Time-dependent Volatility Multi-stage Compound Real Option Model and Application*

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Abstract

The simple compound option model has many limitations when applied in practice. The research on compound option theory mainly focuses on two directions. One is the extension from two-stage to multi-stage, and the other is the modification of the stochastic difference equations which describe the movement of underlying asset value. This paper extends the simple compound option model in both two directions and proposes the Time-dependent Volatility Multi-stage Compound Real Option Model. Due to the introduction of time-dependent volatility, it is difficult to derive the closed-form solution by the traditional analytical approach. This paper presents the pricing governing partial differential equation, proposes the boundary conditions and terminal conditions, and then gets the numerical solution by Finite Differential Methods. Finally this paper applies Time-dependent Volatility Multi-stage Compound Real Option Model to evaluate venture capital investment. The numerical results show that the Fixed Volatility Multi-stage Compound Real Option Model underestimates significantly the intrinsic value of venture capital investment as well as exercise threshold of later stages, but overestimates the exercise threshold of earlier stages.

Keywords: Time-dependent volatility, Multi-stage compound real option, Contingent Claim Analysis, Finite Differential Method, Venture capital investment

1 INTRODUCTION

Compound option is option on option[1]. Research about compound option model began with the path breaking work on option pricing of Black and Scholes. They view the stock as option on the firm value, and if the firm value can be viewed as the option of the liability of firm, then the stock can be modelled as the compound option on liability of firm. Since Geske derived the closed form solution for the simple two-stage compound European option model[2][3], compound option model has been widely used. In essence, compound option model reflects one right sequence which compounds each other. So it is suitable to describe problems involved sequential decision. Generally R&D projects have multi-stage nature[4]. Only when the research or management goal of earlier stage is achieved, project can enter the next stage. Venture capital investment is another typical multi-stage investment[5]. If the given goal during the operation is not achieved, the venture capitalist can cancel the investment of next stage. Another multi-stage decision example is the firm strategy decision[6]. In fact, when the management make the strategy, they are not only interested in the direct predictable cash flow, but also the potential future investment opportunities

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accompanying the current investment, which generate considerable cash flow in the future or make the firm stay in favour competition position. The above three examples are full of management flexibility and strategy flexibility. Such multi-stage decision problems can be modelled as compound option.

The simple compound option model is based on the Black-Scholes framework. The assumption that the underlying asset value follows Geometric Brown Motion is very strict. Furthermore, the two-stage setting seems too simple. In practice, the simple compound option model has many limitations. The current research about compound option model mainly focus on the development of simple compound option model in two directions. One is the extension from two-stage to multi-stage. Dixit and Pindyck make use of multi-stage compound option to model the sequential investment[7]. They adopt Dynamic Programming and Contingent Claims Analysis (CCA) to derive the governing partial differential equation(PDE), and then give the analytical solution for value function and exercise threshold of compound option under some specific boundary conditions. However, numerical methods are required to solve such a two dimensions parabolic PDE under most conditions. Alvarez and Stenbacka develop a mathematical approach based on the Green representation of Markovian functionals to find the value function of compound option and the optimal exercise rules[8]. Lin extends directly the conclusion of simple compound option to multi-stage compound real option, presented the closed form solution, and compared several numerical calculating method for the solution[5]. The main shortcoming lies in the existence of nested high-dimension integral. Since the computing complexity and cost increase rapidly with the total stage number, the traditional analytical approach is hardly used, and it is difficult to derive the closed form solution. The calculation costs large computing resource.

The other research direction is the improvement of stochastic differential equations which describe the movement of the underlying asset value. In the simple situation, model only consider single underlying asset, and assume that the movement of underlying asset value can be modelled as Geometric Brownian Motion. However, in the multi-stage situation, the sensitivity of the underlying motion parameters is amplified and then the simple assumption of fixed volatility and fixed return rates appears more unpractical. Buraschi and Dumas derive a solution for the valuation of compound options when the underlying asset value follows a general diffusion process[9]. The solution can be expressed as a forward integral of the price surface of European plain vanilla options. Geman, El Karoui[10] and Rochet, and Elettra and Rossella[11] focus on relaxation of the Geometric Brownian Motion assumption and introduce the time-dependent volatility. In their model they also consider the time-dependent interest rate and extend the model into two underlying asset situation. They derive the analytical solution for two-stage European compound option. However the solution still includes high-dimension integrals. Herath and Park extend the binomial lattice framework to model a multi-stage investment as a compound real option on several uncorrelated underlying variables[12]. However, they don’t consider the accuracy and convergence of the numerical solution, and the calculation of the exercise threshold.

Only in few specific situations we can derive the analytical solution for compound option model. Many researcher adopted modern numerical technique to find the solution. Trigeorgis presents a numerical method called Log-Transformed Binomial Numerical Analysis Method[13], to value complex investments with multiple interacting options, including compound option. The method can achieve good consistency, stability, and efficiency. Breen presents Accelerated Binomial Option Pricing Model based on the binomial and Geske-Johnson models[14], which is faster than the conventional binomial model and applicable to a wide range of option pricing problems. Though the numerical methods based on Binomial Option Pricing Model is easy to use, but in multi-stage situation the convergence of numerical solution is very slow and the computing cost will be very large. In fact, there are many other available numerical techniques, such as Finite Differential Method (FDM), Finite Elements Method, and etc, which are applicable for more complex problems and achieve better accuracy, convergence and stability.

