ &+ \frac{\partial G}{\partial X}\zeta + \frac{1}{2}\frac{\partial^2 G}{\partial \omega_2^2}\omega_1^2\omega_2^2[(\beta_1 - \beta_2)^2 + (\delta_1 - \delta_2)^2] \\ &+ \frac{1}{2}\frac{\partial^2 G}{\partial X^2}(\psi^2 + \phi^2) + \frac{\partial^2 G}{\partial \omega_2 \partial X}\omega_1\omega_2[(\beta_2 - \beta_1)\psi + (\delta_2 - \delta_1)\phi] \\ &+ p_1 \ln\left\{p_1\left[\int_t^T p_1(\tau)d\tau - \omega_1\omega_2\frac{\partial G}{\partial \omega_2}\right]^{-1}\right\} = 0, \end{aligned} \tag{48}$$

$$\frac{\partial Q}{\partial t} + \frac{\gamma-1}{\gamma} Q \left\{ \mu + (\beta_2 - \sigma)\lambda + \delta_2 \eta - p_2^\gamma \left[Q - \frac{\gamma}{\gamma-1} \omega_1 \omega_2 \frac{\partial Q}{\partial \omega_1} \right]^{-\gamma} - \frac{1}{2\gamma}(\beta_2^2 + \delta_2^2) \right\}$$
$$+ \frac{\partial Q}{\partial \omega_1} \omega_1 \omega_2 \left\{ (\beta_1 - \beta_2)(\lambda - \sigma) + (\delta_1 - \delta_2)\eta - p_1 \left[\int_t^T p_1(\tau) d\tau - \omega_1 \omega_2 \frac{\partial G}{\partial \omega_2} \right]^{-1} \right.$$
$$\left. + p_2^\gamma \left[Q - \frac{\gamma}{\gamma-1} \omega_1 \omega_2 \frac{\partial Q}{\partial \omega_1} \right]^{-\gamma} + \frac{\gamma-1}{\gamma} [\beta_2(\beta_1 - \beta_2) + \delta_2(\delta_1 - \delta_2)] \right\}$$
$$+ \frac{\partial Q}{\partial X} \left[\zeta + \frac{\gamma-1}{\gamma}(\beta_2 \psi + \delta_2 \phi) \right] + \frac{1}{2} \frac{\partial^2 Q}{\partial \omega_1^2} \omega_1^2 \omega_2^2 [(\beta_1 - \beta_2)^2 + (\delta_1 - \delta_2)^2]$$
$$+ \frac{1}{2} \frac{\partial^2 Q}{\partial X^2}(\psi^2 + \phi^2) + \frac{\partial^2 Q}{\partial \omega_1 \partial X} \omega_1 \omega_2 [(\beta_1 - \beta_2)\psi + (\delta_1 - \delta_2)\phi]$$
$$+ p_2^\gamma \left[Q - \frac{\gamma}{\gamma-1} \omega_1 \omega_2 \frac{\partial Q}{\partial \omega_1} \right]^{1-\gamma} = 0, \tag{49}$$

subject to final conditions, $G(\omega_2, X, T) = 0$ and $Q(\omega_1, X, T) = 0$.

For investor 1, his optimal consumption strategy is given by

$$c_1 = p_1 \left[\int_t^T p_1(\tau) d\tau - \omega_1 \omega_2 \frac{\partial G}{\partial \omega_2} \right]^{-1} W_1. \tag{50}$$

His optimal investment strategy is given by

$$\beta_1 = \frac{\Delta_{\beta_1}}{\Delta}, \qquad \delta_1 = \frac{\Delta_{\delta_1}}{\Delta}, \tag{51}$$

where

$$\Delta_{\beta_1} = \begin{vmatrix} b_1\psi & a_{12} & a_{13} \\ b_2\psi & a_{22} & a_{23} \\ \sigma & \omega_2 & 0 \end{vmatrix}, \qquad \Delta_{\delta_1} = \begin{vmatrix} b_1\phi & a_{12} & a_{13} \\ b_2\phi & a_{22} & a_{23} \\ 0 & \omega_2 & 0 \end{vmatrix}.$$

His indirect utility function is given by

$$V_1(W_1, \omega_2, X, t) = \int_t^T p_1(\tau) d\tau \ln W_1 + G(\omega_2, X, t). \tag{52}$$

For investor 2, his optimal consumption strategy is given by

$$c_2 = p_2^\gamma \left[Q - \frac{\gamma}{\gamma-1} \omega_1 \omega_2 \frac{\partial Q}{\partial \omega_1} \right]^{-\gamma} W_2. \tag{53}$$

His optimal investment strategy is given by

$$\beta_2 = \frac{\Delta_{\beta_2}}{\Delta}, \qquad \delta_2 = \frac{\Delta_{\delta_2}}{\Delta}, \tag{54}$$

where

$$\Delta_{\beta_2} = \begin{vmatrix} a_{11} & b_1\psi & a_{13} \\ a_{21} & b_2\psi & a_{23} \\ \omega_1 & \sigma & 0 \end{vmatrix}, \qquad \Delta_{\delta_2} = \begin{vmatrix} a_{11} & b_1\phi & a_{13} \\ a_{21} & b_2\phi & a_{23} \\ \omega_1 & 0 & 0 \end{vmatrix}.$$

His indirect utility function is given by

$$V_2(\omega_1, W_2, X, t) = Q(\omega_1, X, t)\frac{W_2^{(\gamma-1)/\gamma}}{\gamma - 1}. \tag{55}$$

Proof. See appendix E.