In this paper we develop the simple compound option both in two directions. We introduce time-dependent volatility into the multi-stage compound real option model, based on the Fixed Volatility Multi-stage Compound Real Option Model (FV-MCROM) presented by Lin[5]. Because of the introduction of the time-dependent volatility, it is hard to derive the analytical solution for the Time-dependent Volatility Multi-stage Compound Real Option Model (TV-MCROM). We use CCA to establish the governing PDE, and then propose the boundary conditions and terminal conditions. But in this paper we don’t try to derive the analytical solution, but apply FDM to find the numerical solution. Finally we present one application of the TV-MCROM in venture capital investment evaluation, in which we can conclude that such a development does really make sense.

In the next section we briefly review the FV-MCROM. And section 3 proposes the TV-MCROM and presents the governing PDE, terminal conditions and boundary conditions. We detail the solving procedure of PDE by FDM in section 4. In Section 5 we apply the TV-MCROM to evaluate the venture capital investment with. Section 6 concludes.

2 REVIEWS OF THE FIXED VOLATILITY MULTI-STAGE COMPOUND REAL OPTION MODEL

The right sequence of investor embedded in a multi-stage project can be viewed as a series of real option. Its value consists of two components: one is value of current direct cash flow; the other is value of following investment right. In each stage the investor receives the cash flow of that stage and at the same time decides whether he will exercise the real option or not. If exercising, the investor will purchase the real option, i.e. investment right, of next stage at a certain exercise price, i.e. investment outlay, thus the project continues. If not, the investor keeps the investment outlay and abandons the following investment right, thus the project is abandoned. The investor repeats this decision procedure till the end of the project. Such right sequence can be modelled as a multi-stage compound real option. Lin uses FV-MCROM to evaluate an investment project involving high-tech industry[5]. In this model Lin supposes that the decision-making time points are given ahead, and decisions can only be made at these time points. Furthermore, Lin supposes that cash flow only occurs at maturity.

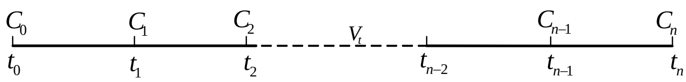

Figure 1. Fixed Volatility Multi-stage Decision Model

Figure 1 shows the fixed volatility multi-stage decision model, in which $t_k, I_k (k = 0,\dots,n)$ represent respectively the given decision-making time point and the planned investment outlay at that time point; V_t represents the underlying project value at t; C_k represents the payoff function of the investor at t_k: $C_k = max(F_k - I_k, 0)\ (k = 0,\dots,n-1)$, where F_k represents real option value, i.e. investment right, at k-*th* stage, i.e. between t_k and t_{k+1}. If $F_k \geq I_k$, investor will pay I_k to purchase the investment real option of next stage, with value of F_k; or he will abandon the project. $C_n = max(V_n - I_n, 0)$ represents the terminal payoff of the investor. If $V_n \ge I_n$, he will pay I_n to purchase underlying asset with a value of V_n; or he will abandon the project.

Lin assumes that the underlying project value followed a Geometric Brownian Motion, i.e. $dV_t / V_t = \alpha_V dt + \sigma_V dz$, where α_V and σ_V represent separately the instantaneous expected return rate and the instantaneous volatility rate of V_t, dz represents the standard Wiener produce. Lin assumed that there existed equivalent traded "twin security" in the market, which has the same risk nature as the underlying project. Between the return rate of the "twin security" and that of the project value there existed return rate shortfall, δ. From CAPM, it can be verified that δ must satisfies: $\delta = r + \rho_{VM} \lambda_M \sigma_V - \alpha_V$, where r represents the riskfree interest rate, ρ_{VM} represents the correlation coefficient between the return rate of "twin security" and that of market portfolio, $\lambda_M \equiv (\alpha_M - r)/\sigma_M$ represents the market price of risk of market portfolio, α_M and σ_M represent separately the expected return rate and the instantaneous volatility rate of market portfolio. According to Risk Neutral Pricing Theory, the natural measure was transformed into the risk neutral measure. In the risk neutral world, the expected return rate of any asset (tradable or non-tradable) is exactly the riskfree interest rate[15]. So the current value of any asset is the discounted value of expected future cash flow at the riskfree interest rate. Since all real options, $F_k(\cdot), (k = 0, \ldots, n-1)$, are European style contingent claims written on V_t, we can get the value of real options, by discounting the terminal payoff backward stage by stage. Lin presents the closed form solution:

$$F_i = V_i e^{-\delta(t_n - t_i)} \Phi_{n-i}(H_{n-i}; R_{n-i}) - \sum_{J=1}^{n-i} e^{-r\tau_J} I_{J+i} \Phi_J(K_J; R_J), \quad (i = 0, \ldots, n-1)$$

where $R_J = (R_{mn}) \in \mathbb{R}^{J \times J}$; $R_{mn} = \sqrt{\frac{\tau_m}{\tau_n}} (m = 1, 2, \cdots J; n = 1, 2, \cdots J)$; $\tau_J = t_{i+J} - t_i$;

$H_J \equiv (h_{i,1}, h_{i,2}, \ldots, h_{i,J-1}, h_{i,J})' \in \mathbb{R}^{J \times 1}$; $K_J \equiv (k_{i,1}, k_{i,2}, \ldots, k_{i,J-1}, k_{i,J})' \in \mathbb{R}^{J \times 1}$;

$$h_{i,j} = \begin{cases} \dfrac{ln\left(\dfrac{V_i}{V_{i+j}^*}\right) + (r - \delta + \frac{1}{2}\sigma_V^2)(t_{i+j} - t_i)}{\sigma_V \sqrt{t_{i+j} - t_i}} & (j = 1, 2, \ldots, n-i-1) \\ \dfrac{ln\left(\dfrac{V_i}{I_n}\right) + (r - \delta + \frac{1}{2}\sigma_V^2)(t_n - t_i)}{\sigma_V \sqrt{t_n - t_i}} & (j = n-i) \end{cases};$$

$k_{i,j} = h_{i,j} - \sigma_V \sqrt{t_{i+j} - t_i}$;

V_{i+j}^* is the exercise threshold at t_{i+j}, and satisfies $F_{i+j}(X_{i+j}) = I_{i+j}$;

$$\Phi_J(H_J; R_J) = \frac{1}{(2\pi)^{\frac{J}{2}} |R|^{1/2}} \int_{-\infty}^{h_{i,J}} \cdots \int_{-\infty}^{h_{i,1}} e^{-\frac{1}{2}x^T R^{-1} x} dx_1 \cdots dx_J, (J = 1, \ldots, n).$$

To get the final solution F_0, we need to calculate high dimension integrals: $\Phi_J\ (J = 1, \ldots, n)$, and compute the root of equations: $F_{i+j}(X_{i+j}) - I_{i+j} = 0$, which is a nonlinear nested high dimension integral function. Thus calculation will cost large computing resource. Especially when the total stage number is large, it is difficult to handle the accuracy and convergence of numerical solution.

3 TIME-DEPENDENT VOLATILITY MULTI-STAGE COMPOUND REAL OPTION MODEL: AN GENERALIZATION

Though the FV-MCROM can reflect the multi-stage nature of high-tech project investment, the fixed volatility assumption is still unreasonable. In this section we introduce time-dependent volatility to reflect the fact that the multi-stage project usually has different risk-return nature at different stages. For simplification, we suppose that the underlying asset generate cash flow only at maturity. The analysis about the situation that cash flow occurs before maturity has no essential difference in our framework. Then we also assume that the decision-making time points are given ahead and decision can be made only at those time points, at which when option value is larger than the investment outlay the option will be exercised, i.e. investment continue; or project will be abandoned.

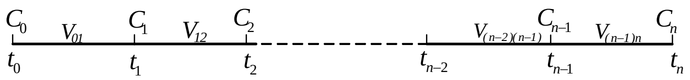

Figure 2. Time-dependent Volatility Multi-stage Decision Model

Figure 2 shows the time-dependent volatility multi-stage decision model, in which $V_{k(k+1)}(t)(k=0,\ldots,n-1)$ represent the company value at time point t at the k-th stage, $t\in[t_k,t_{k+1}]$. Other denotations are the same as above. Different from the fixed volatility assumption of Lin[5], we assume that $V_{k(k+1)}(t)$ obeys the following procedure:

$$dV_{k(k+1)}/V_{k(k+1)}=\alpha_{k(k+1)}dt+\sigma_{k(k+1)}dz_k\,, \tag{1}$$

where $\alpha_{k(k+1)}$ and $\sigma_{k(k+1)}$ represent separately the instantaneous expected return rate and the instantaneous volatility rate of $V_{k(k+1)}(t)$, which vary with k; dz_k represents the standard Wiener produce, and $var(dz_k,dz_{k'})=0$, $(k,k'=0,1,\ldots n-1;k\neq k')$. Again we assume that there exists equivalent traded "twin security" in the market. Then the return rate shortfall $\delta_{k(k+1)}$ between the return rate of the "twin security" and that of the risky company value at the k-th stage is $\delta_{k(k+1)}=r+\rho_{kM}\lambda_M\sigma_{k(k+1)}-\alpha_{k(k+1)}$, where ρ_{kM} represents the relationship coefficient between the return rate of the risky company value at the k-th stage and the market security portfolio. According to the Risk Neutral Pricing Theory, we can transform the natural measure into risk neutral measure and then equation (1) transforms as

$$dV_{k(k+1)}/V_{k(k+1)}=(r-\delta_{k(k+1)})dt+\sigma_{k(k+1)}dz$$

Since the decision-making time points are known ahead, $F_k(V_{k(k+1)},t)$ is European style contingent claim written directly on $F_{k+1}(V_{(k+1)(k+2)},t)$, and indirectly on $V_{k(k+1)}$, with maturity $T_{k+1}=t_{k+1}-t_k$. Though our model is very similar to the fixed volatility model, it is too difficult to derive the closed form solution by means of discounting the terminal payoff backward stage by stage. We firstly present the pricing governing partial differential equation (PDE) which every real option $F_k(V_{k(k+1)},t)$, $(k=0,\ldots,n-1;t\in[t_k,t_{k+1}])$ satisfy, from Contingent Claim Analysis (CCA):

$$\tfrac{1}{2}\sigma_{k(k+1)}^2V_{k(k+1)}^2\frac{\partial^2F_k}{\partial V_{k(k+1)}{}^2}+(r-\delta_{k(k+1)})V_{k(k+1)}\frac{\partial F_k}{\partial V_{k(k+1)}}+\frac{\partial F_k}{\partial t}-rF_k=0 \tag{2}$$

In compound option model the earlier option and the later option compound with each other, so the terminal conditions and the boundary conditions differ with each other. For the last stage, the real option $F_{n-1}(\cdot)$ is similar to European financial call option written on stock with continued dividend. Thus the terminal condition is

$$F_{n-1}(V_n, T_n) = max(V_n - I_n, 0) . \tag{3}$$

When the company value is zero, i.e. bankruptcy occurs, the value of the real option is zero too. Thus the lower boundary condition is

$$F_{n-1}(0, t) = 0 . \tag{4}$$

And when company value is far larger than the exercise threshold, it is almost sure that the option will be exercised. So if we don't consider the effect of dividend, the only difference between the option and the underlying asset is exercise outlay. Thus the upper boundary condition is

$$\frac{F_{n-1}(V_{(n-1)n}, t)}{[V_{(n-1)n} - I_n e^{-r(T_n - t)}] e^{-\delta_{(n-1)n}(T_n - t)}} \to 1, \quad (V_{(n-1)n} \to +\infty) \tag{5}$$