The myopic investor 1's indirect utility function is given by equation (52). Unlike the result in the economy of one representative investor, the level function $G(\omega_2, X, t)$ is now a function of wealth ratio ω_2. The non-myopic investor 2's indirect utility function is given by equation (55), where the coefficient function $Q(\omega_1, X, t)$ is a function of wealth ratio ω_1. Since the sum of ω_1 and ω_2 is 1, only one of them is independent. In order to keep the symmetry of the two PDEs (48, 49) for G and Q functions, we choose to use two wealth ratios ω_1 and ω_2 in the formulation and keep in mind that $\partial/\partial\omega_1 = -\partial/\partial\omega_2$. One may notice that the wealth ratio appears as a new state variable in the equilibrium conditions everywhere, including the market prices of risk and risk-free rate, and the optimal consumption and investment strategies of two investors. In order to describe the dependency of equilibrium conditions on the wealth ratio ω_1, state of technology X and time t, one needs to solve the G and Q functions from the two highly entangled nonlinear PDEs (48, 49). This task can be achieved numerically with the recent rapid development of computer technology and computational science. The details of solving the two PDEs and the financial intuitions of the numerical results will be reported in a subsequent study.

Similarly we can study the equilibrium conditions in a production economy of two non-myopic investors.

Proposition 7 *In a production economy of two investors with utility functions,*

$$U_1(c_1) = \frac{c_1^{(\gamma_1-1)/\gamma_1}}{\gamma_1 - 1}, \qquad U_2(c_2) = \frac{c_2^{(\gamma_2-1)/\gamma_2}}{\gamma_2 - 1}, \qquad \gamma_1 \neq \gamma_2, \tag{56}$$

where γ_1 and γ_2 are positive but not equal to 1, the equilibrium market prices of risk and risk-free rate are given by

$$\lambda = \frac{\Delta_\lambda}{\Delta}, \qquad \eta = \frac{\Delta_\eta}{\Delta}, \qquad r = \mu - \sigma\lambda, \tag{57}$$

where

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \omega_1 & \omega_2 & 0 \end{vmatrix}, \quad \Delta_\lambda = \begin{vmatrix} a_{11} & a_{12} & b_1\psi \\ a_{21} & a_{22} & b_2\psi \\ \omega_1 & \omega_2 & \sigma \end{vmatrix}, \quad \Delta_\eta = \begin{vmatrix} a_{11} & a_{12} & b_1\phi \\ a_{21} & a_{22} & b_2\phi \\ \omega_1 & \omega_2 & 0 \end{vmatrix},$$

and

$$\begin{aligned}
a_{11} &= -\frac{\gamma_1 - 1}{\gamma_1^2} Q_1 + 2\left(\omega_1 - \frac{\gamma_1 - 1}{\gamma_1}\right)\omega_1\omega_2 \frac{\partial Q_1}{\partial \omega_2} + \omega_1^2\omega_2^2 \frac{\partial^2 Q_1}{\partial \omega_2^2}, \\
a_{12} &= \left(2\omega_2 - 1 + \frac{\gamma_1 - 1}{\gamma_1}\right)\omega_1\omega_2 \frac{\partial Q_1}{\partial \omega_2} - \omega_1^2\omega_2^2 \frac{\partial^2 Q_1}{\partial \omega_2^2}, \\
a_{13} &= \frac{\gamma_1 - 1}{\gamma_1} Q_1 - \omega_1\omega_2 \frac{\partial Q_1}{\partial \omega_2}, \qquad b_1 = -\frac{\gamma_1 - 1}{\gamma_1}\frac{\partial Q_1}{\partial X} + \omega_1\omega_2 \frac{\partial^2 Q_1}{\partial \omega_2 \partial X},
\end{aligned}$$

$$\begin{aligned}
a_{21} &= \left(2\omega_1 - 1 + \frac{\gamma_2 - 1}{\gamma_2}\right)\omega_1\omega_2 \frac{\partial Q_2}{\partial \omega_1} - \omega_1^2\omega_2^2 \frac{\partial^2 Q_2}{\partial \omega_1^2}, \\
a_{22} &= -\frac{\gamma_2 - 1}{\gamma_2^2} Q_2 + 2\left(\omega_2 - \frac{\gamma_2 - 1}{\gamma_2}\right)\omega_1\omega_2 \frac{\partial Q_2}{\partial \omega_1} + \omega_1^2\omega_2^2 \frac{\partial^2 Q_2}{\partial \omega_1^2}, \\
a_{23} &= \frac{\gamma_2 - 1}{\gamma_2} Q_2 - \omega_1\omega_2 \frac{\partial Q_2}{\partial \omega_1}, \qquad b_2 = -\frac{\gamma_2 - 1}{\gamma_2}\frac{\partial Q_2}{\partial X} + \omega_1\omega_2 \frac{\partial^2 Q_2}{\partial \omega_1 \partial X}.
\end{aligned}$$

The two functions, $Q_1(\omega_2, X, t)$ and $Q_2(\omega_1, X, t)$, are determined by the two PDEs jointly