Now let's track back and consider the earlier stages ($k = n-2, n-3, \ldots, 1, 0$). As discussed previously, $F_k(\cdot)$ is European style contingent claim written indirectly on $V_{k(k+1)}$, and all impact of the following option $F_{k+1}(\cdot)$ is reflected in the terminal condition:

$$F_k(V_{k+1}, T_{k+1}) = max(F_{k+1}(V_{k+1}) - I_{k+1}, 0) \tag{6}$$

Similar to the plain vanilla option, value of compound option is increase with the underlying assets value. So from (5) we can get the exercise threshold value V_k^* of the real option $F_k(\cdot)$ $(k = 0, 1, \ldots, n-1)$, from

$$F_k(V_k^*, 0) = I_k . \tag{7}$$

Then for $F_k(\cdot)$, the upper boundary condition is

$$F_k(0, t) = 0 . \tag{8}$$

And the lower boundary condition is

$$\frac{F_k(V_{k(k+1)}, t)}{[F_{k+1}(V_{k(k+1)}, 0) - I_{k+1} e^{-r(T_{k+1} - t)}] e^{-\delta_{k(k+1)}(T_{k+1} - t)}} \to 1, \quad (V_{k(k+1)} \to +\infty) . \tag{9}$$

Since the value function of later option enter the terminal condition of earlier option, it is difficult to derive the analytical solution to our problem (2)～(9). Notice that the solving domain is regular half-zonal domain $(t, V_{k(k+1)}) \in \{[t_k, t_{k+1}], [0, +\infty]\}$, and the boundary conditions and terminal conditions are the Dirichlet boundary conditions. So it is simple and efficient to solve the model by Finite Differential Method (FDM).

4 NUMERICAL CALCULATION

In previous section we present the governing PDE, and terminal conditions and boundary conditions for the TV-MCROM. In this section we detail the numerical calculation procedure with FDM.

Firstly, let $z_{k(k+1)} = ln(V_{k(k+1)})$ and divide the domain $(t, z_{k(k+1)})$ into following uniform mesh:

$$T_{k+1} = I \Delta t \ , \quad \overline{z}_{k(k+1)} - \underline{z}_{k(k+1)} = J \Delta z_{k(k+1)} ,$$

where $t \in [t_k, t_{k+1}]$, $z_{k(k+1)} \in [\underline{z}_{k(k+1)}, \overline{z}_{k(k+1)}]$, $\overline{z}_{k(k+1)} = ln(\overline{V}_{k(k+1)})$, $\underline{z}_{k(k+1)} = ln(\underline{V}_{k(k+1)})$,

$\bar{V}_{k(k+1)}$ ($\underline{V}_{k(k+1)}$) represents the maximum (minimum) underlying asset value, which need be predetermined definitely ahead and is generally given as a very large (small) number.

We adopt following implicit differential scheme:

$$\frac{\partial F_k}{\partial t} \approx \frac{F_{k,j}^{i+1} - F_{k,j}^{i}}{\Delta t};\ \frac{\partial F_k}{\partial z_k} \approx \frac{F_{k,j+1}^{i} - F_{k,j-1}^{i}}{2\Delta z_{k(k+1)}};\ \frac{\partial^2 F_k}{\partial z_{k(k+1)}{}^2} \approx \frac{F_{k,j+1}^{i} - 2F_{k,j}^{i} + F_{k,j-1}^{i}}{\Delta z_{k(k+1)}{}^2},$$

where $F_{k,j}^{i} = F_k(\underline{z}_{k(k+1)} + j\Delta z_{k(k+1)}, i\Delta t)$. Applying the scheme into the equation system (2)~(9) and ignoring the higher order term gives

$$\alpha F_{k,j+1}^{i} + \beta F_{k,j}^{i} + \gamma F_{k,j-1}^{i} = F_{k,j}^{i+1},\ (i = 0,\cdots,I-1;\ j = 1,\cdots,J-1;\ k = 0,1,\ldots,n-1) \tag{10}$$

where, $\alpha \triangleq -\frac{\sigma_{k(k+1)}{}^2\Delta t}{2\Delta z_{k(k+1)}{}^2} - \frac{(r - \delta_{k(k+1)} - \frac{1}{2}\sigma_{k(k+1)}{}^2)}{2\Delta z_{k(k+1)}}$; $\beta \triangleq \frac{\sigma_{k(k+1)}{}^2\Delta t}{\Delta z_{k(k+1)}{}^2} + r\Delta t + 1$;

$\gamma \triangleq \frac{(r - \delta_{k(k+1)} - \frac{1}{2}\sigma_{k(k+1)}{}^2)\Delta t}{2\Delta z_{k(k+1)}} - \frac{\sigma_{k(k+1)}{}^2\Delta t}{2\Delta z_{k(k+1)}{}^2}$.

Rewriting (10) into matrix form we obtain

$$\begin{bmatrix} \beta & \gamma & & & & \\ \alpha & \beta & \gamma & & & \\ & \alpha & \beta & \gamma & & \\ & & \ddots & \ddots & \ddots & \\ & & & \alpha & \beta & \gamma \\ & & & & \alpha & \beta \end{bmatrix} \begin{pmatrix} F_{k,1}^{i} \\ F_{k,2}^{i} \\ F_{k,3}^{i} \\ \vdots \\ F_{k,J-2}^{i} \\ F_{k,J-1}^{i} \end{pmatrix} = \begin{pmatrix} F_{k,1}^{i+1} - \alpha F_{k,0}^{i} \\ F_{k,2}^{i+1} \\ F_{k,3}^{i} \\ \vdots \\ F_{k,J-2}^{i+1} \\ F_{k,J-1}^{i+1} - F_{k,J}^{i} \end{pmatrix},\quad (i = 0,\cdots,I-1) \tag{11}$$

The terminal condition (3) and boundary conditions (4)(5) transform as

$$F_{n-1,j}^{I} = max(e^{\underline{z}_{(n-1)n} + j\Delta z_{(n-1)n}} - I_n, 0), (j = 0,\cdots,J), \tag{12}$$

$$F_{n-1,0}^{i} = 0, (i = 0,\cdots,I), \tag{13}$$

$$F_{n-1,J}^{i} = [e^{\bar{z}_{(n-1)n}} - I_n e^{-r(T_n - i\Delta t)}]e^{-\delta_{(n-1)n}(T_n - i\Delta t)}, (i = 0,\cdots,I). \tag{14}$$

And for earlier stages $(k = n-2, n-3,\ldots,1,0)$ terminal condition (6) are changed as

$$F_{k,j}^{I} = max(F_{k+1,j}^{0} - I_{k+1}, 0), (j = 0,\cdots,J). \tag{15}$$

Here, we can calculate the threshold value

$$V_k^* = e^{z_{k(k+1)}^*} = e^{\underline{z}_{k(k+1)} + j^*\Delta z_{k(k+1)}}\ (k = 0,1,\ldots,n-1),$$

where j^* satisfies

$$F_{k,j^*}^{0} = I_k. \tag{16}$$

And boundary conditions (8)(9) transform as

$$F^{i}_{k,0} = 0, (i = 0, \cdots, I), \tag{17}$$

$$F^{i}_{k,J} = F^{0}_{k+1,J} e^{-\delta_{k(k+1)}(T_{k+1} - i\Delta t)} - I_{k+1} e^{-(\delta_{k(k+1)} + r)(T_{k+1} - i\Delta t)}, (i = 0, \cdots, I). \tag{18}$$

Thus equation systems (12)～(18) define the value of our multi-stage compound real option model along the edges of the mesh. Computing the linear equation system (11) backward stage by stage, we can get the value of the TV-MCROM in every grid node $(t, z_{k(k+1)})$. Because the differential scheme adopted in this paper is implicit differential scheme, we can achieve good stability and convergence, and error is almost zero when as $I \to \infty$ and $J \to \infty$. That is to say, the numerical solution will converges to the solution of the PDE if I and J are large enough. The method we adopted is robust[1].

5 EVALUATION OF VENTURE CAPITAL INVESTMENT: AN EXAMPLE

Venture capital investment is a typical multi-stage investment activity with high-risk and high-return. In practice, venture capital investment is generally multi-stage investment. This multi-stage pattern corresponds to the high-risk and high-return characteristic, the irreversibility of investment outlay, and the information asymmetric nature of venture capital investment.

Firstly, the multi-stage investment pattern can greatly reduce the risk of venture capitalist and bring more return. Since investment outlay is generally irreversible, once management fails capitalist can't often recover the initial investment. Under the multi-stage investment pattern, venture capitalist can make proper responses to the new arrival information. If management fails or the unfavourable situation appears, venture capitalist can reduce the investment scale, delay investment, even abandon the project to avoid further loss; when market potentiality of new product appears gradually, venture capitalist can grasp the opportunity by expanding the investment scale. So the multi-stage pattern is full of operation flexibility and strategetic flexibility.

Secondly, the multi-stage investment pattern can reduce the management risk due to asymmetric information between entrepreneur and venture capitalist. In practice, even when the risk company faces bankruptcy, entrepreneur still has incentive to maintain operation, utilizing the information asymmetry, which will bring extra risk to venture capitalist. Under the multi-stage investment pattern, venture capitalist can get more inside information about management of the project, hence reduce the management risk.

Finally the multi-stage investment pattern can give more restriction to entrepreneur. When management fails, venture capitalist will exercise the right to refuse the follow-up investment. This signal passes the inside information that the management fails, which will make it difficult for entrepreneur to win venture capital from other venture capitalists. The company will suffer from the threat that company will go into bankruptcy. That will make entrepreneur to put more effort into management and do his best to reach the management goal. So venture capital investment is generally the multi-stage investment.

Just because venture capital investment is an investment activity with high-risk and high-return and has multi-stage nature, traditional NPV method and other approaches based on discounted cash flow, which cannot reflect the flexibility which the venture capitalist can utilize when new information arrives, are not suitable to evaluate the venture capital investment. Real option