$$\begin{aligned}
&\frac{\partial Q_1}{\partial t} + \frac{\gamma_1 - 1}{\gamma_1} Q_1 \left\{ \mu + (\beta_1 - \sigma)\lambda + \delta_1 \eta - p_1^{\gamma_1} \left[Q_1 - \frac{\gamma_1}{\gamma_1 - 1} \omega_1 \omega_2 \frac{\partial Q_1}{\partial \omega_2} \right]^{-\gamma_1} - \frac{1}{2\gamma_1}(\beta_1^2 + \delta_1^2) \right\} \\
&\quad + \frac{\partial Q_1}{\partial \omega_2} \omega_1 \omega_2 \left\{ (\beta_2 - \beta_1)(\lambda - \sigma) + (\delta_2 - \delta_1)\eta - p_2^{\gamma_2} \left[Q_2 - \frac{\gamma_2}{\gamma_2 - 1} \omega_1 \omega_2 \frac{\partial Q_2}{\partial \omega_1} \right]^{-\gamma_2} \right. \\
&\qquad \left. + p_1^{\gamma_1} \left[Q_1 - \frac{\gamma_1}{\gamma_1 - 1} \omega_1 \omega_2 \frac{\partial Q_1}{\partial \omega_2} \right]^{-\gamma_1} + \frac{\gamma_1 - 1}{\gamma_1} [\beta_1(\beta_2 - \beta_1) + \delta_1(\delta_2 - \delta_1)] \right\} \\
&\quad + \frac{\partial Q_1}{\partial X} \left[\zeta + \frac{\gamma_1 - 1}{\gamma_1}(\beta_1 \psi + \delta_1 \phi) \right] + \frac{1}{2} \frac{\partial^2 Q_1}{\partial \omega_2^2} \omega_1^2 \omega_2^2 [(\beta_2 - \beta_1)^2 + (\delta_2 - \delta_1)^2] \\
&\quad + \frac{1}{2} \frac{\partial^2 Q_1}{\partial X^2}(\psi^2 + \phi^2) + \frac{\partial^2 Q_1}{\partial \omega_2 \partial X} \omega_1 \omega_2 [(\beta_2 - \beta_1)\psi + (\delta_2 - \delta_1)\phi] \\
&\quad + p_1^{\gamma_1} \left[Q_1 - \frac{\gamma_1}{\gamma_1 - 1} \omega_1 \omega_2 \frac{\partial Q_1}{\partial \omega_2} \right]^{1 - \gamma_1} = 0
\end{aligned} \tag{58}$$

$$\begin{aligned}
&\frac{\partial Q_2}{\partial t} + \frac{\gamma_2 - 1}{\gamma_2} Q_2 \left\{ \mu + (\beta_2 - \sigma)\lambda + \delta_2 \eta - p_2^{\gamma_2} \left[Q_2 - \frac{\gamma_2}{\gamma_2 - 1} \omega_1 \omega_2 \frac{\partial Q_2}{\partial \omega_1} \right]^{-\gamma_2} - \frac{1}{2\gamma_2}(\beta_2^2 + \delta_2^2) \right\} \\
&\quad + \frac{\partial Q_2}{\partial \omega_1} \omega_1 \omega_2 \left\{ (\beta_1 - \beta_2)(\lambda - \sigma) + (\delta_1 - \delta_2)\eta - p_1^{\gamma_1} \left[Q_1 - \frac{\gamma_1}{\gamma_1 - 1} \omega_1 \omega_2 \frac{\partial Q_1}{\partial \omega_2} \right]^{-\gamma_1} \right. \\
&\qquad \left. + p_2^{\gamma_2} \left[Q_2 - \frac{\gamma_2}{\gamma_2 - 1} \omega_1 \omega_2 \frac{\partial Q_2}{\partial \omega_1} \right]^{-\gamma_2} + \frac{\gamma_2 - 1}{\gamma_2} [\beta_2(\beta_1 - \beta_2) + \delta_2(\delta_1 - \delta_2)] \right\} \\
&\quad + \frac{\partial Q_2}{\partial X} \left[\zeta + \frac{\gamma_2 - 1}{\gamma_2}(\beta_2 \psi + \delta_2 \phi) \right] + \frac{1}{2} \frac{\partial^2 Q_2}{\partial \omega_1^2} \omega_1^2 \omega_2^2 [(\beta_1 - \beta_2)^2 + (\delta_1 - \delta_2)^2] \\
&\quad + \frac{1}{2} \frac{\partial^2 Q_2}{\partial X^2}(\psi^2 + \phi^2) + \frac{\partial^2 Q_2}{\partial \omega_1 \partial X} \omega_1 \omega_2 [(\beta_1 - \beta_2)\psi + (\delta_1 - \delta_2)\phi] \\
&\quad + p_2^{\gamma_2} \left[Q_2 - \frac{\gamma_2}{\gamma_2 - 1} \omega_1 \omega_2 \frac{\partial Q_2}{\partial \omega_1} \right]^{1 - \gamma_2} = 0,
\end{aligned} \tag{59}$$

subject to final conditions, $Q_1(\omega_2, X, T) = 0$ and $Q_2(\omega_1, X, T) = 0$.

For investor 1, his optimal consumption strategy is given by

$$c_1 = p_1^{\gamma_1} \left[Q_1 - \frac{\gamma_1}{\gamma_1 - 1} \omega_1 \omega_2 \frac{\partial Q_1}{\partial \omega_2} \right]^{-\gamma_1} W_1. \tag{60}$$

His optimal investment strategy is given by

$$\beta_1 = \frac{\Delta_{\beta_1}}{\Delta}, \qquad \delta_1 = \frac{\Delta_{\delta_1}}{\Delta}, \tag{61}$$

where

$$\Delta_{\beta_1} = \begin{vmatrix} b_1\psi & a_{12} & a_{13} \\ b_2\psi & a_{22} & a_{23} \\ \sigma & \omega_2 & 0 \end{vmatrix}, \qquad \Delta_{\delta_1} = \begin{vmatrix} b_1\phi & a_{12} & a_{13} \\ b_2\phi & a_{22} & a_{23} \\ 0 & \omega_2 & 0 \end{vmatrix}.$$

His indirect utility function is given by

$$V_1(W_1, \omega_2, X, t) = Q_1(\omega_2, X, t)\frac{W_1^{(\gamma_1-1)/\gamma_1}}{\gamma_1 - 1}. \tag{62}$$

For investor 2, his optimal consumption strategy is given by

$$c_2 = p_2^{\gamma_2}\left[Q_2 - \frac{\gamma_2}{\gamma_2 - 1}\omega_1\omega_2\frac{\partial Q_2}{\partial \omega_1}\right]^{-\gamma_2} W_2. \tag{63}$$