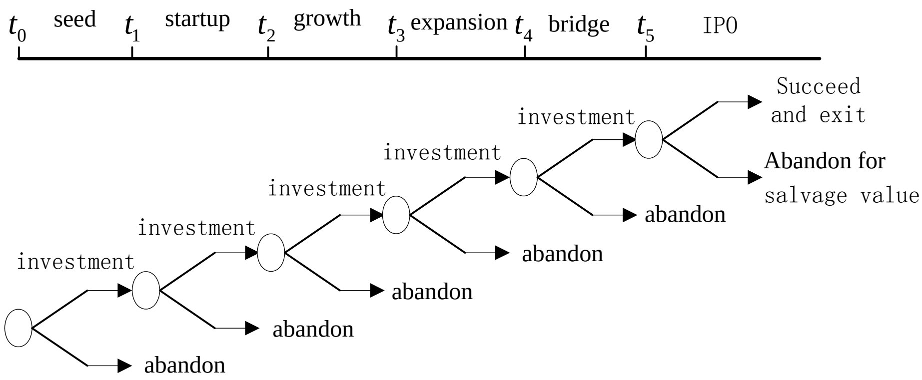

Figure 3. The multi-stage (sequential) decision procedure of venture capital investment

approach becomes a powerful tool, which reflects properly the management flexibility and strategic flexibility[7][15]. To apply real option approach to price venture capital investment, we must adopt one appropriate model which can reflect the aforementioned special nature of venture capital investment.

The existing compound real option models have limitation when they are applied to value the venture capital investment. Venture capital investment is generally divided into the many stages: the seed stage, the start up stage, the growth stage, the expansion stage and the bridge stage, as shown in Figure 3. Since the risk company has different goal and task at different stages, the risk characteristic differs at different stages. Risky company's task at early stages is R&D of new product, so the main uncertainty at initial stages is technology uncertainty. In contrast with the uncertainty in management and market exploration at late stages, that uncertainty is greater and more difficult to control. However correspondingly, once the R&D succeeds, relying on the protection of the intellectual property, company can build the key competitiveness and stay in favorable market competitive position, gain short-term excess monopoly profit and etc. So the real option pricing model of venture capital investment needs not only to reflect the high-risk and high-return and multi-stage nature, but also reflect the fact that venture capital investment has

Table 1 Selection of Parameters

Current risky firm value: V_0	[40,110]
Risk-free interest rate: r	0.0279
Instantaneous expected return rate of risky firm value: α_V	0.05
Instantaneous expected volatility of risky firm value: σ_V	[0.1,0.5,0.9]
Market price of risk of market security portfolio: λ_M	0.4
Correlation coefficient between return rate of "twin security" and that of market portfolio ρ_{VM}	0.1
Investment outlay at t_0: I_0	3
Investment outlay at t_1: I_1	5
Investment outlay at t_2: I_2	5
Investment outlay at t_3: I_3	10
Investment outlay at t_4: I_4	15
Investment outlay at t_5: I_5	30
Total stage number: n	5

different risk-return characteristics at different stages. In this section, we apply the TV-MCROM to evaluate venture capital investment.

The TV-MCROM has better adaptability. If we let all $\alpha_{i(i+1)} \equiv \alpha_V$ and $\sigma_{i(i+1)} \equiv \sigma_V$, the model will be the FV-MCROM of Lin (2002). In following analysis, we firstly compare FDM with the analytical computing approach, letting $\alpha_{i(i+1)} \equiv \alpha_V$ and $\sigma_{i(i+1)} \equiv \sigma_V$. And then we discuss the impact of introduction of time-dependent volatility to the compound real option value and the exercise threshold. Finally we perform sensitivity analysis of volatility.

Since the venture capital investment is generally divided into five stages, we suppose that n is 5, and the interval is 1.5 years. The other parameters are chosen as Table 1. The riskfree interest rate consults the present interest rate of 5 years fixed deposit of Chinese banks.

5.1 Comparison between the FDM and the Analytical Approach

In this subsection we let $\alpha_{i(i+1)} \equiv \alpha_V$ and $\sigma_{i(i+1)} \equiv \sigma_V$, and compare the numerical and analytical approach. As a reference, in analytical approach we choose the Monte-Carlo algorithm for multivariate normal probabilities proposed by Genz [16], and choose the Brent algorithm[17] to search the root, which was originated by T. Dekker[18] and combine bisection, secant and inverse quadratic interpolation methods. In the calculating procedure we make the error of each numerical integral computation less than 10^{-5} and the error of root finding less than 10^{-6}. In FDM we set

$$I=50\,,\quad J=200\,,\quad \underline{V}_{k(k+1)}=0.01\,,\quad \bar{V}_{k(k+1)}=10000\,,\quad (k=0,1,\ldots,n-1)\,.$$

Table 2 presents the results of the of compound real option value and the exercise threshold with FDM approach and analytical approach, in three cases of $\sigma_V = 0.1$, $\sigma_V = 0.5$, and $\sigma_V = 0.9$. From the results we find that the accuracy of the two approaches is very close. The absolute difference has the magnitude of 10^{-1}, and the relative difference is less than 3% on an average.

Table 2. Comparison about the accuracy in the degradation case

Volatility σ_V	Company value at beginning (V_0)	Value of option (F_0)		Exercise threshold		
		Analytical Approach	FDM		Analytical Approach	FDM
0.1	30	0.0001	0.0010	V_0^*	48.5501	47.3151
	40	0.1878	0.2596	V_1^*	50.3619	50.1187
	50	4.0175	4.0947	V_2^*	49.6486	50.1187
	60	13.5713	13.7219	V_3^*	49.1841	50.1187
	70	24.8764	24.9774	V_4^*	42.5977	42.1697
	80	36.3225	36.4129			
	90	47.7378	47.8671			
	100	59.1740	59.3233			
	110	70.6050	70.7797			
0.5	10	0.0899	0.0853	V_0^*	25.8008	25.1189
	20	1.3633	1.3414	V_1^*	31.2752	31.6228
	30	4.6918	4.6478	V_2^*	33.4310	33.4965
	40	9.2247	9.7239	V_3^*	40.6464	39.8107
	50	15.9160	16.0733	V_4^*	39.8662	39.8107
	60	24.4612	23.3145			