His optimal investment strategy is given by

$$\beta_2 = \frac{\Delta_{\beta_2}}{\Delta}, \qquad \delta_2 = \frac{\Delta_{\delta_2}}{\Delta}, \tag{64}$$

where

$$\Delta_{\beta_2} = \begin{vmatrix} a_{11} & b_1\psi & a_{13} \\ a_{21} & b_2\psi & a_{23} \\ \omega_1 & \sigma & 0 \end{vmatrix}, \qquad \Delta_{\delta_2} = \begin{vmatrix} a_{11} & b_1\phi & a_{13} \\ a_{21} & b_2\phi & a_{23} \\ \omega_1 & 0 & 0 \end{vmatrix}.$$

His indirect utility function is given by

$$V_2(\omega_1, W_2, X, t) = Q_2(\omega_1, X, t)\frac{W_2^{(\gamma_2-1)/\gamma_2}}{\gamma_2 - 1}. \tag{65}$$

Proof. See appendix F.

The indirect utility functions of two non-myopic isoelastic investors are given by (62, 65). Both coefficient functions $Q_1(\omega_2, X, t)$ and $Q_2(\omega_1, X, t)$ are functions of wealth ratio. Having in mind that only one of the two wealth ratios is independent, we use ω_2 in Q_1 and ω_1 in Q_2 in order to keep a symmetry in the two PDEs (58, 59). In fact, if one replaces subscript index 1 by 2 and 2 by 1 in equation (58), then the equation becomes equation (59). Once again the quantitative stochastic behavior of the equilibrium market prices of risk and risk-free rate relies on the numerical solution of these two highly entangled nonlinear PDEs.

For a production economy of n heterogeneous investors, the optimal indirect utility function of each investor will be a function of n wealth levels, W_1, W_2, $\cdots$, W_n, state variable X and time t, i.e.,

$$V_i = V_i(W_1, W_2, \cdots, W_n, X, t), \qquad i = 1, 2, \cdots, n.$$

The equilibrium conditions requires us to solve the n utility functions, V_1, V_2, $\cdots$ and V_n from n entangled PDEs. If the investors have isoelastic utility functions, the dimension of the problem can be reduced by 1. The n wealth levels will be replaced by $n-1$ wealth ratios because one of the wealth ratio is not independent due to the fact $\sum_{i=1}^{n} \omega_i = 1$.

We choose not to present the equilibrium conditions for n investors here because of their lengthy mathematical formulation.

4 Conclusion

Building on the work of Cox, Ingersoll and Ross (1985), Dumas (1989), and Vasicek (2005), we have developed a general method of constructing equilibrium asset prices in a production economy controlled by a state variable. We present the equilibrium conditions explicitly for two heterogeneous investors with general utility functions, and demonstrate that how these conditions might be simplified for some specified isoelastic utility functions.

Due to the intertemporal transformation of resources in the production economy, the wealth distribution among investors play an important role in the general equilibrium. The wealth levels/ratios are essentially serving as new state variables in the equilibrium conditions. These extra state variables make the problem of determining the equilibrium much more difficult to solve. As a result, for the production economy of two heterogeneous investors, we have to solve two highly entangled nonlinear partial differential equations (PDEs) in order to determine the equilibrium market prices of risks and risk-free rate. It is unlikely that one can achieve the task analytically. With the recent development of computational tools, it is possible to solve the PDEs numerically. The quantitative results and their financial intuitions will be discussed in a subsequent research.

A Proof of Proposition 1

The investor's wealth process is

$$dW = [W(r + \beta\lambda + \delta\eta) - c]dt + W\beta dy + W\delta dx.$$

The investor's problem is to determine optimal consumption and investment strategies in order to maximize his expect utility function of life time consumption. The optimal indirect utility is

$$V(W, X, t) = \max E_t \int_t^T p(s)U(c(s))ds.$$

The condition of optimality is given by the Bellman equation

$$\begin{aligned} \max_{c,\beta,\delta} \Big[& V_t + V_W(W(r + \beta\lambda + \delta\eta) - c) + V_X\zeta + \frac{1}{2}V_{WW}W^2(\beta^2 + \delta^2) \\ & + V_{WX}W(\beta\psi + \delta\phi) + \frac{1}{2}V_{XX}(\psi^2 + \phi^2) + pU(c) \Big] = 0, \end{aligned} \tag{66}$$

where the subscripts of V stands for partial derivatives. Differentiating equation (66) with respect to c gives

$$-V_W + pU'(c) = 0, \qquad \Longrightarrow \qquad c = U'^{-1}(V_W/p). \tag{67}$$

Differentiating equation (66) with respect to β and δ gives

$$V_W W\lambda + V_{WW}W^2\beta + V_{WX}W\psi = 0,$$

$$V_W W\eta + V_{WW}W^2\delta + V_{WX}W\phi = 0.$$

Applying the constraints,

$$\beta = \sigma, \qquad \delta = 0, \tag{68}$$

gives

$$\lambda = -\frac{V_{WW}}{V_W}W\sigma - \frac{V_{WX}}{V_W}\psi, \qquad \eta = -\frac{V_{WX}}{V_W}\phi.$$

Substituting the optimal consumption rule (67) and optimal investment strategy (68) into the Bellman equation (66) gives

$$V_t + V_W(\mu W - U'^{-1}(V_W/p)) + V_X\zeta + \frac{1}{2}V_{WW}W^2\sigma^2$$
$$+ V_{WX}W\sigma\psi + \frac{1}{2}V_{XX}(\psi^2+\phi^2) + pU(U'^{-1}(V_W/p)) = 0.$$

B Proof of Proposition 2

We consider an investor with a logarithmic utility function, i.e., $U(c) = \ln c$. The indirect utility function is assumed to have the form

$$V(W, X, t) = Q(X, t)\ln W + G(X, t).$$

Substituting the forms of U and V into (67) gives

$$c = \frac{p}{Q}W.$$

Substituting the optimal consumption rule, optimal investment strategy (68) and the forms of U and V into (66) gives

$$Q_t \ln W + G_t + Q\left(\mu - \frac{p}{Q}\right) + Q_X\zeta\ln W + G_X\zeta - \frac{1}{2}Q\sigma^2 + Q_X\sigma\psi$$
$$+\frac{1}{2}Q_{XX}(\psi^2+\phi^2)\ln W + \frac{1}{2}G_{XX}(\psi^2+\phi^2) + p\ln\frac{pW}{Q} = 0. \tag{69}$$

Collecting the coefficients of $\ln W$ gives an equation for $Q(X,t)$

$$Q_t + Q_X\zeta + \frac{1}{2}Q_{XX}(\psi^2+\phi^2) + p = 0.$$

Integrating the equation subject to the condition $Q(X, T) = 0$ gives

$$Q(X,t) = Q(t) = \int_t^T p(\tau)d\tau.$$

Substituting the solution of Q into (69) gives an equation for $G(X,t)$

$$G_t + G_X\zeta + \frac{1}{2}G_{XX}(\psi^2+\phi^2) + \left(\mu - \frac{1}{2}\sigma^2\right)Q - p + p\ln\frac{p}{Q} = 0,$$

subject to a final condition

$$G(X,T) = 0.$$

Since $\mu = \mu(X,t)$ and $\sigma = \sigma(X,t)$, G depends on both X and t in general, but the function G does not affect the optimal consumption strategy and wealth process

$$c = \frac{p}{\int_t^T p(\tau)d\tau}W, \qquad dW = \left[\mu - \frac{p}{\int_t^T p(\tau)d\tau}\right]W dt + \sigma W dy.$$

Substituting the solution of V into (19) gives

$$\lambda = \sigma, \qquad \eta = 0, \qquad r = \mu - \sigma^2.$$

C Proof of Proposition 3

We consider the investor with a non-logarithmic isoelastic utility function, i.e,

$$U(c) = \frac{c^{(\gamma-1)/\gamma}}{\gamma - 1}, \qquad \gamma > 0,\ \gamma \neq 1.$$

The indirect utility is assumed to have the form

$$V(W,X,t) = Q(X,t)\frac{W^{(\gamma-1)/\gamma}}{\gamma - 1}.$$

Substituting the forms of U and V into (67) gives

$$c = \frac{p^\gamma}{Q^\gamma}W.$$

Substituting the optimal consumption rule, optimal investment strategy (68) and the forms of U and V into (66) gives an equation for $Q(X,t)$

$$Q_t + \frac{\gamma-1}{\gamma}\left(\mu - \frac{1}{2\gamma}\sigma^2\right)Q + \left(\zeta + \frac{\gamma-1}{\gamma}\sigma\psi\right)Q_X + \frac{1}{2}(\psi^2 + \phi^2)Q_{XX} + \frac{1}{\gamma}\frac{p^\gamma}{Q^{\gamma-1}} = 0.$$

subject to a final condition

$$Q(X,T) = 0.$$

The wealth process is then written as

$$dW = \left[\mu - \frac{p^\gamma}{Q^\gamma}\right] W dt + \sigma W dy.$$

If $\mu - \frac{1}{2\gamma}\sigma^2$ does not depend on X, then Q, being a function of t only, can be solved

$$Q(t) = \left[\int_t^T p^\gamma(\tau) e^{(\gamma-1)\int_t^\tau (\mu - \frac{1}{2\gamma}\sigma^2) ds} d\tau\right]^{1/\gamma}.$$

Substituting the solution of V into (19) gives the required results of the market prices of risk.

D Proof of Proposition 4

The two investors' wealth processes are

$$dW_1 = [W_1(r + \beta_1\lambda + \delta_1\eta) - c_1]dt + W_1\beta_1 dy + W_1\delta_1 dx,$$
$$dW_2 = [W_2(r + \beta_2\lambda + \delta_2\eta) - c_2]dt + W_2\beta_2 dy + W_2\delta_2 dx.$$

The investor 1's problem is to determine his optimal consumption and investment strategies in order to maximize his expect utility function of life time consumption. The optimal indirect utility is

$$V_1(W_1, W_2, X, t) = \max E_t \int_t^T p_1(s) U_1(c_1(s)) ds.$$

The condition of optimality is given by the Bellman equation

$$\begin{aligned}\max_{c_1,\beta_1,\delta_1} \Bigg\{ & \frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial W_1}[W_1(r + \beta_1\lambda + \delta_1\eta) - c_1] + \frac{\partial V_1}{\partial W_2}[W_2(r + \beta_2\lambda + \delta_2\eta) - c_2] + \frac{\partial V_1}{\partial X}\zeta \\ & + \frac{1}{2}\frac{\partial^2 V_1}{\partial W_1^2} W_1^2(\beta_1^2 + \delta_1^2) + \frac{1}{2}\frac{\partial^2 V_1}{\partial W_2^2} W_2^2(\beta_2^2 + \delta_2^2) + \frac{1}{2}\frac{\partial^2 V_1}{\partial X^2}(\psi^2 + \phi^2) \\ & + \frac{\partial^2 V_1}{\partial W_1 \partial W_2} W_1 W_2(\beta_1\beta_2 + \delta_1\delta_2) + \frac{\partial^2 V_1}{\partial W_1 \partial X} W_1(\beta_1\psi + \delta_1\phi) \\ & + \frac{\partial^2 V_1}{\partial W_2 \partial X} W_2(\beta_2\psi + \delta_2\phi) + p_1 U_1(c_1) \Bigg\} = 0. \end{aligned} \tag{70}$$