	70	32.4575	31.1807			
	80	38.9697	39.4950			
	90	47.6572	48.1470			
	100	57.9423	57.0472			
	110	66.1362	66.1562			
0.9	10	1.6213	1.5998	V_0^*	13.8256	14.1254
	20	6.0006	5.9505	V_1^*	18.6871	18.8365
	30	11.7001	11.687	V_2^*	20.5185	21.1349
	40	18.3880	18.206	V_3^*	29.6695	29.8538
	50	24.9698	25.219	V_4^*	33.5195	33.4965
	60	32.3129	32.573			
	70	39.5358	40.172			
	80	48.3225	47.956			
	90	55.8858	55.886			
	100	64.0545	63.929			
	110	71.7779	72.073			

However, the FDM has much advantage over the analytical approach about computational speed, as shown in Figure 4. With the analytical approach we need to compute the integral of 1 dimension, 2 dimensions, till n dimensions, and need to seek the root of integral function of 1 dimension, 2 dimensions, till $(n\text{-}1)$ dimensions. Thus the analytical approach is much slower and computing time increases rapidly with n. But the FDM need only solve some linear equation systems, the computing time is approximately linear with n. Furthermore, we can get the value of real option in all the grid nodes once. The FDM is much more efficient that the analytical approach.

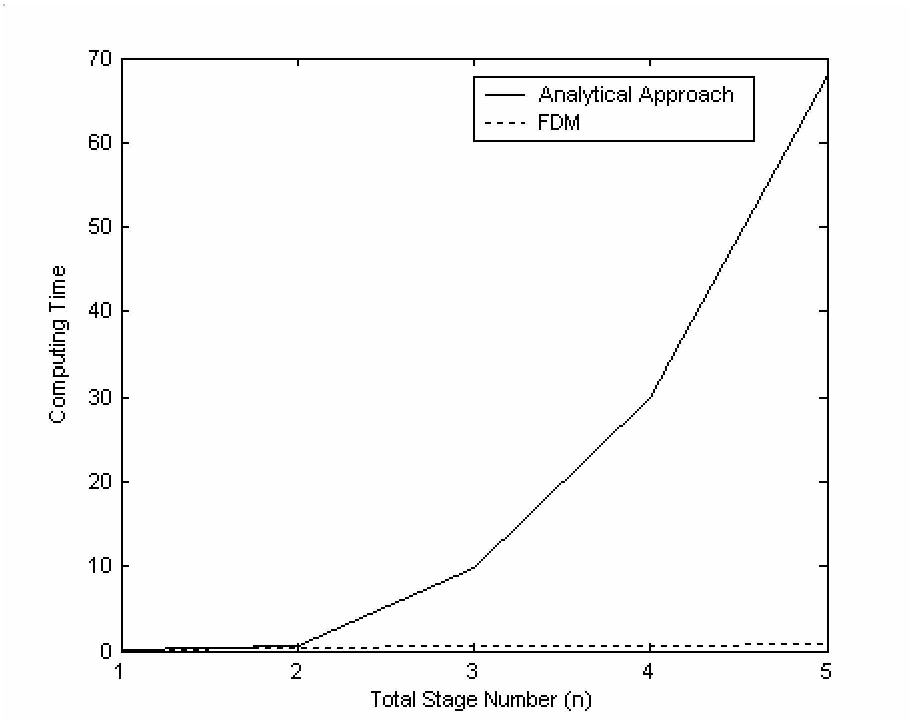

Figure 4. Comparison of computing time

5.2 Impact of Variable Volatility on Value of Real Option and Exercise Threshold

As discussed previously, in practice the risk-return characteristics of venture capital investment differs at different stages. In this subsection we suppose that the volatility of the seed stage is largest, and volatility decreases with *n*: $\sigma_{01} = 0.9$, $\sigma_{12} = 0.5$, $\sigma_{23} = 0.3$, $\sigma_{34} = 0.2$, $\sigma_{45} = 0.1$. Since the FV-MCROM cannot allow the volatility to change with *n*, we select an average volatility: $\sigma_V = 0.4$, as a benchmark. The other parameters are same as the previous subsection. The result is shown in Figure 5 and Table 2. We find that the FV-MCROM underestimates significantly the value of venture capital investment. Furthermore, we find that the FV-MCROM overestimates the exercise threshold of the earlier stages (the seed stage and the startup stage), and underestimates that of the later stages (the growth stage, the expansion stage and the bridge stage). Hence if we select the same volatility, we will set the extra barrier and lose good opportunity at the earlier stages; but at later stages we may lower the threshold and bring extra risk. So the fixed volatility multi-stage compound real option model is not suitable to price venture capital investment.