Differentiating equation (70) with respect to c_1 gives

$$-\frac{\partial V_1}{\partial W_1} + p_1 U_1'(c_1) = 0, \qquad \Longrightarrow \qquad c_1 = U_1'^{-1}\left(\frac{1}{p_1}\frac{\partial V_1}{\partial W_1}\right). \tag{71}$$

Differentiating equation (70) with respect to β_1 and δ_1 gives

$$\frac{\partial V_1}{\partial W_1}W_1\lambda + \frac{\partial^2 V_1}{\partial W_1^2}W_1^2\beta_1 + \frac{\partial^2 V_1}{\partial W_1 \partial W_2}W_1W_2\beta_2 + \frac{\partial^2 V_1}{\partial W_1 \partial X}W_1\psi = 0, \tag{72}$$

$$\frac{\partial V_1}{\partial W_1}W_1\eta + \frac{\partial^2 V_1}{\partial W_1^2}W_1^2\delta_1 + \frac{\partial^2 V_1}{\partial W_1 \partial W_2}W_1W_2\delta_2 + \frac{\partial^2 V_1}{\partial W_1 \partial X}W_1\phi = 0. \tag{73}$$

The investor 2's problem is to determine his optimal consumption and investment strategies in order to maximize his expect utility function of life time consumption. The optimal indirect utility is

$$V_2(W_1, W_2, X, t) = \max E_t \int_t^T p_2(s)U_2(c_2(s))ds.$$

The condition of optimality is given by the Bellman equation

$$\begin{aligned}
\max_{c_2,\beta_2,\delta_2} \Bigg\{ & \frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial W_1}[W_1(r + \beta_1\lambda + \delta_1\eta) - c_1] + \frac{\partial V_2}{\partial W_2}[W_2(r + \beta_2\lambda + \delta_2\eta) - c_2] + \frac{\partial V_2}{\partial X}\zeta \\
& + \frac{1}{2}\frac{\partial^2 V_2}{\partial W_1^2}W_1^2(\beta_1^2 + \delta_1^2) + \frac{1}{2}\frac{\partial^2 V_2}{\partial W_2^2}W_2^2(\beta_2^2 + \delta_2^2) + \frac{1}{2}\frac{\partial^2 V_2}{\partial X^2}(\psi^2 + \phi^2) \\
& + \frac{\partial^2 V_2}{\partial W_1 \partial W_2}W_1W_2(\beta_1\beta_2 + \delta_1\delta_2) + \frac{\partial^2 V_2}{\partial W_1 \partial X}W_1(\beta_1\psi + \delta_1\phi) \\
& + \frac{\partial^2 V_2}{\partial W_2 \partial X}W_2(\beta_2\psi + \delta_2\phi) + p_2U_2(c_2)\Bigg\} = 0.
\end{aligned} \tag{74}$$

Differentiating equation (74) with respect to c_2 gives

$$-\frac{\partial V_2}{\partial W_2} + p_2 U_2'(c_2) = 0, \qquad \Longrightarrow \qquad c_2 = U_2'^{-1}\left(\frac{1}{p_2}\frac{\partial V_2}{\partial W_2}\right). \tag{75}$$

Differentiating equation (74) with respect to β_2 and δ_2 gives

$$\frac{\partial V_2}{\partial W_2}W_2\lambda + \frac{\partial^2 V_2}{\partial W_2^2}W_2^2\beta_2 + \frac{\partial^2 V_2}{\partial W_1 \partial W_2}W_1W_2\beta_1 + \frac{\partial^2 V_2}{\partial W_2 \partial X}W_2\psi = 0, \tag{76}$$

$$\frac{\partial V_2}{\partial W_2}W_2\eta + \frac{\partial^2 V_2}{\partial W_2^2}W_2^2\delta_2 + \frac{\partial^2 V_2}{\partial W_1 \partial W_2}W_1W_2\delta_1 + \frac{\partial^2 V_2}{\partial W_2 \partial X}W_2\phi = 0. \tag{77}$$

The constraints are

$$W_1\beta_1 + W_2\beta_2 = \sigma(W_1 + W_2), \tag{78}$$

$$W_1\delta_1 + W_2\delta_2 = 0. \tag{79}$$

Combining equations (72), (76) and (78) gives a linear system of three equations

$$\begin{pmatrix} \frac{\partial^2 V_1}{\partial W_1^2} W_1^2 & \frac{\partial^2 V_1}{\partial W_1 \partial W_2} W_1 W_2 & \frac{\partial V_1}{\partial W_1} W_1 \\ \frac{\partial^2 V_2}{\partial W_2 \partial W_1} W_2 W_1 & \frac{\partial^2 V_2}{\partial W_2^2} W_2^2 & \frac{\partial V_2}{\partial W_2} W_2 \\ W_1 & W_2 & 0 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \lambda \end{pmatrix} = \begin{pmatrix} -\frac{\partial^2 V_1}{\partial W_1 \partial X} W_1 \psi \\ -\frac{\partial^2 V_2}{\partial W_2 \partial X} W_2 \psi \\ \sigma(W_1 + W_2) \end{pmatrix}, \tag{80}$$

which can be solved to give β_1, β_2 and λ in Proposition 4.

Combining equations (73), (77) and (79) gives a linear system of three equations

$$\begin{pmatrix} \frac{\partial^2 V_1}{\partial W_1^2} W_1^2 & \frac{\partial^2 V_1}{\partial W_1 \partial W_2} W_1 W_2 & \frac{\partial V_1}{\partial W_1} W_1 \\ \frac{\partial^2 V_2}{\partial W_2 \partial W_1} W_2 W_1 & \frac{\partial^2 V_2}{\partial W_2^2} W_2^2 & \frac{\partial V_2}{\partial W_2} W_2 \\ W_1 & W_2 & 0 \end{pmatrix} \begin{pmatrix} \delta_1 \\ \delta_2 \\ \eta \end{pmatrix} = \begin{pmatrix} -\frac{\partial^2 V_1}{\partial W_1 \partial X} W_1 \phi \\ -\frac{\partial^2 V_2}{\partial W_2 \partial X} W_2 \phi \\ 0 \end{pmatrix}, \tag{81}$$

which can be solved to give δ_1, δ_2 and η in Proposition 4.

E Proof of Proposition 6

For the two investors with different utility functions, the two indirect utility functions are assumed to be

$$V_1(W_1, \omega_2, X, t) = \int_t^T p_1(\tau) d\tau \ln W_1 + G(\omega_2, X, t),$$

$$V_2(W_2, \omega_1, X, t) = Q(\omega_1, X, t) \frac{W_2^{(\gamma-1)/\gamma}}{\gamma - 1},$$

where ω_1 and ω_2 are wealth ratios given by

$$\omega_1 = \frac{W_1}{W_1 + W_2}, \qquad \omega_2 = \frac{W_2}{W_1 + W_2}.$$

Computing the partial derivatives, we have

$$\frac{\partial V_1}{\partial t} = -p_1 \ln W_1 + \frac{\partial G}{\partial t}, \qquad \frac{\partial V_1}{\partial W_1} W_1 = \int_t^T p_1(\tau) d\tau - \omega_1 \omega_2 \frac{\partial G}{\partial \omega_2}, \qquad \frac{\partial V_1}{\partial W_2} W_2 = \omega_1 \omega_2 \frac{\partial G}{\partial \omega_2},$$

$$\frac{\partial V_1}{\partial X} = \frac{\partial G}{\partial X}, \qquad \frac{\partial^2 V_1}{\partial W_1^2} W_1^2 = -\int_t^T p_1(\tau) d\tau + 2\omega_1^2 \omega_2 \frac{\partial G}{\partial \omega_2} + \omega_1^2 \omega_2^2 \frac{\partial^2 G}{\partial \omega_2^2},$$

$$\frac{\partial^2 V_1}{\partial W_2^2} W_2^2 = -2\omega_1\omega_2^2 \frac{\partial G}{\partial \omega_2} + \omega_1^2\omega_2^2 \frac{\partial^2 G}{\partial \omega_2^2}, \qquad \frac{\partial^2 V_1}{\partial X^2} = \frac{\partial^2 G}{\partial X^2},$$
$$\frac{\partial^2 V_1}{\partial W_1 \partial W_2} W_1 W_2 = (2\omega_2 - 1)\omega_1\omega_2 \frac{\partial G}{\partial \omega_2} - \omega_1^2\omega_2^2 \frac{\partial^2 G}{\partial \omega_2^2},$$
$$\frac{\partial^2 V_1}{\partial W_1 \partial X} W_1 = -\omega_1\omega_2 \frac{\partial^2 G}{\partial \omega_2 \partial X}, \qquad \frac{\partial^2 V_1}{\partial W_2 \partial X} W_2 = \omega_1\omega_2 \frac{\partial^2 G}{\partial \omega_2 \partial X},$$

Denoting $B = \dfrac{W_2^{(\gamma-1)/\gamma}}{\gamma - 1}$, we have

$$\frac{\partial V_2}{\partial t} = \frac{\partial Q}{\partial t} B, \qquad \frac{\partial V_2}{\partial W_2} W_2 = \left(\frac{\gamma - 1}{\gamma} Q - \omega_1\omega_2 \frac{\partial Q}{\partial \omega_1} \right) B, \qquad \frac{\partial V_2}{\partial W_1} W_1 = \omega_1\omega_2 \frac{\partial Q}{\partial \omega_1} B,$$
$$\frac{\partial V_2}{\partial X} = \frac{\partial Q}{\partial X} B, \qquad \frac{\partial^2 V_2}{\partial W_2^2} W_2^2 = \left[-\frac{\gamma - 1}{\gamma^2} Q + 2\left(\omega_2 - \frac{\gamma - 1}{\gamma} \right) \omega_1\omega_2 \frac{\partial Q}{\partial \omega_1} + \omega_1^2\omega_2^2 \frac{\partial^2 Q}{\partial \omega_1^2} \right] B,$$
$$\frac{\partial^2 V_2}{\partial W_2 \partial W_1} W_2 W_1 = \left[\left(2\omega_1 - 1 + \frac{\gamma - 1}{\gamma} \right) \omega_1\omega_2 \frac{\partial Q}{\partial \omega_1} - \omega_1^2\omega_2^2 \frac{\partial^2 Q}{\partial \omega_1^2} \right] B,$$
$$\frac{\partial^2 V_2}{\partial W_1^2} W_1^2 = \left(-2\omega_1^2\omega_2 \frac{\partial Q}{\partial \omega_1} + \omega_1^2\omega_2^2 \frac{\partial^2 Q}{\partial \omega_1^2} \right) B, \qquad \frac{\partial^2 V_2}{\partial X^2} = \frac{\partial^2 Q}{\partial X^2} B,$$
$$\frac{\partial^2 V_2}{\partial W_2 \partial X} W_2 = \left(\frac{\gamma - 1}{\gamma} \frac{\partial Q}{\partial X} - \omega_1\omega_2 \frac{\partial^2 Q}{\partial \omega_1 \partial X} \right) B, \qquad \frac{\partial^2 V_2}{\partial W_1 \partial X} W_1 = \omega_1\omega_2 \frac{\partial^2 Q}{\partial \omega_1 \partial X} B.$$

Substituting these partial derivatives into Proposition 4 gives us the results in Proposition 6.