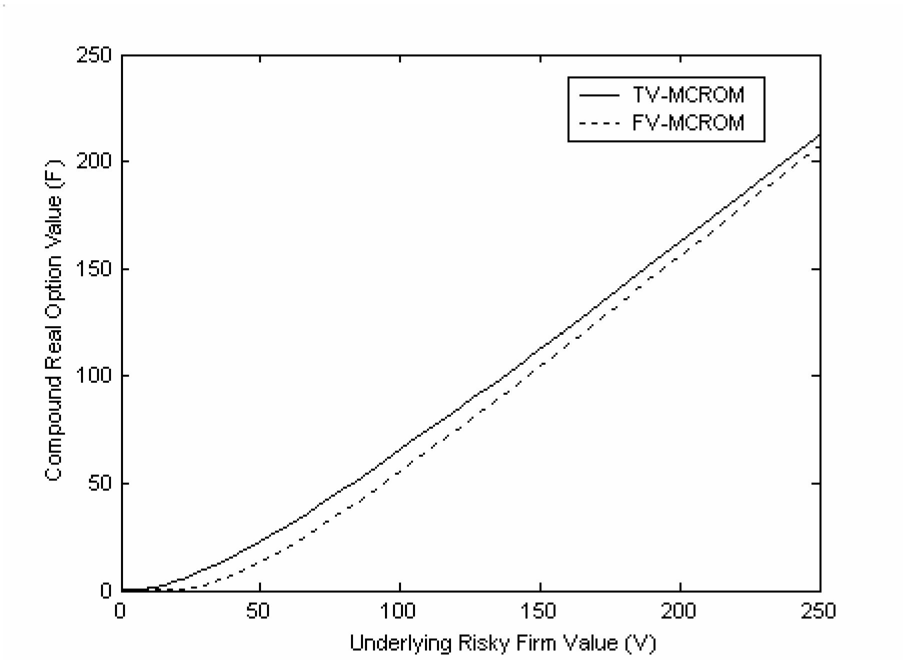

Figure 5. Comparison of TV-MCROM and FV-MCROM

Table 2. Comparison of exercise threshold

exercise threshold at	FV-MCROM	TV-MCROM
t_0	30.9030	16.7880
t_1	36.3078	33.8844
t_2	38.0189	42.6580
t_3	43.6516	48.4172
t_4	41.2098	42.6580

5.3 Sensitivity Analysis of Volatility

Lin (2002) found that the value of the FV-MCROM does not always increase with volatility. When the underlying company value is large enough, option value decreases with volatility, which is different from the conclusion in financial option theory. We also find similar observation. Consider following three cases:

Case I: $\sigma_{01} = 0.6$, $\sigma_{12} = 0.5$, $\sigma_{23} = 0.3$, $\sigma_{34} = 0.2$, $\sigma_{45} = 0.1$;

Case II: $\sigma_{01} = 0.7$, $\sigma_{12} = 0.6$, $\sigma_{23} = 0.4$, $\sigma_{34} = 0.3$, $\sigma_{45} = 0.2$;

Case III: $\sigma_{01} = 0.8$, $\sigma_{12} = 0.7$, $\sigma_{23} = 0.5$, $\sigma_{34} = 0.4$, $\sigma_{45} = 0.3$.

The value of compound real option in the three cases is shown in Figure 6. When the underlying company value is small, value of real option increases with volatility. However, when the underlying company value is very large, value of real option decreases with volatility. It is because that the underlying company is non-tradable. On the one hand, δ increase with the product of ρ and σ . And in the other hand, the greater the company value is, the greater the impact of δ on value of options is. Hence, the volatility effects the value of option in two ways at the same time . Firstly, increase of the volatility make it more possible that the company value reach to the threshold, and hence increase the value of option. Then the increase of volatility also increases δ , and hence decreases the value of the option. The final effect of volatility depends on the contrast. When the company value is small, δ has less effect on value of options. Hence in this case, value of real option increases with volatility. When the company value is very large, δ has main effect on value of options. Hence in this case, value of real option decreases with volatility.

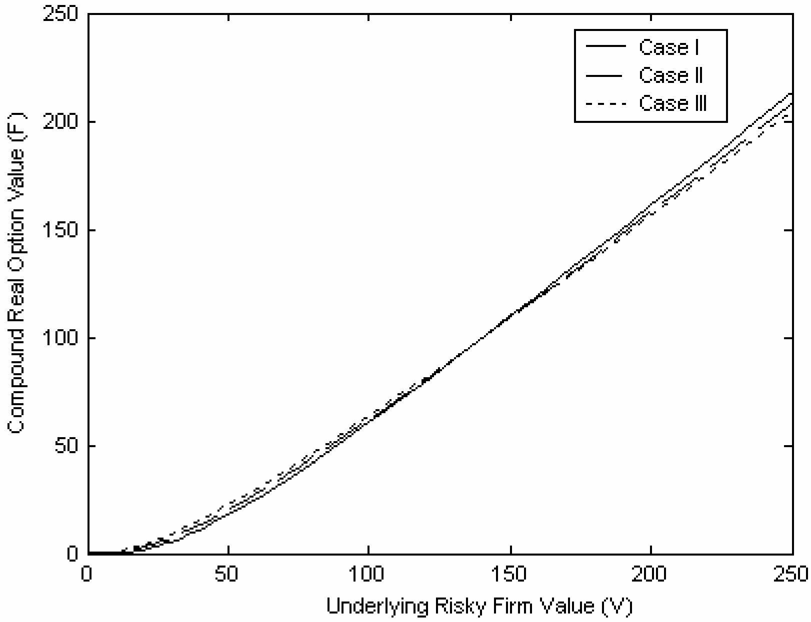

Figure 6. Sensitivity Analysis of Volatility

6 CONCLUSIONS

We compare the FDM and the analytical approach in the degradation case, and find that FDM has close accuracy as the analytical approach, and significant advantage about the computation speed. Then we find that the FV-MCROM underestimates the value of venture capital investment. In the same time it overestimates the exercise threshold of the earlier stages, which set the extra barrier and lose good opportunity in the earlier stages, and underestimates that of the later stages, which lower the threshold and bring extra risk. Finally we perform the sensitivity analysis of volatility, and find that there exists the non-monotone observation which was also emphasized by Lin, which is because that the underlying company is non-tradable asset.

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