F Proof of Proposition 7

For the two investors with different utility functions, the two indirect utility functions are assumed to be

$$V_1(W_1, \omega_2, X, t) = Q_1(\omega_2, X, t) \frac{W_1^{(\gamma_1 - 1)/\gamma_1}}{\gamma_1 - 1},$$
$$V_2(W_2, \omega_1, X, t) = Q_2(\omega_1, X, t) \frac{W_2^{(\gamma_2 - 1)/\gamma_2}}{\gamma_2 - 1},$$

where ω_1 and ω_2 are wealth ratios given by

$$\omega_1 = \frac{W_1}{W_1 + W_2}, \qquad \omega_2 = \frac{W_2}{W_1 + W_2}.$$

Denoting $B_1 = \dfrac{W_1^{(\gamma_1-1)/\gamma_1}}{\gamma_1 - 1}$, and $B_2 = \dfrac{W_2^{(\gamma_2-1)/\gamma_2}}{\gamma_2 - 1}$, we compute the partial derivatives

$$
\begin{aligned}
&\frac{\partial V_1}{\partial t} = \frac{\partial Q_1}{\partial t}B_1, \qquad \frac{\partial V_1}{\partial W_1}W_1 = \left(\frac{\gamma_1-1}{\gamma_1}Q_1 - \omega_1\omega_2\frac{\partial Q_1}{\partial \omega_2}\right)B_1,\\
&\frac{\partial V_1}{\partial W_2}W_2 = \omega_1\omega_2\frac{\partial Q_1}{\partial \omega_2}B_1, \qquad \frac{\partial V_1}{\partial X} = \frac{\partial Q_1}{\partial X}B_1,\\
&\frac{\partial^2 V_1}{\partial W_1^2}W_1^2 = \left[-\frac{\gamma_1-1}{\gamma_1^2}Q_1 + 2\left(\omega_1 - \frac{\gamma_1-1}{\gamma_1}\right)\omega_1\omega_2\frac{\partial Q_1}{\partial \omega_2} + \omega_1^2\omega_2^2\frac{\partial^2 Q_1}{\partial \omega_2^2}\right]B_1,\\
&\frac{\partial^2 V_1}{\partial W_1 \partial W_2}W_1W_2 = \left[\left(2\omega_2 - 1 + \frac{\gamma_1-1}{\gamma_1}\right)\omega_1\omega_2\frac{\partial Q_1}{\partial \omega_2} - \omega_1^2\omega_2^2\frac{\partial^2 Q_1}{\partial \omega_2^2}\right]B_1,\\
&\frac{\partial^2 V_1}{\partial W_2^2}W_2^2 = \left(-2\omega_2^2\omega_1\frac{\partial Q_1}{\partial \omega_2} + \omega_1^2\omega_2^2\frac{\partial^2 Q_1}{\partial \omega_2^2}\right)B_1, \qquad \frac{\partial^2 V_1}{\partial X^2} = \frac{\partial^2 Q_1}{\partial X^2}B_1,\\
&\frac{\partial^2 V_1}{\partial W_1 \partial X}W_1 = \left(\frac{\gamma_1-1}{\gamma_1}\frac{\partial Q_1}{\partial X} - \omega_1\omega_2\frac{\partial^2 Q_1}{\partial \omega_2 \partial X}\right)B_1, \qquad \frac{\partial^2 V_1}{\partial W_2 \partial X}W_2 = \omega_1\omega_2\frac{\partial^2 Q_1}{\partial \omega_2 \partial X}B_1.
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial V_2}{\partial t} = \frac{\partial Q_2}{\partial t}B_2, \qquad \frac{\partial V_2}{\partial W_2}W_2 = \left(\frac{\gamma_2-1}{\gamma_2}Q_2 - \omega_1\omega_2\frac{\partial Q_2}{\partial \omega_1}\right)B_2,\\
&\frac{\partial V_2}{\partial W_1}W_1 = \omega_1\omega_2\frac{\partial Q_2}{\partial \omega_1}B_2, \qquad \frac{\partial V_2}{\partial X} = \frac{\partial Q_2}{\partial X}B_2,\\
&\frac{\partial^2 V_2}{\partial W_2^2}W_2^2 = \left[-\frac{\gamma_2-1}{\gamma_2^2}Q_2 + 2\left(\omega_2 - \frac{\gamma_2-1}{\gamma_2}\right)\omega_1\omega_2\frac{\partial Q_2}{\partial \omega_1} + \omega_1^2\omega_2^2\frac{\partial^2 Q_2}{\partial \omega_1^2}\right]B_2,\\
&\frac{\partial^2 V_2}{\partial W_2 \partial W_1}W_2W_1 = \left[\left(2\omega_1 - 1 + \frac{\gamma_2-1}{\gamma_2}\right)\omega_1\omega_2\frac{\partial Q_2}{\partial \omega_1} - \omega_1^2\omega_2^2\frac{\partial^2 Q_2}{\partial \omega_1^2}\right]B_2,\\
&\frac{\partial^2 V_2}{\partial W_1^2}W_1^2 = \left(-2\omega_1^2\omega_2\frac{\partial Q_2}{\partial \omega_1} + \omega_1^2\omega_2^2\frac{\partial^2 Q_2}{\partial \omega_1^2}\right)B_2, \qquad \frac{\partial^2 V_2}{\partial X^2} = \frac{\partial^2 Q_2}{\partial X^2}B_2,\\
&\frac{\partial^2 V_2}{\partial W_2 \partial X}W_2 = \left(\frac{\gamma_2-1}{\gamma_2}\frac{\partial Q_2}{\partial X} - \omega_1\omega_2\frac{\partial^2 Q_2}{\partial \omega_1 \partial X}\right)B_2, \qquad \frac{\partial^2 V_2}{\partial W_1 \partial X}W_1 = \omega_1\omega_2\frac{\partial^2 Q_2}{\partial \omega_1 \partial X}B_2.
\end{aligned}
$$

Substituting these partial derivatives into Proposition 4 gives us the results in Proposition 7.

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